ENUMERATION OF QUADRANGULAR DISSECTIONS OF THE DISK

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I. Introduction

1. A *dissection* of the disk will be a cell complex (**1**, p. 39) K with polyhedron the closed disk B^2. It will further be required that:

(a) every edge of K be incident with two distinct vertices (called its *ends*);

(b) no two edges have the same ends; and

(c) every vertex be incident with at least two edges.

Cells of K will be said to be *external* or *internal* according as they do or do not lie in the boundary of B^2; $\dot{K}$ will denote the subcomplex formed by the external cells of K.

A dissection is *rooted* if an external oriented edge is designated as its *root.*

A dissection of the disk will be called a *triangulation* or *quadrangulation* according as all of its faces are incident respectively with three or four edges. In (**4**) the author has shown that the number of inequivalent rooted triangulations having $m + 3$ external and n internal vertices is

$$\frac{2\,(2m+3)!\,(4n+2m+1)!}{(m+2)!\,m!\,n!\,(3n+2m+3)!}$$

and has also determined the numbers of such triangulations having various symmetries. It will be shown in this paper that the number of inequivalent rooted quadrangulations having $2p + 4$ external and n internal vertices is

$$\frac{3\,(3p+4)!\,(3n+3p+2)!}{(2p+3)!\,p!\,n!\,(2n+3p+4)!} \qquad \text{(cf. §3).}$$

The numbers of such quadrangulations having various symmetries will also be determined, and asymptotic estimates obtained.

2. Historical note. The problem of enumerating rooted triangulations with no internal vertices appears to have originated with Euler; its history is discussed in (**4, 6, 15**) and references therein cited. The more general problem of enumerating rooted dissections of the disk with no internal vertices but with faces of arbitrary valency seems to have been first posed by Pfaff to N. Fuss. (The valency of a face is the number of edges with which it is incident.) The solution of Fuss (**9**), published in 1793, is similar to that of Segner (**14**) to the triangulation problem: an algorithm for computing the numbers is provided. Closed-form solutions have been given in (**2, 7, 8, 15**). Also, certain extensions to cases of internal vertices may have been known to T. P. Kirkman.

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3. Isomorphisms. The classes of unoriented cells of a quadrangulation K and its boundary $\dot{K}$ will be denoted respectively by $\langle K\rangle$, $\langle\dot{K}\rangle$.

Let K, L be quadrangulations. A *homeomorphism* $(f):K \to L$ will be a homeomorphism $(f):B^2 \to B^2$ carrying i-cells of K onto i-cells of L $(i = 0, 1, 2)$. An *isomorphism* $f:K \to L$ will be a bijection $f:\langle K\rangle \to \langle L\rangle$ mapping i-cells onto i-cells $(i = 0, 1, 2)$ such that any two cells a, b of K are incident in K if and only if fa, fb are incident in L. Conditions (a), (b), and (c) of §1 ensure that the boundary of every face of a quadrangulation is a simple quadrangle (having no singularities) and so we have the following:

(3.1) LEMMA. (a) *A homeomorphism $(f):K \to L$ induces an isomorphism $f:K \to L$ defined by*

$$fa =_{\text{def}} (f)a \qquad (a \in \langle K\rangle). \tag{*}$$

(b) *An isomorphism $f:K \to L$ induces homeomorphisms $(f):K \to L$ satisfying* (*).

The proofs will not be given here.

An *oriented* quadrangulation K^* will be a quadrangulation K for which $\dot{K}$ is assigned a definite orientation; $\dot{K}$ so oriented will be denoted by $\dot{K}^*$. Let K^*, L^* be oriented quadrangulations. An isomorphism $f:K^* \to L^*$ will be an isomorphism $f:K \to L$ preserving the orientation.

A *rooted* quadrangulation will be an ordered pair (K^*, a), where K^* is an oriented quadrangulation of which a is an external vertex. An isomorphism $f:(K^*, a) \to (L^*, b)$ will be an isomorphism $f:K^* \to L^*$ such that $fa = b$.

K, K^*, or (K^*, a) will be said to be *of type* $[n, m]$ if K has $m + 4$ external and n internal vertices.

Two quadrangulations, oriented quadrangulations, or rooted quadrangulations will be said to be *inequivalent* if there exists no isomorphism from one to the other. In §II inequivalent rooted quardangulations of type $[n, m]$ will be enumerated; in §III inequivalent oriented quadrangulations of type $[n, m]$ will be enumerated; and in §IV methods will be described for enumerating inequivalent quadrangulations of type $[n, m]$.

II. Rooted quadrangulations

4. The number of rooted quadrangulations of type $[n, m]$ will be denoted by $U_{n,m}$. We define the generating function U as a formal power series in indeterminates x and y,

$$U = U(x, y) = \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} U_{n,m}\, x^n y^m;$$

$U(x, 0)$ will be abbreviated to U_0.

The methods used to determine $U_{n,m}$ will be similar to those of **(3, 4, 17)**; a functional equation satisfied by U will be found and solved.

5. An equation for U. Let a rooted quadrangulation (K^*, a) be given; wherein b is the vertex of $\dot{K}^*$ following a, and the boundary of the face incident with a and b contains the vertices a, b, c, and d in cyclic order. By considering the various possible cases where either or both of c and d are external or internal we can obtain an equation satisfied by U. Proofs are analogous to those of **(4)** (familiarity with which is assumed) and will not be given here.

In Table I the possible forms of (K^*, a) are tabulated with the series enumerating each. The subtracted term $x^2U_0\,U$ in the first case is analogous to a similar term in **(4**, §3, Case 2**)**; it ensures that condition (b) of §1 will be satisfied (cf. Fig. 1(*b*)).

TABLE I

$c \in \langle\dot{K}\rangle$	$d \in \langle\dot{K}\rangle$	Enumerating series	Figure
No	No	$x^2y^{-2}(U - U_0) - x^2U_0U$	1(*a*)
No	Yes	$xU(1 + y^2U)$	1(*c*)
Yes	No	$xU(1 + y^2U)$	1(*d*)
Yes	Yes	$(1 + y^2U)^3$	1(*e*)

Thus U satisfies the equation

$$U = x^2y^{-2}(U - U_0) - x^2U_0U + 2xU(1 + y^2U) + (1 + y^2U)^3. \tag{5.1}$$

6. Solution of equation (5.1). It can be shown by rewriting (5.1) in the form of a recurrence for $U_{n,m}$ that the equation has a unique solution U which is a formal series in non-negative powers of x and y **(4**, §3**)**.

We define a formal power series

$$S(x) = \sum_{n=1}^{\infty} \frac{(3n-2)!}{n!\,(2n-1)!} x^n$$

and note, by Lagrange's theorem **(18**, p. 132**)**, that $S(x)[1 - S(x)]^2 = x$. We note further that, for positive integers t,

$$[1 - S(x)]^{-t} = \sum_{n=0}^{\infty} \binom{n+t-1}{n} S(x)^n$$

is a well-defined power series **(5**, (2.2)**)**; the coefficient of each power of x is expressible as a finite sum of finite products of integers. We abbreviate $S(x)$, $1 - S(x)$, respectively to u, v, and define z, w to be respectively the formal power series in x and y, $S(y^2v^{-3})$ and $1 - S(y^2v^{-3})$, which, by virtue of the foregoing remark, are also well defined. Thus

$$x = uv^2, \tag{6.1}$$

$$y^2 = zw^2v^3. \tag{6.2}$$

We now *conjecture* that

$$U = (1 - 2u - vz)v^{-4}w^{-4} \tag{6.3}$$

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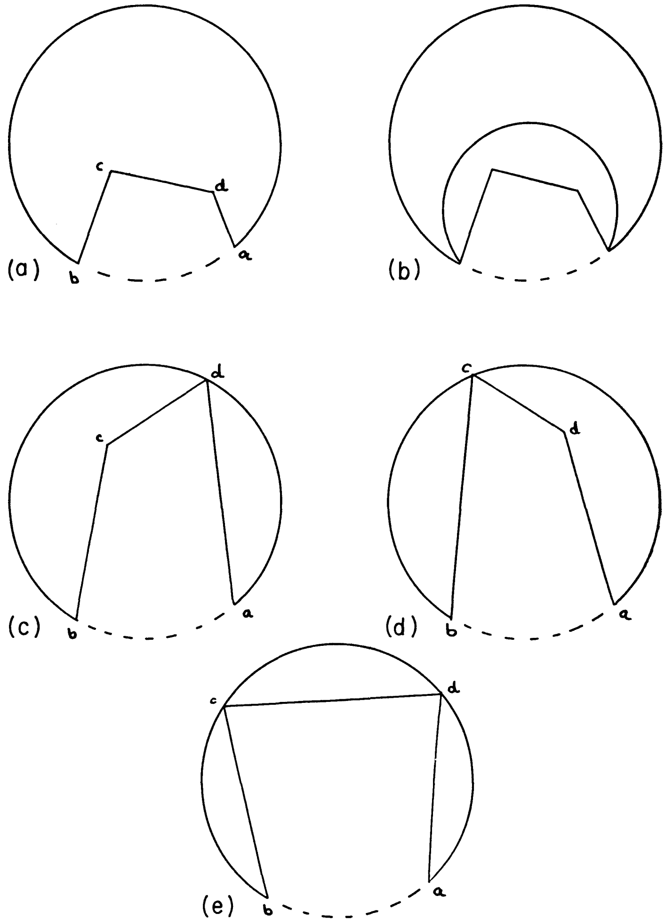

FIGURE 1

and, consequently, that $U_0 = (1-2u)v^{-4}$. If we define W to be $1 + y^2 U$, then (6.3) implies that $W = (v-z)v^{-1}w^{-2}$ and $y^2 W + x = v^2 w(u+z)$. The right side of (5.1) can be seen to be equal to

$$y^{-4}[W(y^2W+x)^2 - (y^2W+x)(x^2U_0+2x) + (x^3U_0+x^2)],$$

which becomes, under substitution (6.3),

$$\begin{aligned} &y^{-4}[v^3(v-z)(u+z^2) - uv^2w(u+z)(2-3u) + u^2v^2(1-u-u^2)] \\ &\qquad = y^{-4}[v^2z^2(1-2u-vz)] = (1-2u-vz)v^{-4}w^{-4}. \end{aligned}$$

Thus (6.3) is indeed a solution of (5.1), and hence the only solution expressible as a series of non-negative powers of x and y.

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To determine the coefficients in U we note that by Lagrange's theorem applied to (6.1), (6.2) or otherwise,

$$v^{-t} = t\sum_{n=0}^{\infty} \frac{(3n+t-1)!}{n!\,(2n+t)!} x^n, \tag{6.4}$$

$$w^{-t} = t\sum_{m=0}^{\infty} \frac{(3m+t-1)!}{m!\,(2m+t)!} y^{2m} v^{-3m} \tag{6.5}$$

for positive integers t (3, (4.14)). Thus

$$\begin{aligned} U &= v^{-4}w^{-4} - 2xv^{-6}w^{-4} - y^2v^{-6}w^{-6} \\ &= 4\sum_{m=0}^{\infty} \frac{(3m+3)!}{m!\,(2m+4)!} y^{2m}v^{-3m-4} - 8x\sum_{m=0}^{\infty} \frac{(3m+3)!}{m!\,(2m+4)!} y^{2m}v^{-3m-6} \\ &\qquad - 6\sum_{m=1}^{\infty} \frac{(3m+2)!}{(m-1)!\,(2m+4)!} y^{2m}v^{-3m-3} \qquad \text{by (6.5)} \\ &= 6\sum_{m=0}^{\infty} \frac{(3m+2)!}{m!\,(2m+4)!} [(m+1)(2v^{-3m-4} - 4xv^{-3m-6}) - mv^{-3m-3}]y^{2m}. \end{aligned}$$

Now, by (6.4),

$$\begin{aligned} &(m+1)(2v^{-3m-4} - 4xv^{-3m-6}) - mv^{-3m-3} \\ &\quad = (m+1)\Big\{2(3m+4)\sum_{n=0}^{\infty} \frac{(3n+3m+3)!}{n!\,(2n+3m+4)!} x^n \\ &\qquad\qquad - 4(3m+6)\sum_{n=0}^{\infty} \frac{n\,(3n+3m+2)!}{n!\,(2n+3m+4)!} x^n \\ &\qquad\qquad\qquad - 3m\sum_{n=0}^{\infty} \frac{(3n+3m+2)!}{n!\,(2n+3m+3)!} x^n\Big\} \\ &\quad = 3(m+1)(m+2)(3m+4)\sum_{n=0}^{\infty} \frac{(3n+3m+2)!}{n!\,(2n+3m+4)!} x^n. \end{aligned}$$

Hence

$$U = 18\sum_{m=0}^{\infty}\sum_{n=0}^{\infty} \frac{(m+1)(m+2)(3m+4)\,(3m+2)!\,(3n+3m+2)!}{(2m+4)!\,m!\,n!\,(2n+3m+4)!} x^n y^{2m}$$

and so

$$U_{n,2p} = \frac{3\,(3p+4)!\,(3n+3p+2)!}{(2p+3)!\,p!\,n!\,(2n+3p+4)!}, \tag{6.6}$$

$$U_{n,2p+1} = 0 \qquad (n = 0, 1, 2, \ldots; p = 0, 1, 2, \ldots). \tag{6.7}$$

Of course, (6.7) could have been proved by other means. In fact (6.7) was assumed in the development of (5.1).

As in **(4)**, we can apply Stirling's formula to obtain an asymptotic estimate for $U_{n,m}$. For fixed p, as $n \to \infty$,

$$U_{n,2p} \sim \frac{(3p+4)!}{2\,(2p+3)!\,p!}\left(\frac{27}{8}\right)^{p}\left(\frac{27}{4}\right)^{n+1} n^{-5/2}\sqrt{\frac{3}{\pi}}. \tag{6.8}$$

Another method for solving (5.1) is described in **(5**, §4**)**. There it is shown how algebraic equations satisfied by each of U, U_0 alone can be obtained from (5.1).

7. Relations between quadrangulations and non-separable planar maps. For this section familiarity with **(16,** §§ 1–6**)** is assumed.

It is not coincidental that x^2U_0 is the same series as $B(x) - 2x = B(x, 1)$ **(3, 16)**, which enumerates rooted non-separable planar maps; i.e., that the number of such maps having $n + 2$ edges is equal to $U_{n,0}$. In what follows a one-to-one correspondence between the two classes will be described.

Let M be a non-separable planar map having $n + 2$ edges. Corresponding to M there is a unique *derivable* map M' **(16,** §2**)**, i.e. a triangular map whose vertex set is the disjoint union of three sets, W, W^*, W^+, with the following properties:

(i) no edge of M' has both ends in the same class W, W^*, W^+; and

(ii) each vertex of W^+ has valency 4.

Moreover, we can assert that

(iii) $||W^+|| = n + 2$.

(For any set X, $||X||$ will denote its cardinality.) If the vertices in W^+ and all their incident edges are erased, the resulting dissection $\bar{M}$ of the sphere has $n + 2$ faces, all of valency 4; hence $2(n + 2)$ edges, and

$$2 + 2(n+2) - (n+2) = n+4$$

vertices. A rooting of M induces a rooting in $\bar{M}$; the complement of the face to the left of the root in $\bar{M}$ is a rooted quadrangulation of type $[(n + 4) - 4, 4 - 4]$. (It can be shown that the non-separability of M implies that conditions (a), (b), and (c) of §1 are satisfied.)

Conversely, let a rooted quadrangulation (K^*, a) be given, of type $[n, 0]$. By attaching to $\dot{K}$ a quadrangular face one obtains a rooted map $\bar{M}$ on the sphere, all of whose faces have valency 4. Any simple polygon formed by edges of this map determines two quadrangulations; hence, by (6.7), the polygon contains an even number of vertices. By **(11**, p. 151, Satz 12**)** the vertices of the map can be separated into two classes W, W^* so that no edge has both ends in the same class. Incident with each face F are two vertices of W. Let these be joined by an edge in F; then let the vertices in W^* and all incident edges be erased. The resulting map M can be shown to be non-separable. A rooting of K induces one of M, and M contains $n + 2$ edges (since $\bar{M}$ contained $n + 2$ faces).

8. Rooted triangulations: a simplified solution of equation (4, (3.7)). The solution given for (5.1) in §6 suggests a simple method for solving (**4**, (3.7)). Define formal power series

$$T(x) = \sum_{n=1}^{\infty} \frac{(4n-2)!}{n!\,(3n-1)!} x^n,$$

$$R(x) = \sum_{n=1}^{\infty} \frac{(2n-2)!}{n!\,(n-1)!},$$

and set u, v, z, w respectively equal to $T(x)$, $1 - T(x)$, $R(yv^{-2})$, $1 - R(yv^{-2})$; then $x = uv^3$ and $y = zwv^2$. After proving that the equation has a unique solution, it can be easily verified that

$$D(x, y) = (1 - 2u - vz)v^{-3}w^{-3}.$$

The coefficients can then be obtained as in §6.

III. Quadrangulations with rotational symmetry

9. Automorphisms. Let K be a quadrangulation of type $[n, m]$ of which a is an external vertex. The isomorphisms of K, K^*, or (K^*, a) with themselves will be called *automorphisms*. If the product of two automorphisms is defined in the obvious way, these form groups, which will be denoted respectively by $\mathfrak{A}(K)$, $\mathfrak{A}(K^*)$, $\mathfrak{A}(K^*, a)$.

An *automorphism* of $\dot{K}$ or $\dot{K}^*$ will be a bijection from $\langle \dot{K} \rangle$ to itself preserving incidences, and, in the case of $\dot{K}^*$, orientation. These automorphisms also form groups, which will be denoted by $\mathfrak{A}(\dot{K})$ and $\mathfrak{A}(\dot{K}^*)$ respectively. It is well known that $\mathfrak{A}(\dot{K}) \cong \mathfrak{D}_{m+4}$, the dihedral group of order $2(m + 4)$; and $\mathfrak{A}(\dot{K}^*) \cong \mathfrak{C}_{m+4}$, the cyclic group of order $m + 4$.

Automorphisms $f: K \to K$, $g: K^* \to K^*$ induce automorphisms $\partial f: \dot{K} \to \dot{K}$, $\partial^* g: \dot{K}^* \to \dot{K}^*$ defined by $\partial f = f|\langle \dot{K} \rangle$, $\partial^* g = g|\langle \dot{K} \rangle$. Moreover, as an automorphism of K is determined by its action on $\dot{K}$ (**3**, (5.1)), the mappings $\partial: \mathfrak{A}(K) \to \mathfrak{A}(\dot{K})$, $\partial^*: \mathfrak{A}(K^*) \to \mathfrak{A}(\dot{K}^*)$ are monomorphisms. Thus $\mathfrak{A}(K)$, $\mathfrak{A}(K^*)$ are, respectively, isomorphic to subgroups of $\mathfrak{D}_{m+4}$, $\mathfrak{C}_{m+4}$; $\mathfrak{A}(K^*)$ is of index 1 or 2 in $\mathfrak{A}(K)$. K, K^*, or (K^*, a) will be said to be *of type* $[n, m; r]$ if r divides the order of $\mathfrak{A}(K^*)$; and *of type* $[n, m]^-$ or *of type* $[n, m]^+$ according as the order of $\mathfrak{A}(K)/\mathfrak{A}(K^*)$ is 2 or 1.

10. Oriented quadrangulations. Let ${}_rU_{n,m}$ be the number of rooted quadrangulations of type $[n, m; r]$. Then it can be shown by the theorem of Pólya (**13**, pp. 131 ff.) or otherwise (**3**, §6; **4**, §6) that the number, $Q_{n,m}$, of oriented quadrangulations of type $[n, m]$ is given by

$$Q_{n,m} = \frac{1}{m+4} \sum_r \phi(r)\, {}_rU_{n,m}, \tag{10.1}$$

where ϕ is the Euler function, and summation is effected over all r such that $r|(m + 4)$.

rU is defined to be the formal power series

$$\sum_{n=0}^{\infty} \sum_{m=0}^{\infty} {}_rU_{n,m}.$$

${}_rU_{.t}$ is defined to be

$$\sum_{n=0}^{\infty} {}_rU_{n,t}\, x^n$$

if $t \geqslant 0$ and 0 otherwise.

Clearly ${}_rU_{n,m} \neq 0$ implies that $r|(m + 4)$. It follows that ${}_rU\, y^4$ is of the form

$$ {}_rU\, y^4 = y^{2r}\, {}_rG + y^r\, {}_rH, \tag{10.2}$$

where ${}_rG = {}_rG(x, y^{2r})$ and ${}_rH = {}_rH(x, y^{2r})$ are formal power series in x and y^{2r}. By (6.7) ${}_rH = 0$ for r odd.

As in **(3, 4)** we define for any set theoretic mapping $f:X \to X$ and any integer $r > 0$, a multivalued mapping $\tilde{f}_r$ which associates with each x in X the set $\tilde{f}_r\, x = \{x, fx, f^2x, \ldots, f^{r-1}x\}$.

11. An equation for ${}_rU$. Clearly ${}_1U = U$. We assume that $r > 1$.

Let (K^*, a) be a rooted quadrangulation of type $[n, m; r]$, and let b, c, d be defined as in §5. By considering the possible cases where c and d are external or internal, and various possible coincidences, we shall obtain an equation for ${}_rU$. The method is analogous to that of **(4**, §7**)** and proofs will be omitted; the cases will be tabulated with their respective enumerating series in Table II. f will be the generator of $\mathfrak{A}(K^*)$ which rotates a through $(m + 4)/r$ edges in the direction of K^*.

The subtracted term $x^{2r}\bar{U}_0\, {}_rU$ in the first case is analogous to a similar term in Case 2 of **(4**, §7**)** (cf. Fig. 2(*b*)). $\bar{U}$, $\bar{U}_0$, and $\bar{W}$ will respectively denote $U(x^r, y^r)$, $U(x^r, 0)$, and $1 + y^{2r}U(x^r, y^r)$.

Adding the enumerating series for ${}_rU$ we obtain, after reduction,

$$\begin{aligned} {}_rF\, {}_rUy^4 = {} & [-x^{2r}y^{-2r} - 2x^r\bar{W} + x^r\bar{W}]_rU_{.r-4}\, y^r \\ & - x^{2r}\, {}_rU_{.2r-4} + [\delta_{r,4}\, y^4 + 2xy^{2r}\, {}_rE](r > 2), \end{aligned} \tag{11.1}$$

$$ {}_2F\, {}_2Uy^4 = -\, x^4\, {}_2U_{.0} - x^4y^2\, {}_2U_{.2} + [3y^2 + 2x]y^4\, {}_2E \tag{11.2}$$

where

$$\begin{aligned} {}_rF &= 1 - x^{2r}y^{-2r} + x^{2r}\bar{U}_0 - 2x^r\bar{W} - 3y^{2r}\bar{W}^2 - 2x^ry^{2r}\bar{U}, \\ {}_rE &= x^r\bar{U} + \bar{W}^2. \end{aligned}$$

Applying (10.2) we obtain

$$\left.\begin{aligned} y^{2r}\, {}_rF\, {}_rG &= -x^{2r}\, {}_rU_{.2r-4} + 2x\, {}_rE, && (11.3) \\ y^{2r}\, {}_rF\, {}_rH &= -\, [x^ry^{2r}\bar{W} + x^{2r}]_rU_{.r-4} + \delta_{r,4}\, y^4, && (11.4) \end{aligned}\right\} r > 2,$$

$$ y^4\, {}_2F\, {}_2G = -\, x^4\, {}_2U_{.0} + 2xy^4\, {}_2E, \tag{11.5}$$

$$ y^4\, {}_2F\, {}_2H = -\, x^4\, {}_2U_{.2} + 3y^4\, {}_2E. \tag{11.6}$$

TABLE II

$c \in \langle \dot{K} \rangle$	$d \in \langle \dot{K} \rangle$	$\lvert\lvert \tilde{f}_r c \rvert\rvert$	$\lvert\lvert \tilde{f}_r d \rvert\rvert$	$\lvert\lvert \tilde{f}_r c \cup \tilde{f}_r d \rvert\rvert$	Enumerating series ($r > 2$)	Enumerating series ($r = 2$)	Figure
(i) No	No	r	r	$2r$	$x^{2r}y^{-2r}[{}_rU - {}_rU_{.r-4}\, y^{r-4} - {}_rU_{.2r-4}\, y^{2r-4}] - x^{2r}\bar{U}_0\, {}_rU$	$x^4y^{-4}[{}_2U - {}_2U_{.0} - {}_2U_{.2}\, y^2] - x^4\bar{U}_0\, {}_2U$	$2(a)$
(ii) No	Yes	r	r	$2r$	$x^r\bar{W}[{}_rU - {}_rU_{.r-4}\, y^{r-4}] + x^ry^{2r}\bar{U}\, {}_rU$	$x^2\bar{W}_2U + x^2y^2\bar{U}[1 + y^2\, {}_2U]$	$2(c)$ $2(d)$
(iii) Yes	No	r	r	$2r$	Same as (ii)	Same as (ii)	
(iv) Yes	Yes	r	r	Unrestricted	$3y^{2r}\bar{W}^2{}_rU + \delta_{r,4}$	$3y^2\bar{W}^2[1 + y^2\, {}_2U]$	$2(e), 2(f)$
(v) No	No	1	r	$r + 1$	$x^{r+1}y^{2r-4}\bar{U}$	$x^3\bar{U}$	$2(g)$
(vi) No	No	r	1	$r + 1$	Same as (v)	Same as (v)	
(vii) Yes	No	r	1	$r + 1$	$xy^{2r-4}\bar{W}^2$	$x\bar{W}^2$	$2(h)$
(viii) No	Yes	1	r	$r + 1$	Same as (vii)	Same as (vii)	
(ix) No	No	r	r	r	$x^ry^{r-4}\bar{W}\, {}_rU_{.r-4}$	$x^2y^2\bar{U}$	$2(i)$

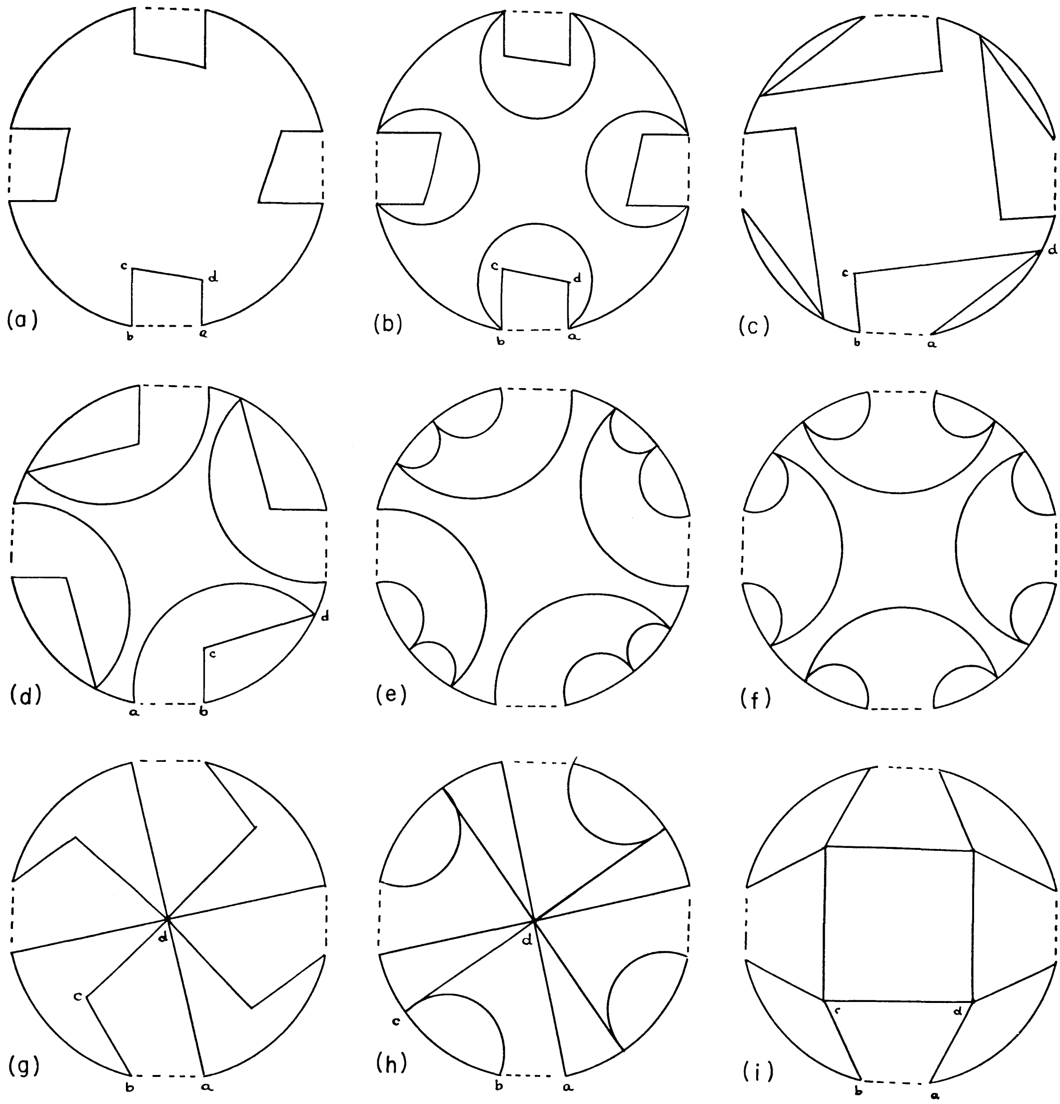

FIGURE 2

If u, v, z, w are respectively defined to be the series $S(x^r)$, $1 - S(x^r)$, $S(y^{2r}v^{-3})$, and $1 - S(y^{2r}v^{-3})$ it can be shown that

$$(11.7) \qquad y^{2r}\,{}_rF = -v^4w(1 - 3z)(u^2 - vz + z^2)$$

and

$${}_rE = v^{-2}w^{-3}(1 - u - u^2 - z).$$

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The series obtained by setting

$$y^{2r} = u^2v^3(1+u) \tag{11.8}$$

in $\bar{U}$ is well defined (**5**, (2.2)). Moreover, (11.8) implies that

$$(1+u-z)(u^2-vz+z^2) = 0.$$

But $S\{[u^2v^3(1+u)]v^{-3}\} \neq 1+u$. Hence under the substitution (11.8), ${}_rF = 0$. Applying (11.8) to (11.3)–(11.6) we obtain after reduction

$$\begin{cases} {}_rU_{.2r-4} = 2xv^{-3}, \\ {}_rU_{.r-4} = \delta_{r,4}\, v^{-1}, \end{cases} \quad r > 2, \qquad \begin{aligned} {}_2U_{.0} &= 2xv^{-3}, \\ {}_2U_{.2} &= 3v^{-3}. \end{aligned} \tag{11.9}$$

These expressions, substituted in (11.1) and (11.2), yield, again after reduction,

$$\begin{cases} {}_rU = (1-3z)^{-1}(\delta_{r,4}\, v^{-1} + 2xzy^{-4}) \qquad (r > 2), \\ {}_2U = (1-3z)^{-1}(2xy^{-4}+3y^{-2})z. \end{cases} \tag{11.10}$$

To compute the coefficients in ${}_rU$, we note that, by (6.4) and (6.5),

$$\begin{aligned} z(1-3z)^{-1} &= \sum_{m=0}^{\infty} 3^m y^{2(m+1)r} w^{-2(m+1)} v^{-3(m+1)} \\ &= 2\sum_{m=0}^{\infty}\sum_{s=0}^{\infty} 3^m(m+1)\frac{(3s+2m+1)!}{s!\,(2s+2m+2)!} v^{-3(m+s+1)} y^{2(m+s+1)r} \\ &= 6\sum_{m=0}^{\infty}\sum_{s=0}^{\infty}\sum_{n=0}^{\infty} 3^m(m+1)(m+s+1) \\ &\qquad \times \frac{(3s+2m+1)!\,(3n+3m+3s+2)!}{s!\,(2s+2m+2)!\,n!\,(2n+3m+3s+3)!} x^{nr} y^{2(m+s+1)r} \\ &= 6\sum_{p=1}^{\infty}\sum_{n=0}^{\infty}\sum_{m=0}^{p-1} 3^m(m+1)p\frac{(3p-m-2)!\,(3p+3n-1)!}{(p-m-1)!\,(2p)!\,n!\,(2n+3p)!} x^{nr} y^{2pr}. \end{aligned}$$

Similarly, it can be shown that

$$\begin{aligned} z(1-3z)^{-1}v^{-1} = 6\sum_{p=1}^{\infty}\sum_{n=0}^{\infty}\sum_{m=0}^{p-1} 3^m(m+1)(p+1) \\ \times \frac{(3p-m-2)!\,(3p+3n)!}{(p-m-1)!\,(2p)!\,n!\,(2n+3p+1)!} x^{nr} y^{2pr}. \end{aligned}$$

Noting that $(1-3z)^{-1}v^{-1} = v^{-1} + 3z(1-3z)^{-1}v^{-1}$, we see that

$${}_rU_{nr+1,2pr-4} = 12p\frac{(3p+3n-1)!}{n!\,(2p)!\,(2n+3p)!}\lambda_p, \tag{11.11}$$

$${}_2U_{2n,4p-2} = 18p\frac{(3p+3n-1)!}{n!\,(2p)!\,(2n+3p)!}\lambda_p, \tag{11.12}$$

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$$(11.13) \qquad {}_4U_{4n,0} = \frac{(3n)!}{n!\,(2n+1)!},$$

$$(11.14) \qquad {}_4U_{4n,4p} = 18\,\frac{(p+1)\,(3p+3n)!}{n!\,(2p)!\,(2n+3p+1)!}\,\lambda_p,$$

where

$$\lambda_p =_{\text{def}} \sum_{m=0}^{p-1} \frac{3^m(m+1)\,(3p-m-2)!}{(p-m-1)!},$$

$$(n = 0, 1, 2, \ldots ; p = 1, 2, 3, \ldots ; r = 2, 3, \ldots)$$

all other values of ${}_rU_{s,t}$ being zero.

12. Asymptotic behaviour of ${}_rU_{n,m}$. For fixed p as $n \to \infty$ it can be shown by Stirling's formula that

$$\frac{(3p+3n-1)!}{n!\,(2n+3p)!} \sim c_p\left(\frac{27}{4}\right)^{n+1} n^{-3/2},$$

where c_p is a constant which depends on p. Hence, for fixed p and r $(r > 1)$, as $n \to \infty$,

$$(12.1) \qquad {}_rU_{n,2p} \lesssim c_{r,p}(27/4)^{n/2},$$

where $c_{r,p}$ is a constant depending upon p and r. Hence

$$\sum_{r=2}^{2p+4} {}_rU_{n,2p} \lesssim d_p\left(\frac{27}{4}\right)^{n/2},$$

where d_p is a constant depending upon p. It follows from (10.1) that, for fixed p, as $n \to \infty$,

$$(12.2) \qquad Q_{n,2p} \sim \frac{U_{n,2p}}{2p+4} = \frac{1}{4}\binom{3p+4}{p}\left(\frac{27}{8}\right)^p\left(\frac{27}{4}\right)^{n+1} n^{-5/2}\sqrt{\frac{3}{2\pi}}\,.$$

Of course, $Q_{n,2p+1} = 0$, by (6.7).

It has thus been shown that, for fixed m, almost all quadrangulations of type $[n, m]$ are rotationally asymmetrical (**3**, §9; **4**, §9).

IV. Quadrangulations with reflectional symmetry

13. Quadrangulations of type $[n, m]^-$. Let (K^*, a) be a rooted quadrangulation of type $[n, m]^-$ and let b be the vertex following a in $\dot{K}^*$. (K^*, a) will be called a *Y-rooting* or *Z-rooting* of K if respectively a, or the edge with ends a and b, is invariant under some automorphism of K other than the identity. It can be shown (**6**, (6.1)) that K has, up to automorphisms, exactly two Y-rootings or two Z-rootings or one Y-rooting and one Z-rooting. Let $Y_{n,m}$, $Z_{n,m}$ respectively denote the numbers of inequivalent Y-rooted or Z-rooted quadrangulations of type $[n, m]^-$. If $R_{n,m}$ is the number of quadrangulations of type $[n, m]$, it can be shown **(4**, §10**)** that

$$R_{n,m} = (1/2)Q_{n,m} + (1/4)(Y_{n,m} + Z_{n,m}). \tag{13.1}$$

The generating functions Y, Z, Y_0, Z_0 will be defined analogously to U, U_0. In what follows we shall develop an equation for Z, from which we shall determine Z_0 and show that almost all quadrangulations of type $[n, 0]$ have no Z-rootings. We shall also state an equation for Y, together with a possible method for solving it.

14. An equation for Z**.** Let (K^*, a) be a Z-rooting for a quadrangulation K, with b, c, d as described in §5. By considering the possibilities of c and d being internal or external, we obtain an equation for Z **(4**, §11**)**. The cases are tabulated with the corresponding enumerating series in Table III. $\bar{W}$ here denotes $1 + y^4U(x^2, y^2)$.

TABLE III

$c \in \langle\dot{K}\rangle$	$d \in \langle\dot{K}\rangle$	Enumerating series	Figure
No	No	$x^2y^{-2}(Z - Z_0) - x^2Z_0Z$	1(a)
Yes	Yes	$\bar{W}(1 + y^2Z)$	1(e)

Thus Z satisfies

$$Z = x^2y^{-2}(Z - Z_0) - x^2Z_0 Z + \bar{W}(1 + y^2Z),$$

i.e.

$$(y^2 - x^2 + x^2y^2Z_0 - y^4\bar{W})Z = y^2\bar{W} - x^2Z_0. \tag{14.1}$$

15. Solution of equation (14.1). If

$$y^2\bar{W} = x^2Z_0 \tag{15.1}$$

has a solution y of the form $y = xT$, where T is a formal power series in x, then $Z(x, xT)$ will be a well-defined power series **(5**, (2.2)**)** having constant term 1 (hence different from 0). It follows that $y = xT$ must also satisfy

$$y^2 - x^2 + x^2y^2Z_0 - y^4\bar{W} = 0 \tag{15.2}$$

and hence also

$$y^2 = x^2. \tag{15.3}$$

But a solution y of the desired form *does* exist for (15.1), and can be found, for example, by Lagrange's theorem. Thus Z_0 must satisfy the equation obtained by eliminating y between (15.1) and (15.3), and so

$$Z_0 = 1 + x^4U(x^2x^2). \tag{15.4}$$

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Thus

$$(15.5)\qquad \begin{cases} Z_{0,0} = 1, \qquad Z_{2,0} = 0, \\ Z_{2q+4,0} = 3 \displaystyle\sum_{p=0}^{[q/2]} \frac{(3p-4)!\,(3q-3p+2)!}{(2p+3)!\,p!\,(q-2p)!\,(2q-p+4)!} \\ \qquad\qquad\qquad\qquad\qquad\qquad (q = 0, 1, 2, \ldots). \end{cases}$$

The coefficients in Z could now be computed from (14.1), apparently with some difficulty.

A parametric representation for Z_0 can be obtained as follows. Replacing x and y each by x^2 in (5.1) yields, by (15.4),

$$(15.6)\qquad Z_0[2 + x^2U(x^2, 0) - 2Z_0 - x^2Z_0^2] = 0.$$

Thus, setting $u = S(x^2)$ (where S is as defined in §6) and $v = 1 - u$, it follows from (6.3) that

$$(15.7)\qquad (1 + x^2Z_0)^2 = 1 + 2u - 3u^2,$$

whence Z_0 can be expressed in terms of u.

16. Asymptotic behaviour of $Z_{n,0}$. For two power series

$$M = \sum_{n=0}^{\infty} M_n x^n, \qquad N = \sum_{n=0}^{\infty} N_n x^n$$

we write $M \ll N$ if $M_n \leqslant N_n$ for all n (**4**, §14).

Clearly $1 \ll 1 + x^2Z_0$. Hence

$$\begin{aligned} 1 + x^2Z_0 \ll (1 + x^2Z_0)^2 &= 1 + 2u - 3u^2 \qquad \text{by (15.7)} \\ &= 1 + 2x^2v^{-2} - 3x^4v^{-4} \\ &= 1 + 4\sum_{n=1}^{\infty} \frac{(3n-3)!}{(n-1)!\,(2n)!} x^{2n} \text{ by (6.4).} \end{aligned}$$

It follows that

$$Z_0 \ll \sum_{n=0}^{\infty} \frac{(3n)!}{n!\,(2n+1)!} x^{2n}$$

and

$$(16.1)\qquad \begin{cases} Z_{2n,0} \leqslant \dfrac{(3n)!}{n!\,(2n+1)!}, \\ Z_{2n+1,0} = 0. \end{cases}$$

By Stirling's formula, as $n \to \infty$,

$$(16.2)\qquad Z_{n,0} \lesssim \left(\frac{27}{4}\right)^{n/2} n^{-3/2} \sqrt{\frac{3}{2\pi}};$$

hence by comparison with (6.8) we see that almost all quadrangulations of type $[n, 0]$ have no Z-rootings. It is conjectured that, for fixed m, almost all quadrangulations of type $[n, m]$ have no Z-rootings.

17. An equation for Y. By a method analogous to that of **(4,** § 12**)** the following equation was developed:

$$
\begin{aligned}
(17.1)\quad Y = {} & x^4y^{-4}(Y - Y_0 - y^2Y_1) - x^4\bar{U}_0\, Y \\
& + x^2y^2\bar{U}(1 + y^2Y) + y^2\bar{W}^2(1 + y^2Y) + x^2\bar{W}Y \\
& + x^3Y_0\, Y + x^3y^{-2}(Y - Y_0) + x\bar{W}(1 + y^2Y) + y^2Y + \bar{W} + xY,
\end{aligned}
$$

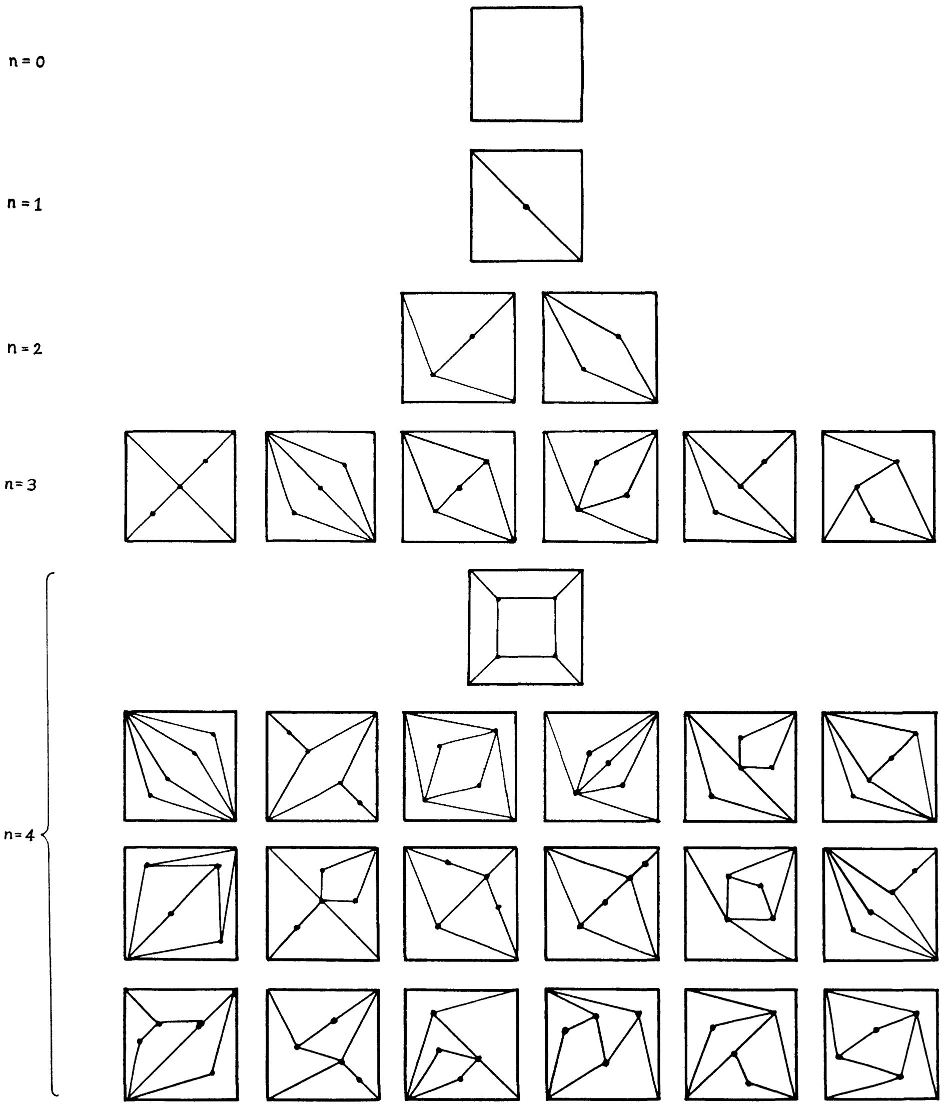

FIGURE 3

where $\bar{U} = U(x^2, y^2)$, $\bar{U}_0 = U(x^2, 0)$, $\bar{W} = 1 + y^4\bar{U}$, and Y_1 is the coefficient of x^1 in Y.

The method employed in the solution of (14.1) is inadequate here, as it yields only one equation for two unknown series, Y_0 and Y_1. (Such an equation might, however, yield an upper bound for Y_0, as it can easily be shown that $Y_0 \ll Y_1$.)

Another possibility is as follows. Define Y^e and Y^o to be respectively the y-even and y-odd parts of Y. Then (17.1) can be separated into two equations which are linear in Y^e and Y^o. There exists a solution for this system of the form

$$Y^e : Y^o : 1 :: L : M : N$$

where L, M, N are formal power series in x and y. It can be shown that the equation $N = 0$ has two distinct solutions of the form $y = xT$, where T is a formal power series in x. It is *conjectured* that the eliminants of y between M and N, and between L and N, are such that Y_1 can be eliminated between them, to yield an algebraic equation for Y_0.

The quadrangulations of type $[n, 0]$ for $n < 5$ are shown in Figure 3.

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