

On the Möbius function of a non-singular binary relation

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Some of the results in the theory of Möbius functions of finite partially ordered sets are extended to arbitrary non-singular binary relations on finite sets.

1. Introduction

A systematic theory of inversion formulae on a partially ordered set was developed in Rota [2]. The purpose of this note is to indicate how certain of the key ideas in that theory, for instance Möbius function, incidence algebra and Galois connection, can be discussed in the context of a general binary relation and not just a partial order. For reasons of simplicity we discuss only finite sets and restrict ourselves to just a few instances of results analagous to those established by Rota.

2. The incidence algebra of a binary relation

Let $\rho \subseteq S \times S$ be a binary relation on a non-empty finite set S . We often write $x\rho y$ instead of $(x, y) \in \rho$. For (x, y) in $S \times S$ we write

$$[x, y] = \{z : z \in S, x\rho z \text{ \& } z\rho y\}$$

and we call $[x, y]$ a ρ -interval of S . We note the trivial statements of

LEMMA 2.1.

(i) $[x, y] \neq \square$ (the empty set) if and only if there is z in S

Received 4 May 1970.

with $x\rho z$ and $z\rho y$,

(ii) if ρ is transitive and $[x, y] \neq \square$ then $x\rho y$,

(iii) if ρ is transitive and reflexive then $[x, y] \neq \square$ whenever $x\rho y$.

We denote the (real) vector space of real-valued functions on $S \times S$ by $A = A(S)$. In fact A is a linear associative algebra with multiplication defined by

$$fg(x, y) = \sum_{z \in S} f(x, z)g(z, y)$$

for any f and g in A . The algebra A has a unique identity, namely the Kronecker delta, $\delta(x, y) = 1$ or 0 according as $x = y$ or $x \neq y$. We define the zeta-function of ρ , $\zeta = \zeta_\rho$ by the equation

$$\zeta(x, y) = \begin{cases} 1 & \text{if } (x, y) \in \rho, \\ 0 & \text{otherwise.} \end{cases}$$

We note the obvious

LEMMA 2.2.

$$\zeta^2(x, y) = \sum_{z \in S} \zeta(x, z)\zeta(z, y) = |[x, y]|.$$

We say that f in A is ρ -restricted when

$$(x, y) \notin \rho \Rightarrow f(x, y) = 0.$$

Note that the zeta-function is ρ -restricted. We write $A_\rho = A_\rho(S)$ for the subset of A consisting of its ρ -restricted elements. The set A_ρ is a vector subspace of A but it is not, in general, a subalgebra of A .

LEMMA 2.3. A_ρ is a subalgebra of A if and only if ρ is transitive.

Proof. Suppose that ρ is transitive, $(x, y) \notin \rho$ and that f and g are in A_ρ . Then for any z in S at least one of (x, z) and (z, y) is not in ρ and since f and g are in A_ρ we obtain

$$fg(x, y) = \sum_{z \in S} f(x, z)g(z, y) = 0,$$

that is fg belongs to A_ρ . Conversely suppose that A_ρ is a subalgebra of A , then ζ^2 is in A_ρ and so, by Lemma 2.2

$$[x, y] \neq \square \Rightarrow (x, y) \in \rho.$$

From this implication it follows at once that ρ is transitive.

To facilitate comparison with Rota [2] introduce another binary composition in A by writing

$$f \circ g(x, y) = \begin{cases} \sum_{z \in [x, y]} f(x, z)g(z, y), & \text{if } (x, y) \in \rho, \\ 0, & \text{otherwise,} \end{cases}$$

where we interpret the sum as 0 when (x, y) is in ρ but $[x, y] = \square$. Then $f \circ g$ is in A_ρ for each f and g in A ; in particular this is so whenever f and g are both in A_ρ . Indeed when f and g belong to A_ρ one has

$$(x, y) \in \rho \Rightarrow f \circ g(x, y) = fg(x, y),$$

and if, in addition, ρ is transitive then $fg(x, y) = 0$ when $(x, y) \notin \rho$ and so, under this additional assumption, $f \circ g = fg$ for any f and g in A_ρ . Thus when ρ is a partial order, what Rota calls the incidence algebra is just the subalgebra A_ρ . In the general case, however, A_ρ is not a subalgebra of A and the binary composition ' \circ ' is not associative.

For our purposes it is convenient to work in the algebra A and, although we shall not make further reference to the term, it is that algebra which we would call the incidence algebra of ρ .

3. Non-singular relations

If $|S| = n$ then each element f of A may be thought of as an $n \times n$ matrix, $\|f(x, y)\|$, the multiplication in A is just matrix

multiplication and the algebra A is just that of the corresponding algebra of $n \times n$ matrices. We say that the binary relation ρ is non-singular when the matrix $\zeta = \|\zeta(x, y)\|$ is a non-singular matrix, and correspondingly that ρ is singular when the same is true of the matrix ζ . When ρ is non-singular the matrix ζ has an inverse; we denote it by $\mu = \|\mu(x, y)\|$ and call the corresponding element of A the Möbius function of ρ . The well-known Möbius inversion formula is, in this context, nothing more, of course, than a matrix inversion applied to a system of linear equations. Thus if k is a real-valued function on S and

$$h(y) = \sum_{x:(x,y) \in \rho} k(x) = \sum_{x \in S} k(x)\zeta(x, y)$$

then, in matrix notation,

$$h' = k'\zeta$$

where h' , k' are row vectors (h , k being interpreted as column vectors). If ρ is non-singular then $k' = h'\mu$ where $\mu = \zeta^{-1}$, that is

$$k(x) = \sum_{y \in S} h(y)\mu(y, x).$$

This is just the Möbius inversion formula. It is to be noted that when ρ is a partial order the zeta-function is not only invertible but invertible in A_ρ ; this is no longer true in the general case, that is the Möbius function may not be ρ -restricted.

Again when ρ is a partial order the Möbius function of a ρ -interval is just the restriction to it of the Möbius function of ρ . This result is not generally true; however we note, without proof, the following

LEMMA 3.1. *If ρ is non-singular and transitive with Möbius function μ then the Möbius function of ρ restricted to a ρ -interval is the restriction of μ to that interval.*

If $\tilde{\rho}$ is the converse of the relation ρ , that is $x\tilde{\rho}y$ when $y\rho x$ then $\tilde{\rho}$ is non-singular when ρ is non-singular and its Möbius function $\tilde{\mu}$ is given by $\tilde{\mu}(x, y) = \mu(y, x)$.

Finally we remark here, and make use of the fact in Section 5, that to establish that ρ is non-singular it is sufficient to exhibit a left-inverse μ of the zeta-function ζ , for $\mu\zeta = \delta$ implies $\zeta\mu = \delta$. This is, of course, obvious in matrix notation; if A and B are $n \times n$ matrices with $AB = I$ (the identity matrix) and x is a vector with $Bx = 0$ then $x = ABx = 0$ and so B is non-singular and $A = B^{-1}$.

4. Galois connections

Let ρ_1, ρ_2 be binary relations on non-empty finite sets S_1, S_2 respectively. Correspondences $x_1 \rightarrow x_1^*$ and $x_2 \rightarrow x_2^\dagger$, from S_1 to S_2 and from S_2 to S_1 respectively are said to be a Galois connection when

$$x_1\rho_1x_2^\dagger \text{ if and only if } x_2\rho_2x_1^* .$$

When ρ_1, ρ_2 are partial orders this definition of a Galois connection is equivalent to the standard one in, for example, Rota [2]. We prove now our main result; it is

THEOREM 4.1. *Let ρ_1, ρ_2 be non-singular binary relations on non-empty finite sets S_1, S_2 respectively, and let μ_1, μ_2 be their respective Möbius functions. Suppose that $x_1 \rightarrow x_1^*$ and $x_2 \rightarrow x_2^\dagger$ is a Galois connection, then for any x_1 in S_1 and any x_2 in S_2 one has*

$$\sum_{y_1: y_1^* = x_2} \mu_1(x_1, y_1) = \sum_{y_2: y_2^\dagger = x_1} \mu_2(x_2, y_2) .$$

Proof. Denote the zeta-functions of ρ_1, ρ_2 by ζ_1, ζ_2 respectively. Since $y_1\rho_1y_2^\dagger$ if and only if $y_2\rho_2y_1^*$ we have $\zeta_2(y_2, y_1^*) = \zeta_1(y_1, y_2^\dagger)$ and hence

$$\sum_{y_1 \in S_1} \mu_1(x_1, y_1)\zeta_2(y_2, y_1^*) = \delta(x_1, y_2^\dagger) .$$

Thus

$$\begin{aligned}
 \sum_{y_1: y_1^* = x_2} \mu_1(x_1, y_1) &= \sum_{y_1 \in S_1} \mu_1(x_1, y_1) \delta(y_1^*, x_2) \\
 &= \sum_{y_1 \in S_1} \mu_1(x_1, y_1) \sum_{y_2 \in S_2} \mu_2(x_2, y_2) \zeta_2(y_2, y_1^*) \\
 &= \sum_{y_2 \in S_2} \left\{ \sum_{y_1 \in S_1} \mu_1(x_1, y_1) \zeta_2(y_2, y_1^*) \right\} \mu_2(x_2, y_2) \\
 &= \sum_{y_2 \in S_2} \delta(x_1, y_2^\dagger) \mu_2(x_2, y_2) \\
 &= \sum_{y_2: y_2^\dagger = x_1} \mu_2(x_2, y_2) .
 \end{aligned}$$

This is the desired result.

From this result one recovers Theorem 1 of Rota [2] in the following way. Suppose that ρ_1, ρ_2 are partial orders, and hence non-singular, and that S_1 has a least element 0_1 . Since

$$0_1 \rho_1 x_2^\dagger \text{ if and only if } x_2 \rho_2 0_1^*$$

it follows that S_2 has a greatest element, $1_2 = 0_1^*$.

Now suppose, in addition, that the Galois connection has the property

$$y_2^\dagger = 0_1 \text{ if and only if } y_2 = 0_1^* ,$$

then Theorem 4.1 gives

$$\mu_2(x_2, 1_2) = \sum_{y_1: y_1^* = x_2} \mu_1(0_1, y_1) ,$$

and, in particular, when S_2 also has a least element 0_2 ,

$$\mu_2(0_2, 1_2) = \sum_{y_1: y_1^* = 0_2} \mu_1(0_1, y_1) .$$

This is Rota's Theorem 1.

Replacing ρ_2 by $\tilde{\rho}_2$ one obtains the following dual version of Theorem 4.1.

THEOREM 4.2. *Let ρ_1, ρ_2 be non-singular binary relations on non-empty finite sets S_1, S_2 respectively and let μ_1, μ_2 be their respective Möbius functions. Suppose that $x_1 \rightarrow x_1^*$ and $x_2 \rightarrow x_2^\dagger$ are*

correspondences, from S_1 to S_2 and S_2 to S_1 respectively, such that

$$x_1 \rho_1 x_2^\dagger \text{ if and only if } x_1^* \rho_2 x_2 ,$$

Then for any x_1 in S_1 and any x_2 in S_2 one has

$$\sum_{y_1: y_1^* = x_2} \mu_1(x_1, y_1) = \sum_{y_2: y_2^\dagger = x_1} \mu_2(y_2, x_2) .$$

5. Closure operators

Let ρ be a binary relation on a non-empty finite set S . A function $J : S \rightarrow S$ is said to be a ρ -closure operator on S when

$$x \rho J(y) \text{ if and only if } J(x) \rho J(y) .$$

We note that when ρ is a partial order our definition of a ρ -closure operator is equivalent to the standard one in, for instance, Rota [2] where, however, the term closure relation is used. We write

$$J(S) = \{J(x) : x \in S\}$$

and

$$\rho_J = \rho \cap \{J(S) \times J(S)\} .$$

If $\rho_J \neq \square$ is a binary relation on $J(S)$, we denote its zeta-function by ζ_J . We prove

THEOREM 5.1. *Let ρ be a non-singular binary relation on a non-empty finite set S and let J be a ρ -closure operator. If ρ_J is not empty it is non-singular and its Möbius function μ_J is given by*

$$\mu_J(a, b) = \sum_{y: J(y)=b} \mu(a, y)$$

for any a, b in $J(S)$. Moreover for any b in $J(S)$ and any x in $S - J(S)$ one has

$$\sum_{y: J(y)=b} \mu(x, y) = 0 .$$

Proof. Since $x \rho J(y)$ if and only if $J(x) \rho J(y)$ one has

$$\zeta(x, J(y)) = \zeta_J(J(x), J(y)) .$$

Let a, b and c be in $J(S)$, then

$$\begin{aligned} \sum_{b \in J(S)} \left[\sum_{y: J(y)=b} \mu(a, y) \right] \zeta_J(b, c) &= \sum_{b \in J(S)} \sum_{y \in S} \mu(a, y) \delta(J(y), b) \zeta_J(b, c) \\ &= \sum_{y \in S} \mu(a, y) \zeta_J(J(y), c) \\ &= \sum_{y \in S} \mu(a, y) \zeta(y, c) \\ &= \delta(a, c) . \end{aligned}$$

This proves that

$$\sum_{y: J(y)=b} \mu(a, y)$$

is a left inverse of ζ_J and hence, because of the remark at the end of Section 3, establishes the first part of the theorem. The second part of the theorem follows from Theorem 4.2 by taking $S_1 = S$, $S_2 = J(S)$, $x_1^* = J(x_1)$ and $x_2^\dagger = x_2$. Indeed had we been willing to assume the non-singularity of ρ_J the first part of the theorem could also have been established by an appeal to Theorem 4.2.

As particular cases of Theorem 5.1 one obtains easily Theorem 4 of Crapo [1] and Proposition 2 of Section 5 of Rota [2].

References

- [1] Henry H. Crapo, "The Möbius function of a lattice", *J. Combinatorial Theory* 1 (1966), 126-131.
- [2] Gian-Carlo Rota, "On the foundations of combinatorial theory I. Theory of Möbius functions", *Z. Wahrscheinlichkeitstheorie und Verw. Gebiete* 2 (1964), 340-368.

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