ON VALUATION SUBALGEBRAS AND THEIR CENTRES

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1. Introduction, preliminaries. We shall extend results of Samuel [**19**] and Griffin [**8, 9**] about conditions which generalise the notion of valuation domain in a field. Let U be a commutative ring with identity, R a subring of U and L an R-submodule of U. The conditions we study have in common the property (EV), that the submodules $L{:}x$ $(x \in U)$ form a chain. We pay particular attention to the strongest of the conditions, viz, that L be a Manis valuation (MV) subring, i.e. having a prime ideal P such that (L, P) is a maximal pair in U (see [**19**], [**16**] and e.g. [**4**]). Such P is unique, being the union of all $L{:}x$ such that $x \notin L$, which we call $P_+(L)$ the centre of L. This set P_+ plays a key role in the study of all our valuation conditions.

The main definitions are in Section 1. In Section 2 basic results on localisation and factoring by a U-ideal are followed by applications. Samuel's result [**19**, Théorème 5] that if (R, P) is (MV) then R satisfies the Bourbaki condition (BV), defined in Section 1.2, is extended to the case where P is replaced by a finite chain of prime ideals. In Section 3 we show that if R is (BV) then prime R-ideals containing $P_+(R)$ are also (BV). The method gives new criteria for a submodule to be (BV). In Section 4, when L is (EV), we discuss the evaluation map v_L given by $v_L(x) = L{:}x$. Our main interest is in the cases $L = P_+(R)$, used in [**16**] to characterise (MV) rings, and $L = R$ mentioned in [**8**]. At the end we discuss the relation, given in [**8**], between (BV) rings and evaluations with cancellative image.

1.1. Conventions. The term subring is to imply possession of 1, whilst a subalgebra need not contain 1. Unless otherwise indicated, the term subalgebra [subring, submodule] means R-subalgebra [subring, R-submodule] of U. The notation $M{:}N$ means all x in U such that $xn \in M$ for all $n \in N$. We write $M{:}\{x\}$ as $M{:}x$. The sign $\subset$ means strict inclusion. In the context of a homomorphism $A \to B$, if $Y \subseteq B$, the contraction (i.e. inverse image) of Y is denoted $Y \cap A$, and we say that Y extends $Y \cap A$.

1.2. The valuation conditions. We define here the notions which are our main concern.

Let $f = f(X_1, \ldots, X_k)$ be the polynomial

$$\sum_{i_1=0}^{1} \cdots \sum_{i_k=0}^{1} a_{i_1 \ldots i_k} X_1^{i_1} \ldots X_k^{i_k}$$

over U. We shall denote the coefficient $a_{1 \ldots 1}$ by $c(f)$. Call f an *L-polynomial* if $a_{i_1 \ldots i_k} \in L$ when $(i_1, \ldots, i_k) \neq (1, \ldots, 1)$, and an *L-power polynomial* if $a_{i_1 \ldots i_k} \in L^{k-i_1-\cdots-i_k}$ when $(i_1, \ldots, i_k) \neq (1, \ldots, 1)$.

Let $C(L, U)$ be the set of all $c(f)$ such that f is an L-polynomial and $f(s_1, \ldots, s_k) =$

0 for some $s_1, \ldots, s_k$ in $U\backslash L$. Extending the terminology of [**11**], call L a *Bourbaki valuation* (BV) submodule if $1 \notin C(L, U)$. We remark that (BV) subrings were introduced in [**19**], and that (BV) subalgebras, studied in [**17**, **12**], were called Rand valuation subalgebras in [**18**].

A *Kirby valuation* (KV) submodule is defined by replacing "L-polynomial" by "L-power polynomial" in the above definition of (BV) submodule. (KV) submodules were introduced in [**12**].

We say that a subset N of U is (CM) if its complement $U\backslash N$ is multiplicatively closed.

Call L an *evaluation* (EV) submodule if the submodules $L{:}x$ $(x \in U)$ are totally ordered by inclusion.

Denote by Γ_L the monoid of all distinct $L{:}x$, where $x \in U$, with ordering $\leqslant$ taken to be $\subseteq$ and operation $\circ$ given by $(L{:}x)\circ(L{:}y) = L{:}xy$. Call L a *Manis valuation* (MV) submodule if $\Gamma_L\backslash\{L{:}0\}$ is a totally ordered group; by Section 1.6, this extends the usual notion for a subring.

Clearly (BV)$\Rightarrow$(CM) and, for L a subalgebra, (BV)$\Rightarrow$(KV). If L is (KV), or is a (CM) subalgebra, then it is (EV) by an argument as in the proof of [**14**, Lemma 2] (cf. [**8**] for the case when L is a (CM) subring). That (MV)$\Rightarrow$(KV) is by a standard argument as in the proofs of [**8**, Proposition 2] and [**18**, Lemma 6.4]. To summarise, we have

$$\begin{array}{ccccc} & & (\mathrm{BV}) & \Rightarrow & (\mathrm{CM}) \\ & & \Downarrow & & \Downarrow \\ (\mathrm{MV}) & \Rightarrow & (\mathrm{KV}) & \Rightarrow & (\mathrm{EV}) \end{array}$$

(assuming L a subalgebra for the downward implications). We shall see in Corollary 4.5 that (KV) with (CM) implies (BV). Note, as easy exercises, that if $1 \in L$ then (KV)$\Rightarrow$(BV) and (EV)$\Rightarrow$(CM). Also, extending a comment from [**12**, p. 95], we see that if $1 \notin L$ and L is (EV) then L is a subalgebra.

It is easy to see that $L{:}U$ is the unique U-ideal maximal in respect of being contained in L, and (cf. [**20**, Lemma 1.7]) if L is (CM) then the ideal $L{:}U$ is prime.

1.3. Components. Let S be a non-empty subset of U. The *S-component* of a subset N of U is the set $N_{[S]}$ of all x in U such that $xs \in N$ for some $s \in S$. If S is multiplicatively closed and $S^{-1}U$ is the ring of fractions of U with respect to S, denote by N_S the set of elements of $S^{-1}U$ representable in the form $s^{-1}n$ $(s \in S,\ n \in N)$. Thus $U_S = S^{-1}U$.

LEMMA 1.3.1. *If $S \subseteq R$, if S is multiplicatively closed and if N is closed under multiplication by R then $N_{[S]}$ is the inverse image $N_S \cap U$ with respect to the natural map $U \rightarrow U_S$.*

LEMMA 1.3.2. *If P is a prime ideal of R, if $S \subseteq R\backslash P$ and if S is multiplicatively closed then $P_{[S]}$ is a prime ideal of $R_{[S]}$ such that $P_{[S]} \cap R = P$.*

1.4. Centres. The component $N_{[U\backslash N]}$ is denoted by $P_+(N, U)$ or just $P_+(N)$. When R is a proper valuation domain in a field, $P_+(R)$ is the maximal ideal of R. Defining

Rad N to be all x in U such that $x^n \in N$ for some integer $n \geqslant 1$, it is easy to check the following lemma.

LEMMA 1.4.1.

(1) *$P_+(N)$ is empty if $N = U$.* (2) $1 \notin P_+(N)$.
(3) $P_+(N) \subseteq N \Leftrightarrow N$ *is (CM).* (4) $P_+(N) \supseteq N \Leftrightarrow 1 \notin N$.
(5) $P_+(N) \supseteq (\mathrm{Rad}\, N)\backslash N$. (6) *$P_+(N)$ is a (CM) subset.*
(7) $P_+(P_+(N)) = P_+(N)$. (8) $P_+(N) \supseteq (N{:}U) \Leftrightarrow N \neq U$.
(9) *If N is closed under multiplication by R then so is $P_+(N)$.*

1.5. Evaluation maps. An *evaluation* on U will mean a map $v: U \to \Gamma$, taken (for convenience only) to be onto, where Γ is a totally ordered abelian monoid with operation $\circ$ and neutral element 0, such that $v(xy) = v(x) \circ v(y)$, $v(x+y) \geqslant \min\{v(x), v(y)\}$ and $v(1) = 0$. We denote $v(0_U)$ by ∞, and call v *cancellative* if $\Gamma\backslash\{\infty\}$ is a cancellative monoid (or is empty).

For evaluations v and w on U, we write $v \gtrsim w$ when $w = \theta v$ for some epimorphism $\theta: v(U) \to w(U)$, and write $v \cong w$ if θ is an isomorphism. We put

$$\mathcal{R}(v) = \{x \in U \mid v(x) \geqslant 0\} \quad \text{and} \quad \mathcal{P}(v) = \{x \in U \mid v(x) > 0\}.$$

Note that $\Re(v)$ is (cf. [**8**]) a (CM) subring of which $\mathcal{P}(v)$ is, if not empty, a (CM) ideal. A non-empty subset J of U is *v-closed* if $j \in J$ and $v(x) \geqslant v(j)$ imply $x \in J$. The v-closed subsets form a chain of $\mathcal{R}(v)$-submodules with smallest member $v^{-1}(\infty)$.

Suppose that the submodule L is (EV). Then the map $v_L: U \to \Gamma_L$ defined by $v_L(x) = L{:}x$ is an evaluation.

LEMMA 1.5.1. *The set of all y such that $v_L(y) \geqslant v_L(x)$ is $L{:}(L{:}x)$.*

Taking in turn $x = 1$ and $x = 0$, we obtain $\mathcal{R}(v_L) = L{:}L$ and $v_L^{-1}(\infty) = L{:}U$. Also, note that $\mathcal{P}(v_L) = P_+(L)$.

LEMMA 1.5.2. *The subring $L{:}L$ and, for $L \neq U$, its ideal $P_+(L, U)$ are both (CM), hence (EV). Also, $P_+(L, U) \supseteq P_+((L{:}L), U)$.*

Proof. If $xy \in L{:}L$ and $y \notin L{:}L$ then $v_L(x) > 0$. Thus $P_+(L{:}L) \subseteq P_+(L)$. The rest is clear.

1.6. Conditions for a submodule to be (MV). For a prime R-ideal P, by [**16**, Proposition 1], (R, P) is a maximal pair in U if and only if, for all y in $U\backslash R$, there exists x in P such that $xy \in R\backslash P$. These conditions imply $P = P_+(R, U)$ by [**15**, Corollary 10.5].

LEMMA 1.6.1. (i) *The submodule L is (MV) if and only if it is (EV) and, for all y in $U\backslash L$, there exists x in U such that $xy \in (L{:}L)\backslash P_+(L, U)$.*

(ii) *If L is (MV) then $L{:}L$ is (MV) and, unless L is a proper U-ideal, $P_+(L, U) = P_+((L{:}L), U)$ and $v_L \cong v_{L:L}$.*

(iii) *When L is a ring, the following are equivalent:* (1) *L is (MV)*, (2) *$(L, P_+(L, U))$ is a maximal pair*, (3) *for all y in $U \backslash L$, there exists x in L such that $xy \in L \backslash P_+(L, U)$.*

Proof. (i) is clear since $L{:}xy = L{:}1$ precisely when $xy \in (L{:}L) \backslash P_+(L, U)$. For (ii), by [**16**, Proposition 1], v_L determines the maximal pair $((L{:}L), P_+(L))$ (note that this gives (1)$\Rightarrow$(2) of (iii)) and so $v_{L:L}$ determines the maximal pair $((L{:}L), P_+(L{:}L))$. By uniqueness, $P_+(L) = P_+(L{:}L)$ and so $v_L \cong v_{L:L}$ by [**16**, Proposition 2]. For (3)$\Rightarrow$(1) of (iii), let $ab \in L$ and $a \in U \backslash L$. Then $xa \in L \backslash P_+(L, U)$ for some x in L. But $xab \in L$; hence $b \in L$. Thus L is (CM) and so, by (i), is (MV).

Clearly, if L is a U-ideal it is (MV) precisely if it is prime.

THEOREM 1.6.2. *Suppose L is not a U-ideal. Then L is (MV) if and only if it is a v_A-closed A-submodule for some (MV) subring A.*

Proof. Clearly L is v_L-closed; so "only if" follows from Lemma 1.6.1(ii). For "if", let $a \notin L{:}U$. Since L is v_A-closed, $L \supseteq A{:}U$; whence $A{:}U = L{:}U$ and there exists b such that $v_A(ba) = 0$. From $v_A(bax) \geqslant v_A(x)$ for all x in L we obtain $baL \subseteq L$. Also, if $bay \in L$ then $y \in L$ since $v_A(y) \geqslant v_A(bay)$. Thus $ba \in (L{:}L) \backslash P_+(L)$ or $L{:}ba = L{:}1$. Finally, suppose $c \in (L{:}a) \backslash (L{:}b)$. Since L is v_A-closed, $v_A(cb) < v_A(ca)$; whence $v_A(b) < v_A(a)$. Then $v_A(xb) \leqslant v_A(xa)$ for all x, and so $x \in L{:}b$ implies $x \in L{:}a$.

2. Localisation, factoring by an ideal and applications. Let S be a non-empty multiplicatively closed subset of R.

LEMMA 2.1. *Write $P_+(L_{[S]}, U) = N$. Then S does not meet N and $N_{[S]} = N$.*

Proof. If $s \in N$, we have $sy \in L_{[S]}$ for some $y \notin L_{[S]}$, and $s \in S$ would give $y \in L_{[S]}$. So $N \cap S$ is empty and $N_{[S]} \subseteq P_+(N)$. But $P_+(N) = N$ by Lemma 1.4.1(7), and, applying 1.4.1(9) to $L_{[S]}$, we obtain $N \subseteq N_{[S]}$, whence $N = N_{[S]}$.

In [**3**] it was shown that (R, P) is (MV) in the total quotient ring T of R if and only if R_S is (MV) in T_S, where $S = R \backslash P$. More generally we have the following result.

PROPOSITION 2.2. (1) *$P_+(L_S, U_S) \subseteq (P_+(L, U))_S$ with equality if S does not meet $P_+(L, U)$, i.e. if $L_{[S]} = L$. Also, $P_+(L_S, U_S) = (P_+(L_{[S]}, U))_S$ and $U \cap P_+(L_S, U_S) = P_+(L_{[S]}, U)$.*

(2) *Let $\mathscr{C}$ denote one of the terms:* (CM), (BV), (EV), (MV). *Amongst the conditions*

$$\text{(i) } L \text{ is } \mathscr{C} \text{ in } U, \text{ (ii) } L_{[S]} \text{ is } \mathscr{C} \text{ in } U, \text{ (iii) } L_S \text{ is } \mathscr{C} \text{ in } U_S,$$

we have the implications (i)$\Rightarrow$(iii) *and* (ii)$\Leftrightarrow$(iii).

Proof. For (1), let $xs^{-1} \notin L_S$ and $xs^{-1}yt^{-1} \in L_S$, where $s, t \in S$. Then $xys_1 \in L$ for some $s_1 \in S$. But $x \notin L$; hence $ys_1 \in P_+(L)$ and so $yt^{-1} \in (P_+(L))_S$. Thus $P_+(L_S) \subseteq (P_+(L))_S$. Note also, for (2), that if L is (CM) then $P_+(L) \subseteq L$; so that $ys_1 \in L$ and $yt^{-1} \in L_S$; whence L_S is (CM).

Let $s \in S$ and $m \in P_+(L)$. Then $bm \in L$ for some $b \in U \backslash L$. When $L = L_{[S]}$, the image of b in U_S is not in L_S. Since $bms^{-1} \in L_S$, we then have $ms^{-1} \in P_+(L_S)$. Thus $P_+(L_S) = (P_+(L))_S$. Replacing L by $L_{[S]}$ gives $P_+(L_S) = (P_+(L_{[S]}))_S$ since S does not meet $P_+(L_{[S]})$. By 1.4.1(9), 1.3.1 and 2.1, we now obtain $U \cap P_+(L_S) = P_+(L_{[S]})$.

If L is (BV) then so is L_S by a variant of the proof of [**13**, Proposition 4]. From the identity $(L:Rx)_S = L_S:(Rx)_S$, we deduce that if L is (EV) then so is L_S. Let $\Gamma_L\backslash\{\infty\}$ be a group, and let $L_S:xy^{-1} \subset U$. Then $L:x \subset U$ so that, for some a, $L:ax = L$ and $L_S:(axy^{-1}) = L_S$. We have now proved (i)$\Rightarrow$(iii), and (ii)$\Rightarrow$(iii) follows.

(iii)$\Rightarrow$(ii). Clearly the (CM) and (BV) properties are preserved upon taking inverse images with respect to $U \to U_S$. Note that $L_S:(Rx)_S \subseteq L_S:(Ry)_S$ implies, by 1.3.1, that $(L:Rx)_{[S]} \subseteq (L:Ry)_{[S]}$ or $L_{[S]}:Rx \subseteq L_{[S]}:Ry$. Hence $L_{[S]}$ is (EV) if L_S is. Proof of the (MV) case is in the same vein as the foregoing.

Proposition 2.3. *Let K be an ideal of U for which $K \subseteq L$ (so that $K \subseteq L:U$).*

(i) *$P_+(L/K, U/K)$ is the set of cosets $(P_+(L, U))/K$.*

(ii) *Let $\mathscr{C}$ be one of: (CM), (BV), (EV), (MV). Then L is $\mathscr{C}$ in U if and only if L/K is $\mathscr{C}$ in U/K.*

Proof. This is straighforward. The (CM), (BV) and (MV) subring cases of (ii) are part of [**8**, Lemma 1].

In (ii) below we extend [**7**, Proposition 9], integral closure being taken in the sense of [**12**, Corollary 4]. Note that if, in (i), L is (CM) it will be (BV) by Corollary 3.5.

Corollary 2.4. *Let U have Krull dimension* 0. (i) *L is (MV) if it is (CM) or (KV).* (ii) *L is integrally closed if and only if it is an intersection of (MV) submodules.*

Proof. (i) Since 0 is not a product of elements of $U\backslash L$, there is a prime, hence maximal, U-ideal N contained in L. Since L is (EV), by Proposition 2.3, L/N is (EV) in the field U/N. Clearly L/N is (MV), hence L is. From (i) and [**12**, loc. cit.] we deduce (ii).

Next, building on Griffin's [**9**, Proposition 5], we obtain criteria for a subring to be (MV). In particular (1)$\Leftrightarrow$(3a) indicates the close relation between (MV) rings and classical valuation rings.

Theorem 2.5. *Put $R\backslash P_+(R, U) = T$ and $R_T:U_T = Q$. The following conditions are equivalent.*

(1) *$R \neq U$ and R is (MV) in U.*

(2a) *R is (CM) in U and (U_T, Q) is a local ring.*

(2b) *R is (CM) in U and Q is a maximal ideal of U_T.*

(3a) *U_T/Q is a field of which R_T/Q is a valuation domain.*

(3b) *The R_T-submodules of the non-zero ring U_T/Q form a chain.*

(3c) *R_T-submodules of U_T are comparable with Q, those containing Q form a chain and $Q \neq U_T$.*

Proof. The implications (2a)$\Rightarrow$(2b)$\Rightarrow$(3a)$\Rightarrow$(3b) and (3c)$\Rightarrow$(3b) are clear. For (1)$\Rightarrow$(2a), if R is (MV) then R is (CM). Also, by Proposition 2.2, the local ring $(R_T, (P_+(R))_T)$ is (MV) in U_T; hence, by [**9**, loc. cit.], (U_T, Q) is local. For (3b)$\Rightarrow$(1), put $R_T/Q = A$ and $U_T/Q = F$. Let $y \in F\backslash A$. Then $A \subseteq Ay$; whence $1 = ay$ for some $a \in A$. Thus, by Lemma 1.6.1(iii), A is (MV) in F and so, by Propositions 2.2 and 2.3, R is (MV) in U. Finally, (1) with (2a) implies (3c) by Proposition 2.2 and [**9**, loc. cit.].

On the theme of Section 2 in [**4**], we have the following corollary from (3b)$\Rightarrow$(1).

COROLLARY 2.6. *If the R-submodules of U form a chain then R is (MV) in U.*

COROLLARY 2.7. *R is (MV) with $P_+(R, U)$ the unique maximal ideal of R if and only if R is (CM) and $(U, (R{:}U))$ is local.*

Proof. The "only if" part follows from [**9**, loc. cit.]. For "if", one finds that $(U_T, (R_T{:}U_T))$ is local which implies, by (2a)$\Leftrightarrow$(1), that R is (MV). So $(R, P_+(R, U))$ is local by [**9**, loc. cit.].

Before generalising a result of Samuel on maximal pairs we need some preliminaries. From Proposition 2.3(ii) and [**17**, Section 4] we obtain the next lemma.

LEMMA 2.8. *Let $f{:}U_1 \to U_2$ be a homomorphism of rings. (Contraction) If J is a (BV) subalgebra of U_2 then $U_1 \cap J$ is a (BV) subalgebra of U_1. (Extension) If I is a (BV) subalgebra of U_1 then there exists a (BV) subalgebra J of U_2 such that $I = U_1 \cap J$.*

The next lemma is implicit in [**17**, Section 8].

LEMMA 2.9. *Let U be a field. Then U contains a valuation domain of which P is a prime ideal if and only if P is a (BV) subalgebra of U.*

The case of Theorem 2.10 in which U is a field follows from Gilmer [**6**, Corollary 19.7] and Lemma 2.9.

THEOREM 2.10. *Let $R \supset P_1 \supset \dots \supset P_k$, where each P_i is a prime ideal of R. Then there is a subring Q_0 containing a chain $Q_1 \supset \dots \supset Q_k$ of prime ideals such that $Q_0, Q_1, \dots, Q_k$ are (BV) in U, $Q_0 \supseteq R$ and $Q_i \cap R = P_i$ for $i = 1, \dots, k$.*

Proof. By Lemma 2.8, $P_k = P \cap R$ for some (BV) subalgebra P of U. Let N be the prime ideal $P{:}U$ of U. Then, for all i, $P_i \supseteq P \cap R \supseteq N \cap R$ and $P_i/(N \cap R)$ is prime in $R/(N \cap R)$. Let F be the field of fractions of U/N. In the subring $(R+N)/N$ of F there is a chain of prime ideals $(P_i + N)/N \cong P_i/(N \cap R)$ which extend (by [**6**, loc. cit.]) to a chain of prime ideals H_i $(i = 1, \dots, k)$ in a valuation domain H_0 of F. Each H_i $(i = 0, \dots, k)$ is (BV) in F (by Lemma 2.9). Hence, each $H_i \cap (U/N)$ is (BV) in U/N and has the form Q_i/N where, by Proposition 2.3(ii), Q_i is (BV) in U. Evidently $Q_0/N \supseteq (R+N)/N$ and, for $i = 1, \dots, k$, Q_i is an ideal of the subring Q_0 such that

$$(Q_i/N) \cap ((R+N)/N) = (P_i + N)/N.$$

From $Q_i \cap (R+N) = P_i + N$ we obtain $Q_i \cap R \subseteq (P_i + N) \cap R = P_i + (N \cap R) \subseteq P_i \subseteq Q_i \cap R$; hence $P_i = Q_i \cap R$.

Samuel's construction of maximal pairs [**19**, Lemma 2] may be generalised in two ways as follows. If A_δ $(\delta \in \Delta)$ is a family of prime ideals of R, and B_δ $(\delta \in \Delta)$ a family of prime ideals of a subring Q of U, we say that the system $(Q, B_\delta\ (\delta \in \Delta))$ extends the system $(R, A_\delta\ (\delta \in \Delta))$ if $Q \supseteq R$ and $R \cap B_\delta = A_\delta$ for all δ. Using Zorn's Lemma, one shows the existence of maximal extensions of a given system. Similarly, restricting attention to *chain* systems, i.e. where the prime ideals form a chain, we see that each chain system has a maximal chain system extension.

THEOREM 2.11. (i) *If a system* $(R, B_\delta\ (\delta \in \Delta))$ *is maximal* (*i.e. is the only extension of itself*) *then R is an intersection of* (*MV*) *rings and* $\bigcup_\delta B_\delta \supseteq R \cap P_+(R)$.

(ii) *If the system consisting of R with a finite chain of prime ideals* $B_1 \supset \ldots \supset B_k$ *is a maximal chain extension then* $R, B_1, \ldots, B_k$ *are each* (*BV*) *and* $B_1 \supseteq P_+(R)$.

Proof. (i) For each δ, let (R_δ, M_δ) be a maximal extension of the pair (R, B_δ). Putting $\bigcap_\delta R_\delta = R'$, we find that the system $(R', M_\delta \cap R'\ (\delta \in \Delta))$ extends $(R, B_\delta\ (\delta \in \Delta))$ and so $R' = R$. Put $R \backslash \left(\bigcup_\delta B_\delta\right) = S$. By Lemma 1.3.2, for each δ, $(B_\delta)_{[S]}$ is a prime ideal of $R_{[S]}$ and $(B_\delta)_{[S]} \cap R = B_\delta$. Maximality of R now gives $R_{[S]} = R$. Thus $R \cap P_+(R)$ does not meet S.

(ii) By Theorem 2.10, $(R, B_1 \supset \ldots \supset B_k)$ extends to a (BV) subring with a chain of (BV) ideals which, by maximality, are just $R, B_1, \ldots, B_k$. Hence $P_+(R) \subset R$, and so, as in (i), we obtain $P_+(R) \subseteq B_1$.

3. Prime ideals containing P_+. By Theorem 2.11(ii), if R is (MV), prime R-ideals comparable with P_+ are (BV). In Theorem 3.3 we shall extend this observation.

Let the submodule $L\ (\neq U)$ be (CM) and put $U \backslash L = S$. Define $L_{\langle S \rangle}$ to be the set, in U_S, of all finite sums of terms bs^{-1} with $b \in L$, $s \in S$. Letting $T = (L{:}L) \backslash P_+(L, U)$, we shall consider the natural homomorphisms $U \to U_T \to U_S$, where the second map is defined to take xt^{-1} to $xs(ts)^{-1}$ for arbitrary $s \in S$. We note that U_S was used, when $L = R$, in [**19**] and [**8**]. Denoting images in U_S by $(\ )'$ and writing $C(L, U)$ (see 1.2) as C, we have the following lemma.

LEMMA 3.1. (i) *If* $t \in T$, $x \in U$ *and* $(xt^{-1})' \in L_{\langle S \rangle}$ *then* $x \in C$.

(ii) *If* $x \in C$ *then* $x' \in L_{\langle S \rangle}$ *and, for some* $b_i \in L$, $s_i \in S$,

$$xs_1 \ldots s_k + \sum_i b_i s_1 \ldots \hat{s}_i \ldots s_k = 0,$$

where $\wedge$ *indicates an omitted term.*

(iii) $L_{\langle S \rangle} \cap U = C$.

Proof. For (i), $(xt^{-1})' = (s_1 s_2 \ldots s_k)^{-1} \sum b_i s_1 \ldots \hat{s}_i \ldots s_k$, where $b_i \in L$, $s_i \in S$, implies $x \in C$ since, for some s, s_0 in S,

$$xss_0 s_1 \ldots s_k - \sum_{i=1}^{k} (tb_i) ss_0 s_1 \ldots \hat{s}_i \ldots s_k = 0.$$

If x is $c(f)$, where f is an L-polynomial such that $f(q_1, \ldots, q_h) = 0$ $(q_i \in S)$, then division by $q_1 q_2 \ldots q_h$ leads to $x' \in L_{\langle S \rangle}$. Now, (ii) and (iii) are straightforward.

LEMMA 3.2. *If L is a* (*BV*) *subalgebra then* $C = P_+(L, U)$.

Proof. Writing $L:L=A$, we note that $L_{\langle S\rangle}$ and, hence, $L_{\langle S\rangle}\cap U_T$ have natural structures as A_T-modules. If $t\in T$ and $x\in P$ $(=P_+(L,U))$ then $xs\in L$ for some $s\in S$; whence $(xt^{-1})'\in L_{\langle S\rangle}$. Thus $P_T\subseteq L_{\langle S\rangle}\cap U_T$. Since L is (BV), $C\subseteq L$. But $L\subseteq A$; so $C\subseteq A$. By Lemma 3.1(i) and since $1\notin C$, we obtain $L_{\langle S\rangle}\cap U_T\subset A_T$. Since P_T is a maximal A_T-ideal, $P_T=L_{\langle S\rangle}\cap U_T$. Hence, by Lemma 3.1(iii), $C=L_{\langle S\rangle}\cap U=P_T\cap U=P_{[T]}=P$.

Theorem 3.3. *Let H be an (EV) submodule and let P be a prime ideal of $H{:}H$ such that $P\supseteq P_+((H{:}H),U)$. Then P (in particular $P_+(H,U)$) is (CM), and is (BV) if H is (KV).*

Proof. That P is (CM) is a simple variant of what follows. Write $H{:}H=A$ and let H be (KV). Then A is (BV) by [**12**, Proposition 10]. Suppose $f(s_1,\ldots,s_k)=0$, where $s_1,\ldots,s_k\in U$ and $f(X_1,\ldots,X_k)$ is a P-polynomial for which $c(f)=1$. We may arrange that $s_1,\ldots,s_h\in A$ and $s_{h+1},\ldots,s_k\notin A$, where $1\leqslant h\leqslant k$. For the case $h<k$, taking $g(X_{h+1},\ldots,X_k)$ to be the P-polynomial $f(s_1,\ldots,s_h,X_{h+1},\ldots,X_k)$, we have $g(s_{h+1},\ldots,s_k)=0$. Then $c(g)\in C(A)$ and, by Lemma 3.2, $C(A)\subseteq P_+(A)\subseteq P$. But $c(g)=s_1s_2\ldots s_h+p$ for some $p\in P$. Hence, for $h<k$ or $h=k$, $s_1s_2\ldots s_h\in P$. So $s_i\in P$ for some i, and P is (BV).

Condition (iii) below simplifies the usual (BV) definition. Concerning (ii), note that if H is (CM) and $1\notin H$ then $P_+(H,U)=H$.

Theorem 3.4. *For the following conditions on a submodule H,*
(i) *H is (BV),*
(ii) *$C(H,U)=P_+(H,U)$,*
(iii) *H is (CM) and, for all $k\geqslant 1$, $s_i\in U\backslash H$, $b_i\in H$,*

$$s_1s_2\ldots s_k+\sum b_is_1\ldots\hat{s}_i\ldots s_k\neq 0,$$

we have (ii)$\Rightarrow$(i)$\Leftrightarrow$(iii) *and, when H is a subalgebra,* (ii)$\Leftarrow$(i).

Proof. (ii)$\Rightarrow$(i) since $1\notin P_+$, (i)$\Rightarrow$(iii) is trivial, (iii)$\Rightarrow$(i) is by Lemma 3.1(ii), and (i)$\Rightarrow$(ii) is Lemma 3.2.

From (iii)$\Rightarrow$(i) we obtain following corollary.

Corollary 3.5. *For a submodule, the conjunction (CM) and (KV) implies (BV). The converse is true (trivially) for a subalgebra.*

4. Evaluations associated with a (CM) subring. For most of this section we tacitly assume $R\neq U$. Our first aim is to give, in Proposition 4.3, a condition for R and $P_+(R,U)$ to determine isomorphic evaluations.

Lemma 4.1. *For evaluations v and w on U, the following conditions are equivalent.*
(1) *$v\gtrsim w$.*
(2) *$v(x)\geqslant v(y)$ implies $w(x)\geqslant w(y)$ for all x, y in U.*
(3) *Every w-closed set is v-closed.*
Further, $w\cong v$ if and only if v-closed and w-closed sets coincide.

Proof. For (3)$\Rightarrow$(2), observe that if $w(y)>w(x)$ then $x \notin \{b \in U \mid w(b) \geqslant w(y)\}$ which is w-closed, and hence v-closed; so that $v(x)<v(y)$. For (2)$\Rightarrow$(1), define $\theta(v(x))=w(x)$ for all x. The rest is straightforward.

LEMMA 4.2. *Let L and M be (EV) submodules.*
(i) (Griffin [**8**]). *If v is an evaluation on U for which $\mathcal{R}(v)=R$ then $v \gtrsim v_R$.*
(ii) *If $L{:}L=R$ then $v_L \gtrsim v_R$.*
(iii) *$v_L \gtrsim v_M$ if and only if $L{:}(L{:}x) \subseteq M{:}(M{:}x)$ for all x in U.*

Proof. (i) If $R{:}y \supset R{:}x$ then, for some a, we have $v(ay) \geqslant 0 > v(ax)$ and so $v(y)>v(x)$. Since $L{:}L=\mathcal{R}(v_L)$, we deduce (ii) from (i). Finally, (iii) follows from Lemmas 1.5.1 and 4.1.

PROPOSITION 4.3. *Suppose that R is (CM), and denote by P its (CM) ideal $P_+(R, U)$. Then $v_R \gtrsim v_P$. Further, $v_R \cong v_P$ if and only if $P{:}P=R$.*

Proof. Let $y \in R{:}(R{:}x)$ and $a \in P{:}x$. Then $axz \in R$ for some z in $U \backslash R$. So $az \in R{:}x$ and $yaz \in R$. Hence $ya \in P$; so $y \in P{:}(P{:}x)$. Lemma 4.2(iii) now gives $v_R \gtrsim v_P$, and Lemmas 4.2 and 4.1 give the "further" part.

Next, when R is (CM) we aim to associate with it two (BV) overrings (see Proposition 4.8). We start by considering an evaluation v such that the U-ideal $v^{-1}(\infty)$ $(=B$, say) is prime. Let θ be the natural map $U_B \rightarrow U'$, where U' denotes U_B/BU_B. Denote by $\mathcal{R}'(v)$ the set of all $\theta(xy^{-1})$ for which there exist a, b such that $\theta(xy^{-1})=\theta(ab^{-1})$ and $v(a) \geqslant v(b)$ (with y, b in $U \backslash B$). One verifies that, for given y $(\notin B)$ and x, if $\theta(xy^{-1}) \in \mathcal{R}'(v)$ then $v(xd) \geqslant v(yd)$ for some $d \notin B$. Since $\theta(ab^{-1}) \notin \mathcal{R}'(v)$ precisely when $\theta(ba^{-1})=(\theta(ab^{-1}))^{-1}$ is a non-zero non-unit of $\mathcal{R}'(v)$, it is straightforward to verify the following lemma.

LEMMA 4.4. *$\mathcal{R}'(v)$ is a valuation domain of the field U'. The maximal ideal $\mathcal{M}'(v)$ of $\mathcal{R}'(v)$ is the set of all $\theta(ab^{-1})$ such that $v(ad)>v(bd)$ for all $d \notin B$.*

The evaluation on U' naturally determined by $\mathcal{R}'(v)$ gives, when composed with $U \rightarrow U'$, a cancellative evaluation on U which we denote by w_v.

LEMMA 4.5. (i) $v \gtrsim w_v$.
(ii) *$v \cong w_v$ if and only if v is cancellative.*

Proof. For (i), use (2)$\Rightarrow$(1) of Lemma 4.1. For (ii), suppose that v is cancellative. If $v(y)>v(x)$ then $v(yd)>v(xd)$ for all $d \notin B$, whence $\theta(yx^{-1}) \in \mathcal{M}'(v)$ and so $w_v(y)>w_v(x)$. Now, Lemma 4.1 and (i) lead to (ii).

We note that $\mathcal{R}(w_v)$, the inverse image in U of $\mathcal{R}'(v)$, is

$$\{x \in U \mid v(xd) \geqslant v(d) \text{ for some } d \notin B\}.$$

Clearly $\mathcal{R}(w_v) \supseteq \mathcal{R}(v) \supset \mathcal{P}(v) \supseteq \mathcal{P}(w_v)$. Using Lemmas 4.4, 2.9 and 2.8, we obtain the following result.

PROPOSITION 4.6. *The ring $\mathcal{R}(w_v)$ and its ideal $\mathcal{P}(w_v)$ are both (BV) in U.*

Using Lemma 4.1, and noting that $w_v(x) \geqslant w_v(y)$, where $y \notin v^{-1}(\infty)$, entails $\theta(xy^{-1}) \in \mathcal{R}'(v)$ and $v(xd) \geqslant v(yd)$ for some $d \notin v^{-1}(\infty)$, one obtains the next lemma.

LEMMA 4.7. *Let u and v be evaluations for which there is the common prime ideal $u^{-1}(\infty) = v^{-1}(\infty)$. If $v \gtrsim u$ then $w_v \gtrsim w_u$; whence $\mathcal{R}(w_v) \subseteq \mathcal{R}(w_u)$.*

Now let the submodule L be (EV) and such that $v_L^{-1}(\infty) = L{:}U$ is prime. Take $v = v_L$ in the foregoing and denote w_v by w_L. Then, by Proposition 4.6, $\mathcal{R}(w_L)$ and $\mathcal{P}(w_L)$ are (BV) in U.

PROPOSITION 4.8. *Let R be (CM), and write $P_+(R, U) = P$. Then $\mathcal{R}(w_R)$ and $\mathcal{R}(w_P)$ are (BV) in U, and $\mathcal{R}(w_P) \supseteq \mathcal{R}(w_R) \supseteq P{:}P \supseteq R$. Also, $P \supseteq \mathcal{P}(w_I) \supseteq P_+(\mathcal{R}(w_I), U)$ when $I = R$ or $I = P$. If R is (MV), each of the foregoing inclusions is an equality.*

Proof. That $\mathcal{R}(w_P) \supseteq \mathcal{R}(w_R)$ follows from Proposition 4.3 and Lemma 4.7. Let $a \in P{:}P$. If $a \in R$ then $R{:}a1 \supseteq R{:}1$; whence $a \in \mathcal{R}(w_R)$. If $a \notin R$ then $R{:}a \supseteq P{:}a \supseteq P \supseteq R{:}a$; whence equalities which give $R{:}a = (R{:}a){:}a$; so that $a \in \mathcal{R}(w_R)$.

Let $\theta(xy^{-1}) \in \mathcal{M}'(v_P)$, where $y \notin P{:}U$. Then, for all $d \notin P{:}U$, $P{:}xd \supset P{:}yd$ by Lemma 4.4, and so $R{:}xd \supset R{:}yd$ by Proposition 4.3; whence $\theta(xy^{-1}) \in \mathcal{M}'(v_R)$. If $r \in \mathcal{P}(w_I)$ then $\theta(r) \in \mathcal{M}'(v_I) \subseteq \mathcal{M}'(v_R)$ and so $R{:}r \supset R{:}1$ or $r \in P$. Further, if $x \in P_+(\mathcal{R}(w_I), U)$ then $xy \in \mathcal{R}(w_I)$ for some $y \notin \mathcal{R}(w_I)$. Taking images in U' gives $\theta(x) \in P_+(\mathcal{R}'(v_I), U') = \mathcal{M}'(v_I)$, and so $x \in \mathcal{P}(w_I)$.

Let R be (MV). Then $w_R \cong v_R$ by Lemma 4.5(ii). Also, $P{:}P = R$ and so $v_P \cong v_R$ by Proposition 4.3. The required equalities follow since $\mathcal{R}(w_R) = \mathcal{R}(v_R) = R$ and $w_P \cong w_R$.

The equivalence of (2) and (3) in Theorem 4.9 is closely related, when R is (MV), to [**16**, Proposition 3] and [**9**, Proposition 4].

THEOREM 4.9. *When R is (CM) and K is a submodule, consider the conditions*

(1) *K is w_R-closed,*
(2) *K is v_R-closed,*
(3) *$K = K_{[T]} \supseteq R{:}U$, where $T = R \backslash P_+(R, U)$.*

Then (1)$\Rightarrow$(2)$\Rightarrow$(3), and R is (MV) if and only if (3)$\Rightarrow$(1) for all submodules K.

Proof. (1)$\Rightarrow$(2) is clear by Lemma 4.5(i). For (2)$\Rightarrow$(3), let $xt \in K$ with $t \in U \backslash P_+(R, U)$. From $xt(R{:}xt) \subseteq R$, we have $x(R{:}xt) \subseteq R$ and $R{:}xt \subseteq R{:}x$. So $x \in K$ by (2). Also, $K \supseteq v_R^{-1}(\infty) = R{:}U$.

Suppose that (3)$\Rightarrow$(1) for all K. Since the submodules satisfying (3) are w_R-closed, they form a chain. Hence, the R_T-submodules of U_T containing $(R{:}U)U_T$ form a chain, since they are the extensions to U_T of those K satisfying (3). Since $(R{:}U)U_T \subseteq R_T{:}U_T$, Theorem 2.5(3b) holds and so R is (MV).

For (3)$\Rightarrow$(1), when R is (MV), use the argument after Proposition 4 in [**9**] (replace Q, v therein by K, w_R) noting that $R = \mathcal{R}(w_R)$ by use of Lemma 4.5(ii).

We end with a discussion on cancellative evaluations.

PROPOSITION 4.10. *When R is (CM) the following are equivalent:*
(1) v_R is cancellative, (2) $w_R \cong v_R$, (3) $\mathcal{R}(w_R) = R$.

Proof. If (3) holds when $w_R \gtrsim v_R$ by Lemma 4.2(i), but $v_R \gtrsim w_R$ by Lemma 4.5(i), and so (2) holds. The implications $(1) \Leftrightarrow (2) \Rightarrow (3)$ are clear.

REMARKS. (i) By Proposition 4.8, the conditions of Proposition 4.10 imply that R is (BV). If $U = R[X]$ with R a field then R is (BV) (as stated in [**8**]) and $\mathcal{R}(w_R) = U \neq R$. Thus R being (BV) (equivalent to (PV2) in [**8**]) does not, in general, imply condition (1) of Proposition 4.10 ((PV3) in [**8**]), contrary to a statement in [**8**].

(ii) As in [**8**, Example 5], let v be the evaluation of $k[X, Y]$, where k is a field, defined by extending the rule $v(X^n Y^m) = n - m\sqrt{2}$. Then $R = \mathcal{R}(v)$ is not (MV) by [**8**]. However, one may verify that v_R is cancellative. Thus, the condition on R that v_R be cancellative is strictly intermediate between (BV) and (MV). Nevertheless we do have the following result. The $R = \mathcal{R}(v)$ part of it is in [**8**], but Griffin's proof does not give the P_+ part.

THEOREM 4.11. *When R is (BV), there is a cancellative evaluation v on U such that $R = \mathcal{R}(v)$ and $P_+(R, U) = \mathcal{P}(v)$.*

Proof. Put $U \backslash R = S$ and $P_+(R, U) = P_+$. Let Q be the subring of U_S consisting of all elements of the type

$$d = (g(s_1, \ldots, s_k))^{-1} f(s_1, \ldots, s_k),$$

where $k \geq 1$, $s_i \in S$ and f, g are R-polynomials such that $c(f) \in R$, $c(g) \in R \backslash P_+$. Let x in U have image x' in Q. Then x' has the form d above, and, for some $s \in S$,

$$(xg(s_1, \ldots, s_k) - f(s_1, \ldots, s_k))s = 0.$$

If $x \notin R$ then $c(g) \in P_+$ by Lemma 3.2. So $x \in R$; whence $Q \cap U = R$. Since $xc(g) - c(f) \in P_+$, we have $x \in P_+$ precisely when $c(f) \in P_+$. Let M be the set of all d in U_S for which $c(f) \in P_+$. One verifies that $M \cap U = P_+$ and M is the unique maximal ideal of Q. The pair (Q, M) extends in U_S to a maximal pair (Q_0, P). By [**16**], there is a cancellative evaluation v_P such that $\mathcal{R}(v_P) = Q_0$ and $\mathcal{P}(v_P) = P$. The evaluation v induced on U by v_P is cancellative. Clearly $\mathcal{R}(v) \supseteq R$ and $\mathcal{P}(v) \supseteq P_+$. We find that $\mathcal{R}(v) = R$ by an argument similar to Griffin's, that if $s \in S$ then $s^{-1} \in M$; whence $v(s) < 0$. To show $\mathcal{P}(v) \subseteq P_+$, let $x \in \mathcal{P}(v)$. Then, for (all) $s \in S$, $x' = (xs)s^{-1} \in Q$. So $x' \in Q \cap P = M$; hence $x \in M \cap U = P_+$.

REFERENCES

1. N. Bourbaki, *Commutative algebra* (Addison-Wesley, 1972).

2. C. H. Brase, Valuation ideals with zero divisors, *J. Reine Angew. Math.* **258** (1973), 192–200.

3. N. Eggert and H. Rutherford, A local characterization of Prüfer rings, *J. Reine Angew. Math.* **250** (1971), 109–112.

4. C. Faith, The structure of valuation rings, *J. Pure Appl. Algebra* **31** (1984), 7–27.

5. C. Faith, The structure of valuation rings II, *J. Pure Appl. Algebra* **42** (1986), 37–43.

6. R. Gilmer, *Multiplicative ideal theory* (Marcel Dekker, 1972).

7. M. Griffin, Prüfer rings with zero divisors, *J. Reine Angew. Math.* **239/240** (1969), 55–67.

8. M. Griffin, *Generalising valuations to commutative rings,* Queen's Mathematical Preprint No. 1970–40 (Queen's University, Kingston, Ontario, 1970).

9. M. Griffin, Valuations and Prüfer rings, *Canad. J. Math.* **26** (1974), 412–429.

10. D. K. Harrison, *Finite and infinite primes for rings and fields* (*Mem. Amer. Math. Soc.* 68), (1966).

11. J. A. Huckaba, On valuation rings that contain zero divisors, *Proc. Amer. Math. Soc.* **40** (1973), 9–15.

12. D. Kirby, Integral dependence and valuation algebras, *Proc. London Math. Soc.* (3) **20** (1970), 79–100.

13. D. Kirby and H. A. Mehran, Homomorphisms and the space of valuation modules, *Quart. J. Math. Oxford Ser.* (2) **21** (1970), 439–443.

14. D. Kirby and H. A. Mehran, Fractional powers of a submodule of an algebra, *Mathematika* **18** (1971), 8–13.

15. M. D. Larsen and P. J. McCarthy, *Multiplicative theory of ideals* (Academic Press, New York, 1971).

16. M. E. Manis, Valuations on a commutative ring, *Proc. Amer. Math. Soc.* **20** (1969), 193–198.

17. D. A. Rand, Non-unitary valuation subalgebras, *Proc. London Math. Soc.* (3) **29** (1974), 485–501.

18. C. P. L. Rhodes, Valuation subalgebras and analogues of the decomposition of a radical ideal into prime ideals, *Proc. London Math. Soc.* (3) **46** (1983), 385–410.

19. P. Samuel, La notion de place dans un anneau, *Bull. Soc. Math. France* **85** (1957), 123–133.

20. J. van Geel, *Places and valuations in noncommutative ring theory* (Marcel Dekker, 1981).

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