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A nonoscillation theorem for a second order sublinear retarded differential equation

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Sufficient conditions are obtained for all solutions of a class of second order nonlinear functional differential equations to be nonoscillatory.

This paper is concerned with the second order functional differential equation

$$(1) \qquad \big(p(t)y'(t)\big)' + q(t)f\big(y(g(t))\big) = r(t) ,$$

where $p, q, r, g : [a, \infty) \to R$ and $f : R \to R$ are continuous functions. In addition, it will be assumed throughout that $p(t) > 0$, $q(t) > 0$, $g(t) \to \infty$ as $t \to \infty$, $f(y)$ is nondecreasing, and $yf(y) > 0$ for $y \neq 0$. Equation (1) is said to be *sublinear* if $\limsup_{|y|\to\infty} f(y)/y < \infty$ and *retarded* if $g(t) \le t$ for all large t.

We shall restrict our attention to solutions $y(t)$ of (1) which are defined on some ray $[T_y, \infty)$ and nontrivial in the sense that

$$\sup\{|y(t)| : t \ge T\} > 0 \text{ for every } T \ge T_y .$$

Such a solution is called nonoscillatory if it is eventually positive or negative. Otherwise the solution is called oscillatory.

The objective of this paper is to obtain sufficient conditions for all solutions of the sublinear retarded equation (1) to be nonoscillatory.

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Since the pioneering work of Atkinson [1] numerous nonoscillation results for nonlinear differential equations have appeared in the literature; see, for example, a survey paper of Wong [5] and the references cited therein. However, most of them pertain to equations without forcing term, and very little is known about the nonoscillation of equations with forcing term even in the case of ordinary differential equations. To our knowledge, the papers of Graef and Spikes [2], [3], [4] are the only references in which nonoscillation criteria for forced nonlinear differential equations can be found.

LEMMA. *(i) No oscillatory solutions of* (1) *are bounded above if*

$$\liminf_{t\to\infty} \int_T^t [r(s)-kq(s)]ds > 0 \quad \text{for any } k > 0 \text{ and } T > a . \tag{2}$$

(ii) No oscillatory solutions of (1) *are bounded below if*

$$\limsup_{t\to\infty} \int_T^t [r(s)+kq(s)]ds < 0 \quad \text{for any } k > 0 \text{ and } T > a . \tag{3}$$

Proof. It suffices to prove the statement *(i)*. Let $y(t)$ be an oscillatory solution of (1) such that $y(t) \le M$ for $t \ge t_0$, where M is a positive constant. Take $t_1 \ge t_0$ so that $g(t) \ge t_0$ for $t \ge t_1$ and let $T \ge t_1$ be a point at which $y'(I) = 0$. An integration of (1) yields

$$p(t)y'(t) = \int_T^t \left[r(s)-q(s)f\left(y\left(g(s)\right)\right)\right]ds$$

$$\ge \int_T^t [r(s)-f(M)q(s)]ds .$$

Letting $t \to \infty$ and using (2), we see that $p(t)y'(t) > 0$ for all large t. But this is impossible for an oscillatory function $y(t)$, and the proof is complete.

REMARK 1. The condition (2) [or (3)] is satisfied, for example, if

$$\int_a^\infty q(t)dt < \infty \quad \text{and} \quad \int_a^\infty r(t)dt = \infty \quad [\text{or } -\infty] ,$$

or if

$$\lim_{t\to\infty} \frac{r(t)}{q(t)} = \infty \quad [\text{or} \quad -\infty\ ].$$

REMARK 2. Suppose that $\int_a^\infty \frac{dt}{p(t)} = \infty$ and replace (2) by a stronger condition

(4) $$\lim_{t\to\infty} \int_a^t [r(s)-kq(s)]ds = \infty \quad \text{for any} \quad k > 0 \ .$$

Then, we have a stronger conclusion that no solutions of (1) are bounded above. A similar remark applies to the case where (3) is replaced by

(5) $$\lim_{t\to\infty} \int_a^t [r(s)+kq(s)]ds = -\infty \quad \text{for any} \quad k > 0 \ .$$

We now state and prove the main result of this paper.

THEOREM. *Let* (1) *be a sublinear retarded equation. Assume that*

(6) $$\int_a^\infty \left(\int_a^t \frac{ds}{p(s)}\right) q(t)dt < \infty \quad \text{if} \quad \int_a^\infty \frac{dt}{p(t)} = \infty ,$$

and

(7) $$\int_a^\infty q(t)dt < \infty \quad \text{if} \quad \int_a^\infty \frac{dt}{p(t)} < \infty .$$

Assume, moreover, that either $r(t) \geq 0$ *and* (2) *holds or* $r(t) \leq 0$ *and* (3) *holds. Then all solutions of* (1) *are nonoscillatory.*

Proof. Consider the case where $r(t) \geq 0$ and (2) holds. Suppose to the contrary that there exists an oscillatory solution $y(t)$ of (1). From *(i)* of the lemma it follows that $y(t)$ is not bounded above. Therefore, it is possible to select two sequences $\{\sigma_n\}$, $\{\tau_n\}$ of zeros of $y(t)$ with the following properties: $\sigma_n < \tau_n$, $\lim_{n\to\infty} \sigma_n = \lim_{n\to\infty} \tau_n = \infty$, $y(t) > 0$ on (σ_n, τ_n) ,

$$M_n = \max_{[\sigma_n,\tau_n]} y(t) = \max_{[\sigma_1,\tau_n]} y(t) \ , \quad n = 1, 2, \dots ,$$

and $\{M_n\}$ tends increasingly to infinity as $n \to \infty$. Let $\{t_n\}$ be a

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sequence such that $t_n \in (\sigma_n, \tau_n)$ and $M_n = y(t_n)$, $n = 1, 2, \ldots$. Integrating (1) from $t \in [\sigma_n, t_n]$ to t_n , we obtain, since $y'(t_n) = 0$ and $r(t) \geq 0$,

$$p(t)y'(t) = \int_t^{t_n} [q(s)f(y(g(s)))-r(s)]ds$$

$$\leq \int_t^{t_n} q(s)f(y(g(s)))ds ,$$

where n is taken so large that $g(t) \geq \sigma_1$ for $t \geq \sigma_n$. Dividing the above inequality by $p(t)$ and integrating from σ_n to t_n , we have

$$(8) \qquad y(t_n) \leq \int_{\sigma_n}^{t_n} \frac{1}{p(t)} \int_t^{t_n} q(s)f(y(g(s)))dsdt .$$

Since $g(t) \leq t$, we have $y(g(t)) \leq M_n$ for $t \leq t_n$. Hence it is true that

$$M_n \leq f(M_n) \int_{\sigma_n}^{t_n} \frac{1}{p(t)} \int_t^{t_n} q(s)dsdt ,$$

from which, observing that (6) or (7) implies

$$(9) \qquad \int_a^{\infty} \frac{1}{p(t)} \int_t^{\infty} q(s)dsdt < \infty ,$$

we obtain

$$(10) \qquad 1 \leq \frac{f(M_n)}{M_n} \int_{\sigma_n}^{\infty} \frac{1}{p(t)} \int_t^{\infty} q(s)dsdt .$$

In view of (9) and the fact that $\{f(M_n)/M_n\}$ is bounded above, the right-hand side of (10) tends to zero as $n \to \infty$. But this is a contradiction. A similar argument leads us to a contradiction if we suppose that $r(t) \leq 0$ and (3) holds. This completes the proof.

REMARK 3. From the theorem and Remark 2 we have the following proposition.

Let (1) *be sublinear and retarded. Suppose* (6) *holds. If* $r(t) \geq 0$ *and* (4) *is satisfied, then all solutions of* (1) *are nonoscillatory and unbounded above. If* $r(t) \leq 0$ *and* (5) *is satisfied, then all solutions of* (1) *are nonoscillatory and unbounded below.*

EXAMPLE 1. Consider the equation

$$y''(t) + t^{-\alpha}y(t) = t^{-1}[1 + \cos(\log t)] , \quad t \geq 1 . \tag{11}$$

If $\alpha > 2$, then (2) and (6) are satisfied, so that all solutions of (11) are nonoscillatory. (These nonoscillatory solutions are unbounded above since (4) is also satisfied.) If $\alpha = 2$, then (6) is violated, and (11) has an oscillatory solution $y(t) = t[1 + \sin(\log t)]$.

EXAMPLE 2. Consider the equation

$$\left(t^{3/2}y'(t)\right)' + t^{-3/2}\left|y\left(t^{\beta}\right)\right|^{\gamma} \operatorname{sgn} y\left(t^{\beta}\right) = (1/2)t^{-1} - \beta^{\gamma}t^{-(\beta\gamma+3)/2} \log^{\gamma} t , \quad t \geq 1 , \tag{12}$$

where $0 < \beta \leq 1$ and $0 < \gamma \leq 1$. Since (2) and (7) are satisfied, all solutions of (12) are nonoscillatory. In fact, (12) possesses a nonoscillatory solution $y(t) = -t^{-1/2} \log t$.

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