ON MAPPINGS WHICH COMMUTE WITH CONVOLUTION

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(Received 9 June, revised 28 October 1969)

Communicated by J. B. Miller

1. Notation

The symbol $\boldsymbol{D}$ will be written for the space of indefinitely differentiable functions on the n-dimensional Euclidean space R^n which have compact support and $\boldsymbol{D}'$ will denote the space of Schwartz distributions on R^n, the topological dual of $\boldsymbol{D}$. Except where the contrary is explicitly stated, it will be assumed that $\boldsymbol{D}'$ is equipped with the strong topology $\beta(\boldsymbol{D}', \boldsymbol{D})$ induced by $\boldsymbol{D}$.

2. Definitions

We shall always use the term *space of distributions* (or, more briefly, *distribution space*) to mean a vector subspace of $\boldsymbol{D}'$ which contains the subspace $\boldsymbol{D}$. This convention will help us to avoid much tedious repetition.

2.1. DEFINITION. Suppose that E and E' are spaces of distributions and that $\langle\ ,\ \rangle$ is a bilinear form on $E \times E'$ such that the relations

$$\langle u, \varphi \rangle = u * \varphi(0) \qquad (u \in E) \tag{2.1}$$

$$\langle \varphi, v \rangle = \varphi * v(0) \qquad (v \in E') \tag{2.2}$$

hold whenever φ is an element of $\boldsymbol{D}$. Then the ordered pair (E, E') together with the bilinear form $\langle\ ,\ \rangle$ is called a *dual pair of distribution spaces*. We shall usually omit explicit reference to the bilinear form $\langle\ ,\ \rangle$ and speak simply of the dual pair (E, E').

The next definition was introduced by Yoshinaga and Ogata in [1]. We restate it here for the sake of completeness.

2.2. DEFINITION. (Yoshinaga and Ogata [1]). Suppose that E is a locally convex space of distributions which possesses the following two properties:

(i) $\boldsymbol{D}$ is dense in E.

(ii) The injection mappings $\boldsymbol{D} \to E$ and $E \to \boldsymbol{D}'$ are continuous.

Then E is said to be an *admissible* space.

REMARK. The topological dual E' of an admissible space E will always be identified with a space of distributions in such as way that (E, E') (with the bilinear form arising from the natural pairing of E and E')) is a dual pair of distribution spaces. This (unique) embedding of E' in $\mathbf{D}'$ will be called the *natural* embedding.

We now introduce the spaces with which we shall be concerned in this note.

2.3. DEFINITION. Let E be an admissible space. We say that E is of *type* (c) (or a (c)-*space*) if it is a module over $\mathbf{D}$ (with respect to convolution) and has the following properties:

(i) For each $u \in E$, the mapping $\varphi \to u * \varphi$ $(\varphi \in \mathbf{D})$ of $\mathbf{D}$ into E is continuous.

(ii) For each $\varphi \in \mathbf{D}$, the mapping $u \to u * \varphi$ $(u \in E)$ of E into itself is continuous.

2.4. DEFINITION. Let E be a (c)-space. A linear mapping T of E into $\mathbf{D}'$ is said to commute with convolution if

$$T(u * \varphi) = (Tu) * \varphi \qquad (\varphi \in \mathbf{D},\ u \in E) \tag{2.3}$$

The space of all such *continuous* mappings is denoted by $H_c(E, \mathbf{D}')$.

REMARK. Let E be a (c)-space. If $u \in E$ and $T \in H_c(E, \mathbf{D}')$, then we shall denote by $u \mathbin{\bar{*}} T$ the image of u under the mapping T. With this notation, the convolution commutativity of T is expressed by

$$(u * \varphi) \mathbin{\bar{*}} T = (u \mathbin{\bar{*}} T) * \varphi \qquad (\varphi \in \mathbf{D},\ u \in E) \tag{2.4}$$

$H_c(E, \mathbf{D}')$ will always be identified with a space of distributions in such a way that the relations

$$u \mathbin{\bar{*}} \varphi = u * \varphi \qquad (u \in E) \tag{2.5}$$

$$\varphi \mathbin{\bar{*}} w = \varphi * w \qquad (w \in H_c(E, \mathbf{D}')) \tag{2.6}$$

hold whenever φ is an element of $\mathbf{D}$. This (unique) embedding of $H_c(E, \mathbf{D}')$ in $\mathbf{D}'$ will be called the *natural* embedding.

3. Some preliminary results

3.1. PROPOSITION. *Suppose that (E, E') is a dual pair of distribution spaces and that T is a continuous linear mapping of $\mathbf{D}'$ into itself which commutes with convolution (by elements of $\mathbf{D}$). Then the following assertions are equiavlent to one another*:

(1) *E is invariant under T and the mapping $u \to Tu$ $(u \in E)$ of E into itself is weakly continuous.*

(2) *Both E and E' are invariant under T and*

$$\langle Tu, v\rangle = \langle u, Tv\rangle \qquad (u \in E,\ v \in E')$$

https://doi.org/10.1017/S1446788700008405 Published online by Cambridge University Press

PROOF. If we bear in mind the theorem in Section 5.11.3 of Edwards [2], it is simple to show that the mapping $v \to Tv$ $(v \in E')$ is the adjoint of the mapping $u \to Tu$ $(u \in E)$ of E into itself.

3.2. PROPOSITION. *Suppose that E is a (c)-space and that T is a continuous linear mapping of $\mathbf{D}'$ into itself which commutes with convolution (by elements of $\mathbf{D}$). Then $H_c(E, \mathbf{D}')$ is invariant under T and*

$$u \mathbin{\bar{*}} Tw = T(u \mathbin{\bar{*}} w) \qquad (u \in E,\ w \in H_c(E, \mathbf{D}'))$$

PROOF. Once again using the theorem in Section 5.11.3 of Edwards [2], it is not difficult to demonstrate that the mapping $u \to T(u \mathbin{\bar{*}} w)$ $(u \in E)$ defines an element of $H_c(E, \mathbf{D}')$. The latter is readily shown to be just Tw.

As an adjunct to Proposition 3.2, we have the following result. Its proof is along the same lines as that of Proposition 3.2.

3.3. PROPOSITION. *Let T be a continuous linear mapping of $\mathbf{D}'$ into itself which commutes with convolution (by elements of $\mathbf{D}$). Suppose that E is a (c)-space with the following property:*

(i) *E is invariant under T and the mapping $u \to Tu$ $(u \in E)$ of E into itself is continuous.*

It then follows that

$$(Tu) \mathbin{\bar{*}} w = T(u \mathbin{\bar{*}} w) = u \mathbin{\bar{*}} (Tw) \qquad (u \in E,\ w \in H_c(E, \mathbf{D}'))$$

The foregoing three propositions easily yield the following facts about spaces of type (c). These will be needed later.

3.4. PROPOSITION. *Suppose that E is a (c)-space. Then the following assertions are true:*

(1) *E' is a module over $\mathbf{D}$ and for each $\varphi \in \mathbf{D}$*

$$\langle u * \varphi, v \rangle = \langle u, v * \varphi \rangle \qquad (u \in E,\ v \in E')$$

(2) *$H_c(E, \mathbf{D}')$ is a module over $\mathbf{D}$ and for each $\varphi \in \mathbf{D}$*

$$(u * \varphi) \mathbin{\bar{*}} w = (u \mathbin{\bar{*}} w) * \varphi = u \mathbin{\bar{*}} (w * \varphi) \qquad (u \in E,\ w \in H_c(E, \mathbf{D}'))$$

4. A criterion for mappings which commute with convolution

In this section we shall take a look at a condition which determines whether a distribution $w \in \mathbf{D}'$ is an element of $H_c(E, \mathbf{D}')$. We need the next result to do this.

4.1. PROPOSITION. *Suppose that E is a (c)-space and that $w \in H_c(E, \mathbf{D}')$. Then the following assertion is true:*

(1) *$w * \varphi \in E'$ for each $\varphi \in \mathbf{D}$ and the mapping $\varphi \to w * \varphi$ $(\varphi \in \mathbf{D})$ of $\mathbf{D}$ into E' is weakly continuous.*

Moreover, we have the identity

$$(4.1) \qquad \langle u, w * \varphi \rangle = (u \mathbin{\overline{*}} w) * \varphi(0) \qquad (\varphi \in \boldsymbol{D},\ u \in E)$$

PROOF. The mapping $\varphi \to w * \varphi$ $(\varphi \in \boldsymbol{D})$ is just the adjoint of the continuous linear mapping $u \to u \mathbin{\overline{*}} w$ $(u \in E)$ of E into $\boldsymbol{D}'$.

If the space E has the topology $\tau(E, E')$, then the preceding result has a converse. We have then the following criterion for determining whether a distribution w belongs to $H_c(E, \boldsymbol{D}')$.

4.2. THEOREM. *Suppose that E is a (c)-space which has the topology $\tau(E, E')$ and that $w \in \boldsymbol{D}'$. Then the following two conditions are equivalent:*

(1) $w \in H_c(E, \boldsymbol{D}')$

(2) $w * \varphi \in E'$ *for all* $\varphi \in \boldsymbol{D}$ *and the mapping* $\varphi \to w * \varphi$ $(\varphi \in \boldsymbol{D})$ *of* $\boldsymbol{D}$ *into* E' *is weakly continuous.*

PROOF. We need only show that (2) implies (1). Consider the adjoint of the mapping $\varphi \to w * \varphi$ $(\varphi \in \boldsymbol{D})$, which is a linear mapping of E into $\boldsymbol{D}'$. This adjoint is continuous (because E has the topology $\tau(E, E')$) and commutes with convolution (because of Proposition 3.4). It is therefore represented by a distribution in $H_c(E, \boldsymbol{D}')$ and this distribution is easily shown to be precisely w.

4.3. COROLLARY. *Suppose that E is a (c)-space which has the topology $\tau(E, E')$. Then E' is contained in $H_c(E, \boldsymbol{D}')$.*

An additional restriction on E, which in itself is not too severe, enables us to strengthen considerably the content of Theorem 4.2. We now turn our attention to this task, beginning with a couple of lemmas.

4.4. LEMMA. *Suppose that E is a (c)-space and that $u \in E$. Let $w \in \boldsymbol{D}'$ be a distribution which has the following property:*

(i) $w * \varphi \in H_c(E, \boldsymbol{D}')$ *for each* $\varphi \in \boldsymbol{D}$.

Then there exists a distribution $s \in \boldsymbol{D}'$ such that

$$u \mathbin{\overline{*}} (w * \varphi) = s * \varphi \qquad (\varphi \in \boldsymbol{D}).$$

PROOF. Let u and w be as in the statement of the Lemma. Denote by L the mapping of $\boldsymbol{D}$ into $\boldsymbol{D}'$ which is defined by

$$(4.7) \qquad L\varphi = u \mathbin{\overline{*}} (w * \varphi) \qquad (\varphi \in \boldsymbol{D}).$$

We claim that L is continuous from $\boldsymbol{D}$ into $\boldsymbol{D}'$. To verify this, notice that, since $\boldsymbol{D}$ has its Mackey topology $\tau(\boldsymbol{D}, \boldsymbol{D}')$, the continuity of the mapping L will be established if we demonstrate that it is weakly continuous. Now, in view of the hypothesis about the distribution $w \in \boldsymbol{D}'$, Proposition 4.1 tells us that $w * \varphi * \psi \in E'$ for all $\varphi, \psi \in \boldsymbol{D}$; and that

$$(4.8) \qquad \langle u, w * \varphi * \psi \rangle = (u \mathbin{\overline{*}} (w * \varphi)) * \psi(0) \qquad (\varphi, \psi \in \boldsymbol{D}).$$

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Relations (4.7) and (4.8) entail that for each $\psi \in \boldsymbol{D}$

$$L\varphi * \psi(0) = (u \mathbin{\bar{*}} (w * \psi)) * \varphi(0) \qquad (\varphi \in \boldsymbol{D}) \tag{4.9}$$

and the weak continuity of L (as a mapping of $\boldsymbol{D}$ into $\boldsymbol{D}'$) is now evident.

Next, we notice that L commutes with convolution; this is an immediate consequence of Proposition 3.4 (2).

Having established the continuity and convolution commutativity of L, we infer the existence of a distribution $s \in \boldsymbol{D}'$ such that

$$L\varphi = s * \varphi \qquad (\varphi \in \boldsymbol{D}). \tag{4.10}$$

In view of (4.7), we see that (4.10) expresses the desired result.

4.5. Lemma. *Suppose that E is a (c)-space which has the topology $\tau(E, E')$ and whose dual E' is sequentially complete for the topology $\beta(E',E)$. Let $w \in \boldsymbol{D}'$ be a distribution which has the following property*:

(i) *$w * \varphi \in H_c(E, \boldsymbol{D}')$ for each $\varphi \in \boldsymbol{D}$.*

Then w is an element of $H_c(E, \boldsymbol{D}')$.

Proof. Let w be a distribution which has property (i) above. Choose a countable approximate identity (k_m) in $\boldsymbol{D}'$ consisting of functions in $\boldsymbol{D}$. By Proposition 4.1, $w * \varphi * \psi \in E'$ for all $\varphi, \psi \in \boldsymbol{D}$. Therefore, for each positive integer m, we may define a mapping L_m of $\boldsymbol{D}$ into E' by setting

$$L_m\varphi = w * \varphi * k_m \qquad (\varphi \in \boldsymbol{D}). \tag{4.11}$$

Our first claim is that, for each m, the mapping L_m is strongly continuous from $\boldsymbol{D}$ into E'. This is easy to verify. For, in view of the hypothesis about w, reference to Proposition 4.1 assures us that, for each m, L_m is weakly, and hence also strongly, continuous. (The assertion about the strong continuity of each L_m is justified by Proposition 8.6.5 in Edwards [2]). We notice also that relations (4.11) and (4.1) entail that for each m and each $u \in E$

$$\langle u, L_m\varphi\rangle = (u \mathbin{\bar{*}} (w * k_m)) * \varphi(0) \qquad (\varphi \in \boldsymbol{D}). \tag{4.12}$$

We next remark that (L_m), as a sequence of mappings of $\boldsymbol{D}$ into E', is bounded at each point of $\boldsymbol{D}$ when E' has the topology $\beta(E', E)$. To show this, it is sufficient to demonstrate that, for each $\varphi \in \boldsymbol{D}$, $(L_m\varphi)$ is uniformly bounded on each bounded subset of E. Thus consider an arbitrary (but fixed) element $\varphi \in \boldsymbol{D}$; and let B be a bounded subset of E. For each $u \in E$, let s_u be the distribution in $\boldsymbol{D}'$ which satisfies

$$u \mathbin{\bar{*}} (w * \psi) = s_u * \psi \qquad (\psi \in \boldsymbol{D}) \tag{4.13}$$

The existence of such distributions s_u is guaranteed by Lemma 4.4. Observe that, for each $\psi \in \boldsymbol{D}$, the set $\{s_u * \psi : u \in B\}$ is the image in $\boldsymbol{D}'$ of the set B under the continuous mapping $u \to u \mathbin{\bar{*}} (w * \psi)$ $(u \in E)$ of E into $\boldsymbol{D}'$. Since B is bounded in E

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we conclude that $\{s_u * \psi : u \in B\}$ is bounded in $\boldsymbol{D}'$ for each $\psi \in \boldsymbol{D}$. Théorème XXII in Chapitre VI of Schwartz [3] now ensures that the set $\{s_u : u \in B\}$ is bounded in $\boldsymbol{D}'$. Thus, since the sequence $(\varphi * k_m)$ is convergent in $\boldsymbol{D}$ and therefore uniformly bounded on each bounded subset of $\boldsymbol{D}'$, we conclude that there exists a constant M such that

$$(4.14) \qquad |\langle s_u * \varphi * k_m(0) \rangle| \leqq M \qquad (k = 1, 2, \cdots)$$

uniformly for $u \in B$. In view of (4.12), (4.13) and (4.14), we may now assert that

$$(4.15) \qquad |\langle u, L_m \varphi \rangle| \leqq M \qquad (k = 1, 2, \cdots)$$

uniformly for $u \in B$. The pointwise boundedness of the sequence (L_m) of mappings of $\boldsymbol{D}$ into E' has now been established.

Now let H_0 be the subspace of $\boldsymbol{D}$ which consists of all elements $\varphi \in \boldsymbol{D}$ for which $(L_m \varphi)$ converges strongly in E'. We shall show that H_0 coincides with the whole of $\boldsymbol{D}$. Since E' is strongly sequentially complete and $\boldsymbol{D}$ is barrelled for its strong topology $\beta(\boldsymbol{D}, \boldsymbol{D}')$, we need only show that H_0 is dense in $\boldsymbol{D}$ (Edwards [2], Corollary 7.1.4). This, in turn, will be established if we succeed in demonstrating that $\varphi * \psi \in H_0$ whenever $\varphi \in \boldsymbol{D}$ and $\psi \in \boldsymbol{D}$. To verify that this in fact true, we proceed as follows. First we notice that if $\psi \in \boldsymbol{D}$, then

$$(4.16) \qquad \lim_m \psi * k_m = \psi \quad \text{strongly in } \boldsymbol{D}.$$

Now, Proposition 4.1 tells us that if $\varphi, \psi \in \boldsymbol{D}$, then $w * \varphi * \psi \in E'$; and that for each fixed $\varphi \in \boldsymbol{D}$, the maping $\psi \to w * \varphi * \psi$ $(\psi \in \boldsymbol{D})$ is weakly, and hence strongly, continuous from $\boldsymbol{D}$ into E'. In view of (4.16) we may therefore conclude that if $\varphi, \psi \in \boldsymbol{D}$, then

$$(4.17) \qquad \lim_m L_m(\varphi * \psi) = \lim_m w * \varphi * \psi * k_m = w * \varphi * \psi$$

the limits in (4.17) being in the strong topology $\beta(E', E)$ on E'. As was explained above, we may now assert that $H_0 = \boldsymbol{D}$.

We can now define a mapping L of $\boldsymbol{D}$ into E' by the relation

$$(4.18) \qquad L\varphi = \lim_m L_m \varphi \qquad (\varphi \in \boldsymbol{D})$$

the limit in (4.18) being once again a strong limit in E'. Then L is strongly continuous from $\boldsymbol{D}$ into E' (we have again used Corollary 7.1.4 in Edwards [2]).

We are now in a position to complete our proof. According to Theorem 4.2, it is sufficient to show that the mapping $\varphi \to w * \varphi$ $(\varphi \in \boldsymbol{D})$ is a weakly continuous linear mapping of $\boldsymbol{D}$ into E'; it will then follow that w is indeed an element of $H_c(E, \boldsymbol{D}')$. Now, since $\boldsymbol{D}'$ is reflexive, each strongly continuous linear mapping of $\boldsymbol{D}$ into E' is weakly continuous (Edwards [2], Corollary 8.6.7). Therefore, we need establish only that the mapping $\varphi \to w * \varphi$ $(\varphi \in \boldsymbol{D})$ is strongly continuous from $\boldsymbol{D}$ into E'. But if $\varphi \in \boldsymbol{D}$ and $\psi \in \boldsymbol{D}$, tl en we see that

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$$\begin{aligned}\psi * L\varphi(0) &= \langle \psi, L\varphi \rangle \\ &= \lim_m \langle \psi, L_m \varphi \rangle \\ &= \lim_m \langle \psi, w * \varphi * k_m \rangle \\ &= \lim_m \psi * w * \varphi * k_m(0) \\ &= \psi * w * \varphi(0).\end{aligned}$$

We infer that $L\varphi = w * \varphi$ for each $\varphi \in \boldsymbol{D}$; whence it follows immediately that the mapping $\varphi \to w * \varphi$ $(\varphi \in \boldsymbol{D})$ is a strongly continuous mapping of $\boldsymbol{D}$ into E'.

With the aid of the above two lemmas, we can prove the following variant of Theorem 4.2.

4.6. THEOREM. *Suppose that E is a (c)-space which has the topology $\tau(E, E')$ and whose dual E' is sequentially complete for the topology $\beta(E', E)$. Let $w \in \boldsymbol{D}'$ be a distribution. Then the following three conditions are equivalent to one another*:

(1) $w \in H_c(E, \boldsymbol{D}')$.

(2) $w * \varphi \in E'$ *for each* $\varphi \in \boldsymbol{D}$.

(3) $w * \varphi * \psi \in E'$ *for all* $\varphi, \psi \in \boldsymbol{D}$.

PROOF. Theorem 4.2 ensures that (2) holds if $w \in \boldsymbol{D}'$ satisfies (1). Proposition 3.4 entails that (2) implies (3). To complete the proof, notice first that E' is contained in $H_c(E, \boldsymbol{D}')$ (Corollary 4.3). Thus if (3) holds, we may appeal to Lemma 4.5 and deduce that $w * \varphi \in H_c(E, \boldsymbol{D}')$ for each $\varphi \in \boldsymbol{D}$; whence it follows (again by Lemma 4.5) that (1) holds.

REMARK. Proposition 4.6 is applicable to (c)-spaces E which are either barrelled or bornological; see Section 8.4.13 in Edwards [2].

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