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APPROXIMATION OF FOLIATIONS

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1. Let $\mathscr{F}$, $\mathscr{F}'$ be two foliations on a C^r manifold M. We say $\mathscr{F}$ and $\mathscr{F}'$ are C^k-conjugate if there exists a C^k diffeomorphism $h:M \to M$ such that h maps the leaves of $\mathscr{F}$ onto the leaves of $\mathscr{F}'$.
We wish to prove the following:

THEOREM. *Let M be an n-dimensional C^r manifold. Let $\mathscr{F}$ be a foliation of class C^k and codimension p on M, $1 \le k \le r \le \infty$. Let δ be a real-valued positive function defined on M. Then there exists an open set U, dense in M, and a foliation $\mathscr{F}'$ of codimension p on M such that*

(1) *$\mathscr{F}'$ is of class C^k*
(2) *$\mathscr{F}' \mid U$ is of class C^r*
(3) *$\mathscr{F}$ and $\mathscr{F}'$ are C^k-conjugate*
(4) *$\mathscr{F}$ and $\mathscr{F}'$ are C^k δ-close.*

Denjoy [2] constructs a foliation of codimension one on $S^1 \times S^1$, of class C^1, such that no foliation of class C^2 on $S^1 \times S^1$ is C°-conjugate to it. This is an example where $U \neq M$ in the theorem (see also Cohen [1]).

Since the theorem and its proof depend only on elementary definitions about foliations, we will provide these in §2. The definitions are a slight modification of the ones in Haefliger [3].

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2. Consider R^n as the Cartesian product $R^{n-p}xR^p$ and denote points by (x, y) with $x \in R^{n-p}$, $y \in R^p$. The simplest example of a foliation of codimension p on R^n is the one whose leaves are the $(n-p)$-planes parallel to the plane $y=0$. Denote this foliation by $\mathscr{F}_0$.

A local homeomorphism h of class C^k of $\mathscr{F}_0$ is a local homeomorphism of R^n which locally preserves the leaves. In the neighborhood of each point (x, y) where h is defined, the homeomorphism $h(x, y)=(x', y')$ is given by

$$(1) \qquad \begin{cases} x' = \phi(x, y) \\ y' = \psi(y) \end{cases}$$

If the map h is of class C^k, ϕ is of class C^k.

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DEFINITION 1. Let M be an n-dimensional topological manifold. A foliated structure or foliation $\mathscr{F}$ of class C^k and codimension p on M is given by a collection $\{U_i, h_i\}$ of charts satisfying

(1) $\{U_i\}$ is an open covering of M.

(2) h_i is a homeomorphism of U_i with an open set in R^n.

(3) The maps $h_j h_i^{-1}$ are local homeomorphisms of R^n of class C^k which are locally of the form (1).

(4) The collection $\{U_i, h_i\}$ is maximal with respect to the preceding properties.

The atlas $\mathscr{A}=\{U_i, h_i\}$ generates a C^k differentiable structure on the manifold M. For this structure the maps h_i are of class C^k.

DEFINITION 2. Let M_α be a manifold with a C^r differentiable structure α. A foliation $\mathscr{F}$ with atlas $\mathscr{A}$ is a C^k foliation on M_α if α is contained in the C^k differentiable structure generated by $\mathscr{A}$. This is equivalent to requiring that the maps of the charts of $\mathscr{A}$ be of class C^k for the structure α.

Let T_0 be the topology on R^n which is the product of the usual topology on R^{n-p} by the discreet topology on R^p. Let $\mathscr{F}$ be a foliation on a manifold M and let $\mathscr{A}=\{U_i, h_i\}$ be the atlas for $\mathscr{F}$. There is a unique topology T on M such that each h_i is a homeomorphism of U_i with $h_i(U_i)$ for the topologies $T \mid U_i$, $T_0 \mid h_i(U_i)$.

DEFINITION 3. The leaves of the foliation $\mathscr{F}$ are the connected components of M relative to the topology T.

The leaves are $(n-p)$-dimensional submanifolds of M which are of class C^k if $\mathscr{F}$ is of class C^k.

Let M be an n-dimensional C^r manifold with tangent bundle TM. Let $\mathscr{F}$ be a foliation of class C^k and codimension p on M. The C^k section σ in the bundle $\mathscr{G}_{n-p}TM$ of $(n-p)$-planes of TM, such that for each $x \in M$, $\sigma(x)$ is tangent to the leaf of $\mathscr{F}$ through x, is called the tangent plane field to $\mathscr{F}$.

An atlas $\mathscr{A}=\{U_i, h_i\}$ is an atlas for a foliation $\mathscr{F}$ if the foliation it defines has the same tangent plane field as $\mathscr{F}$.

For other definitions and basic properties of foliations see Haefliger [3] and Reeb [5].

3. Let M be an n-dimensional C^r manifold and let $\mathscr{S}$ be the space of C^k sections in $\mathscr{G}_{n-p}TM$, with the C^k topology. Let $\mathscr{F}_1$ and $\mathscr{F}_2$ be foliations of class C^k and codimension p on M (as a C^r manifold), $r \geq k \geq 1$, with tangent plane fields σ_1, σ_2, respectively. We have $\sigma_1, \sigma_2 \in \mathscr{S}$.

DEFINITION 4. Let δ be a positive continuous real-valued function on M. We say that $\mathscr{F}_2$ is a C^k δ-approximation to $\mathscr{F}_1$, or that $\mathscr{F}_1$ and $\mathscr{F}_2$ are C^k δ-close, if the sections σ_1 and σ_2 are δ-close in $\mathscr{S}$.

Let M and N be C^r manifolds and $\mathscr{F}'$ a foliation of class C^k and codimension p on N, with atlas $\mathscr{A}'=\{U_i', h_i'\}$. If $h: M \to N$ is a C^r diffeomorphism and $r \geq k$, the inverse image of $\mathscr{F}'$ by h is the foliation $\mathscr{F}=h^{-1}\mathscr{F}'$, of class C^k and codimension p on M defined by the atlas $\mathscr{A}=\{h^{-1}(U_i), h_i\circ h\}$.

The following follows directly from the definitions.

PROPOSITION. *Let M be a manifold, $\mathscr{F}$ a foliation of class C^k on M. Let α, β be C^r differentiable structures on M, with $\mathscr{F}$ a C^k foliation for both M_α and M_β. Let $h:M_\alpha\to M_\beta$ be a C^r diffeomorphism which is C^k δ-close to the identity. Then $\mathscr{F}$ and $\mathscr{F}'=h^{-1}\mathscr{F}$ are two foliations on M_α which are C^k δ-close.*

4. **Proof of the theorem.** Let α be the given C^r differentiable structure on M. Let $\mathscr{F}$ be given by an atlas $\mathscr{A}=\{U_i, h_i\}$ and let β be the C^k differentiable structure generated by $\mathscr{A}$. Then $\alpha\subset\beta$ by definition. Consider pairs $(V, \mathscr{A}_V)$ where V is open in M, $\mathscr{A}_V\subset\mathscr{A}\mid V\subset\mathscr{A}$ and if $\{U_i, h_i\}$, $\{U_j, h_j\}$ are in $\mathscr{A}_V$, then $h_jh_i^{-1}$ is of class C^r, i.e. the changes of coordinates in $\mathscr{A}_V$ are of class C^r. (For example, if $\{U, k\}$ is a chart of $\mathscr{A}$, then $(U, \{U, k\})$ is such a pair, and if $\mathscr{A}_U$ consists of all the charts $\{T, k\mid T\}$ with $T\subset U$, then $(U, \mathscr{A}_U)$ is another such pair.) Define a partial order on the set of such pairs by $(V, \mathscr{A}_V)\leq(V', \mathscr{A}_{V'})$ if $V\subset V'$ and $\mathscr{A}_V\subset\mathscr{A}_{V'}$. If we have a totally ordered chain

$$(V_1, \mathscr{A}_{V_1})\leq\cdots\leq(V_n, \mathscr{A}_{V_n})\leq\cdots$$

then the pair $(\bigcup_1^\infty V_i, \bigcup_1^\infty \mathscr{A}_{V_i})$ is an upper bound for the elements of the chain. The set of pairs $(V, \mathscr{A}_V)$ as above is therefore inductive with $\leq$ and hence by Zorn's lemma there is a maximal element $(W, \mathscr{A}_W)$. Suppose W is not dense in M. Then there is a point x in $M-W$, and a chart $\{U_x, h_x\}\in\mathscr{A}$ such that $W\cap U_x=\varnothing$. But then $\mathscr{A}_W\cup\{U_x, h_x\}\subset\mathscr{A}\mid W\cup U_x$, the changes of coordinates in $\mathscr{A}_W\cup\{U_x, h_x\}$ are of class C^r and $(W, \mathscr{A}_W)\leq(W\cup U_x, \mathscr{A}_W\cup\{U_x, h_x\})$, which contradicts the maximality of $(W, \mathscr{A}_W)$. Hence W is dense in M. Moreover $\mathscr{A}_W$ is a C^r foliation atlas on W. Let α'_W be the C^r differentiable structure on W generated by $\mathscr{A}_W$. We have $\alpha'_W\subset\beta\mid W$. We can extend α'_W to a C^r differentiable structure α' on M with $\alpha'\subset\beta$. Then the foliation $\mathscr{F}$ is a C^k foliation on $M_{\alpha'}$, with $\mathscr{F}\mid W$ a C^r foliation on W considered as a subspace of $M_{\alpha'}$ ($\mathscr{A}_W$ is an atlas for it). Since α and α' are contained in β, the identity map

$$\mathrm{id}:M_\alpha\to M_{\alpha'}$$

is a C^k diffeomorphism. Approximate id by a C^r diffeomorphism $h:M_\alpha\to M_{\alpha'}$, with h C^k δ-close to id (see Munkres [4]). Put $U=h^{-1}W$ and $\mathscr{F}'=h^{-1}\mathscr{F}$. Then U is dense in M, $\mathscr{F}'$ is of class C^k, $\mathscr{F}'\mid U$ is of class C^r. Since α, $\alpha'\subset\beta$, h is a C^k diffeomorphism of M_α, which implies that $\mathscr{F}$ and $\mathscr{F}'$ are C^k conjugate, and by the proposition that $\mathscr{F}$ and $\mathscr{F}'$ are C^k δ-close.

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