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CONVOLUTION WITH ODD KERNELS HAVING A TEMPERED SINGULARITY

BY
R. A. KERMAN

ABSTRACT. Suppose $b(t)$ decreases to 0 on $[1, \infty)$. Define the singular integral operator C_b at periodic f of period 1 in $L^1(T)$, $T = (-1/2, 1/2)$, by

$$(C_b f)(x) = \lim_{\epsilon \to 0+} \int_{\epsilon \leqq |y| \leqq 1/2} f(x-y) b(1/|y|) \cot \pi y dy \qquad x \in T.$$

Then, for a large class of b one has the rearrangement inequality

$$(C_b f)^*(t) \leqq K\left[\frac{b(1/t)}{t} \int_0^t f^*(s) ds + \int_t^1 f^*(s) b(1/s) \frac{ds}{s}\right] \qquad f \in L_1(T).$$

This inequality is used to construct a rearrangement invariant function space X corresponding to a given such space Y so that C_b maps X into Y.

1. **Introduction.** The conjugate function operator C defined at periodic functions f of period 1 by

$$(Cf)(x) = \lim_{\epsilon \to 0+} \int_{\epsilon \leqq |y| \leqq 1/2} f(x-y) \cot \pi y dy \quad x \in T = (-1/2, 1/2),$$

has been studied by many authors.(1) For example, O'Neil and Weiss [11] proved the rearrangement inequality

$$(1.1) \qquad \int_0^t (Cf)^*(s) ds \leqq K \int_0^t (P+Q) f^*(s) ds \qquad 0 < t < 1.$$

Here $K > 0$ is independent of the integrable function f; f^* and $(Cf)^*$ are the nonincreasing rearrangements of f and Cf on (0, 1);

$$(1.2) \qquad (Pg)(s) = s^{-1} \int_0^s g(u) du \quad \text{and} \quad (Qg)(s) = \int_s^1 g(u) \frac{du}{u}.$$

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(1)We have found it convenient to use the kernel $\cot \pi y$ on the interval T rather than the usual kernel $\cot y/2$ on $(-\pi, \pi)$. Certain results usually proved for C on $(-\pi, \pi)$ are stated below on T.

Our purpose in this paper is to show (1.1) can be improved on if the singularity of the kernel cot πy at 0 is tempered somewhat. Thus, we consider operators C_b given at f on T by

$$(C_b f)(x) = \lim_{\epsilon \to 0+} \int_{\epsilon \leqq |y| \leqq 1/2} f(x-y) b(1/|y|) \cot \pi y dy \qquad x \in T,$$

where the nonnegative function b defined on $[1, \infty)$ is, among other things, bounded and decreasing. These operators were introduced by O'Neil in [10]. Later they were used to illustrate a general extrapolation theorem in [7].

Now, one can do slightly better than (1.1) for C.

Using (1.1) itself and the weak-type (1, 1) inequality for C, it was shown in [1] that $K > 0$ exists with

$$(Cf)^*(t) \leqq K(P+Q)f^*(t) \qquad 0 < t < 1, \tag{1.3}$$

for all f which are integrable on T. When $(Cf)^*$ is replaced by the smaller function $(C_b f)^*$ on the left side of (1.3), one can reduce the bounding function on the right. Given certain assumptions on b, which are detailed below, we prove in Theorem 2.2 that

$$(C_b f)^*(t) \leqq K(P^b + Q^b)f^*(t) \qquad 0 < t < 1, \tag{1.4}$$

where

$$(P^b g)(t) = \frac{b(1/t)}{t} \int_0^t g(s) ds \quad \text{and} \quad (Q^b g)(t) = \int_t^1 g(s) b(1/s) \frac{ds}{s}.$$

An inequality such as (1.4) was conjectured in [7].

As might be expected, it is crucial when proving (1.4) to have, for $C_b f$, a strengthened form of Kolmogorov's classical weak-type (1, 1) estimate for Cf:

$$|\{x \in T : |(Cf)(x)| > \lambda\}| \leqq \frac{K}{\lambda} \int_T |f(x)| dx,$$

$K > 0$ being independent of all integrable f and $\lambda > 0$. This is obtained in Theorem 2.1.

We now state the assumptions to be put on $b(y)$ and give some of their consequences. The nonnegative function $b(y)$ on $[1, \infty)$ is said to be slowly varying there if to each $a > 0$ there is a $y_0 > 1$ so that $y^a b(y)$ is increasing and $y^{-a} b(y)$ is decreasing on $[y_0, \infty)$. Such a function is known to satisfy

$$\lim_{y \to \infty} \frac{b(cy)}{b(y)} = 1 \tag{1.5}$$

for all $c > 0$. See [12, Ch. V, (2.4)]. If $b(y)$ is differentiable and $-yb'(y)$, as well as $b(y)$, is slowly varying, then $0 \leqq -b'$ is nonincreasing for large y which yields

$$b(y) = \int_{y/2}^{y} b'(t)dt + b(y/2) \leqq (y/2)b'(y) + b(y/2),$$

so $0 \leqq -yb'(y)/b(y) \leqq 2[b(y/2)/b(y) - 1]$, and hence, by (1.5),

$$\lim_{y \to \infty} \frac{yb'(y)}{b(y)} = 0. \tag{1.6}$$

It was shown in [7] that when $b(y)$ decreases to 0 on $[1, \infty)$ and is, together with $-yb'(y)$, slowly varying there, one has

$$b(1/|y|) \cot \pi y = (Ck)(y) + 0(k(y)) \tag{1.7}$$

on T, where $k(y) = -y^{-2}b'(1/|y|)$. Since $-yb'(y)$ is slowly varying, $k(y)$ is nonincreasing on $(0, x_0]$ for some $x_0 \in (0, 1/2)$, and so there exists $K > 0$ such that

$$K^{-1}k^*(2y) \leqq k(y) \leqq Kk^*(2y) \qquad 0 < y < 1/2. \tag{1.8}$$

The operator of convolution with k will be denoted by T_k; that is,

$$(T_k f)(x) = \int_T k(x - y)f(y)dy.$$

Suppose ϕ is a nonnegative, integrable function on (0, 1) which lies between constant multiples of a nonincreasing one. The Lorentz space $\Lambda(\phi)$ consists of all measurable f on T such that

$$\|f\|_{\Lambda(\phi)} = \int_0^1 f^*(t)\phi(t)dt < \infty.$$

The functions ϕ first considered by Lorentz in [8] were $\phi_p(t) = pt^{1/p-1}$, $1 \leqq p < \infty$. Later, Calderón introduced the notation $L(p, 1)$ for $\Lambda(\phi_p)$ and, indeed, defined spaces $L(p, q)$, also called Lorentz spaces, for other values of p and q. See [4]. An important fact concerning $\Lambda(\phi)$ is that a sublinear operator T from $\Lambda(\phi)$ to a Banach space B satisfies

$$\|Tf\|_B \leqq 2K\|f\|_{\Lambda(\phi)} \qquad f \in \Lambda(\phi)$$

(1.9) if and only if

$$\|T\chi_E\|_B \leqq K\|\chi_E\|_{\Lambda(\phi)}$$

for all measurable $E \subset T$.

2. **The rearrangement inequality.** We begin with a theorem that includes in (2.1) the strengthened form of the weak-type (1, 1) inequality mentioned in the introduction. The estimate (2.2) is a sort of dual to the weak-type (2, 2) result corresponding to (2.1).

THEOREM 2.1. *Let $b(y)$ be a differentiable function on $[1, \infty)$ which decreases to 0 and which, together with $-yb'(y)$, is slowly varying there. Then, there exists $K > 0$ so that for all f integrable on T*

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$$(2.1) \qquad u(C_b f)^*(u) \leqq Kb(1/u) \int_0^1 f^*(y)dy$$

and

$$(2.2) \qquad u^{1/2}(C_b f)^*(u) \leqq K \int_0^1 f^*(y)y^{-1/2}b(1/y)dy$$

when $0 < u < 1$.[2]

Proof. According to (1.7) we need only prove (2.1) and (2.2) for $T_k f$ and $C(T_k f)$ in place of $C_b f$.

We first consider $T_k f$. By an inequality of F. Riesz, [5, p. 279] and (1.7),

$$\begin{aligned}(2.3) \quad (T_k f)^*(u) \leqq P[(T_k f)^*](u) &= u^{-1} \sup_{|E|=u} \int_E (T_k f)(x)dx \\ &\leqq Ku^{-1} \sup_{|E|=u} \int_T \chi_E^+(x)(T_k f)^+(x)dx \\ &\leqq KP[(T_k f^+)^*](u), \qquad \text{(see (1.8))}\end{aligned}$$

where $f^+(t) = f^*(2|t|)$, $t \in T$. Accordng to [9, Lemma 1.5], the convolution, $g * h$, of the functions $g, h \in L^1(T)$ satisfies the inequality

$$u^{-1} \int_0^u (g * h)^*(s)ds \leqq \left[u^{-1} \int_0^u g^*(s)ds\right]\left[\int_0^u h^*(s)ds\right] + \int_u^\infty g^*(s)h^*(s)ds.$$

Combining this with (1.8), it is seen the right hand term in (2.3) is dominated by a constant multiple of

$$u^{-1}b(1/u) \int_0^u f^*(y)dy + \int_u^1 f^*(y)b(1/y)\frac{dy}{y},$$

since

$$\int_0^1 k^*(y)dy < \infty,$$

k is nonincreasing near 0 and, by (1.6), $k(y) \leqq Ky^{-1}b(1/y)$. The desired estimates for $T_k f$ are now easily obtained using the fact that $b(1/y)/y^a$, $a > 0$, decreases for y near zero.

Inequality (2.2) for $C(T_k f)$ amounts to the assertion that the operator $C \circ T_k$ is continuous from the Lorentz space $\Lambda(y^{-1/2}b(1/y))$ to $L(2, \infty)$. But this has been shown to be true for T_k and C is known to preserve the class $L(2, \infty)$; see [1].

According to the estimate (1.3) and the inequality between the second and last terms in (2.3),

[2] Here, as in (2.8) below, the right side of (2.2) may be infinite, in which case there is nothing to prove.

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$$[C(T_kf)]^*(u) \leqq K[P[(T_kf^+)^*](u) + Q[(T_kf)^*](u)], \; 0 < u < 1.$$

Moreover,

$$\begin{aligned} Q[(T_kf)^*](u) &\leqq QP[(T_kf)^*](u) \\ &\leqq KQP[(T_kf^+)^*](u) \qquad \text{(by (2.3))}. \end{aligned}$$

Use of the identity $(QPg)(x) = (Pg)(x) + (Qg)(x) - (Pg)(1)$ now shows a constant times

$$P[(T_kf^+)^*](u) + Q[(T_kf^+)^*](u) \tag{2.4}$$

to be an upper bound of $[C(T_kf)]^*(u)$, $0 < u < 1$. Only the second term in (2.4) has to be dealt with. Because of (1.9), we just have to prove $K > 0$ exists with

$$\frac{u}{s} b(1/u)^{-1} \int_u^1 (T_k\chi_{(-s/2,s/2)})^*(y)\frac{dy}{y} \leqq K \tag{2.5}$$

when $0 < s, u < 1$. Now, $(T_k\chi_{(-s/2,s/2)})^*(y)$ equals

$$\begin{aligned} &b(2/(s+y)) + b(2/(s-y)) && 0 < y < s \\ &b(2/(s+y)) - b(2/(y-s)) && s < y < 1. \end{aligned}$$

For $s/2 < u < 2s$, the left side of (2.5) is no bigger than a constant times

$$\begin{aligned} &b(1/s)^{-1} \int_{s/2}^{2s} [b(2/(s+y)) + b(2/|s-y|)]\frac{dy}{y} \\ &+ b(1/s)^{-1} \int_{2s}^{1} [b(2/(s+y)) - b(2/(y-s))]\frac{dy}{y}. \end{aligned} \tag{2.6}$$

This follows from $b(y) \leqq Kb(2y)$ and $b(1) < \infty$. For the same reason the first term in (2.6) is seen to be bounded. The second term is comparable to

$$sb(1/s)^{-1} \int_{2s}^{1} -y^{-3}b'(1/y)dy, \tag{2.7}$$

as can be seen by applying the mean value theorem to the integrand and recalling that $-yb'(y)$ is slowly varying. L'Hôpital's rule, property (1.5) for $-yb'(y)$, and (1.6) now yield the boundedness of (2.7). The argument used on the second term of (2.6) also shows (2.5) if $u > 2s$.

When $0 < u < s/2$, we break up the range of integration on the left side of (2.5) into the parts $u < y < s/2$ and $s/2 < y < 1$. Since $yb(1/y)^{-1}$ basically increases,

$$\frac{u}{s} b(1/u)^{-1} \int_{s/2}^{1} [T_k\chi_{(-s/2,s/2)}]^*(y)\frac{dy}{y}$$

has its effective maximum value at $s/2$ and so we're back to the case $s/2 < u < s$. Finally,

$$\frac{u}{s}b(1/u)^{-1}\int_u^{s/2}[b(2/(s+y))+b(2/(s-y))]\frac{dy}{y}$$

$$\leqq K\frac{s^{-1/2}b(1/s)}{u^{-1/2}b(1/u)}\left(\frac{u}{s}\right)^{1/2}\log\frac{s}{u},$$

which is a bounded function of u on $0 < u < s/2$.

THEOREM 2.2. *Let $b(y)$ be a differentiable function on $[1, \infty]$ which decreases to 0 and which, together with $-yb'(y)$, is slowly varying. Then,*

$$(C_b f)^*(t) \leqq K[P^b + Q^b]f^*(t) \qquad 0 < t < 1, \tag{2.8}$$

where K is a positive constant independent of integrable f on T.

PROOF. The proof will be given in two stages. We first show $K > 0$ exists, independent of f in $L^1(T)$, so that

$$\int_0^t (C_b f)^*(u)du \leqq K\int_0^t (P^b + Q^b)f^*(u)du \qquad 0 < t < 1; \tag{2.9}$$

then that (2.9) implies (2.8).

Because of (1.9), inequality (2.9) need only be obtained for characteristic functions of measurable sets. Suppose, then, $\chi = \chi_E$, where $E \subset T$ has Lebesgue measure s.

For $u > s$, (2.1) reads

$$(C_b\chi)^*(u) \leqq K\frac{s}{u}b(1/u);$$

while for $u < s$, (2.2) implies

$$(C_b\chi)^*(u) \leqq Ku^{-1/2}\int_0^s y^{-1/2}b(1/y)dy \leqq K\left(\frac{s}{u}\right)^{1/2}b(1/s).$$

Hence,

$$\int_0^t (C_b\chi)^*(u)du \leqq K\int_0^t \min\left[\frac{s}{u}b(1/u), \left(\frac{s}{u}\right)^{1/2}b(1/s)\right]du. \tag{2.10}$$

Now, when $s < t$, (2.10) gives

$$\int_0^t (C_b\chi)^*(u)du \leqq K\left[\int_0^s \left(\frac{s}{u}\right)^{1/2}b(1/s)du + \int_s^t \frac{s}{u}b(1/u)du\right]$$

$$\leqq K\left[sb(1/s) + s\int_s^t b(1/u)\frac{du}{u}\right].$$

However,

$$(P^b + Q^b)\chi_{(0,s)}(u) = \begin{cases} b(1/u) + \int_u^s b(1/y)\frac{dy}{y} & u < s \\ \frac{s}{u}b(1/u) & u > s, \end{cases}$$

which means

$$\int_0^t (P^b + Q^b)\chi_{(0,s)}(u)du \tag{2.11}$$
$$= \int_0^s \left[b(1/u) + \int_u^s b(1/y)\frac{dy}{y}\right]du + \int_s^t \frac{s}{u}b(1/u)du.$$

Further, the right side of (2.11) is no less than

$$\int_0^s b(1/u)du + s\int_s^t b(1/u)\frac{du}{u}.$$

Since b is slowly varying,

$$\int_0^s b(1/u)du \geqq Ksb(1/s)$$

and the proof of (2.9) for $f = \chi$ is complete when $s < t$.

The proof of (2.9) in case $s \geqq t$ can be reduced to the previous case using duality. Thus,

$$\int_0^t (C_b\chi)^*(u)du = \sup_{|F|\leqq t} \int_F |(C_b\chi)(x)|dx \tag{2.12}$$

and (letting $F^+ = F \cap \{C_b\chi \geqq 0\}$, $F^- = F \cap \{C_b\chi < 0\}$)

$$\begin{aligned} \int_F |(C_b\chi)(x)|dx &= \int_{F^+} (C_b\chi)(x)dx - \int_{F^-} (C_b\chi)(x)dx \\ &= -\int_E (C_b\chi_{F^+})(x)dx + \int_E (C_b\chi_{F^-})(x)dx \\ &= \int_E [C_b(\chi_{F^-} - \chi_{F^+})](x)dx \\ &\leqq \int_0^s [C_b(\chi_{F^-} - \chi_{F^+})]^*(u)du. \end{aligned}$$

Since $(\chi_{F^+} - \chi_{F^+})^* = \chi_F^*$, we obtain, from (2.1) and (2.2),

$$\begin{aligned} \int_0^t (C_b\chi)^*(u)du &\leqq \int_0^s [C_b(\chi_{F^-} - \chi_{F^+})]^*(u)du \\ &\leqq K\int_0^s \min\left[\frac{t}{u}b(1/u), \left(\frac{t}{u}\right)^{1/2} b(1/t)\right]du \\ &\leqq K\int_0^s (P^b + Q^b)\chi_{(0,t)}(u)du. \end{aligned} \tag{2.13}$$

So, (2.9) for $s \geqq t$ then follows from (2.13) and the fact that

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$$\int_0^t (P^b + Q^b)\chi_{(0,s)}(u)du = \int_0^s (P^b + Q^b)\chi_{(0,t)}(u)du.$$

Consider now a fixed f and t with $f \in L^1(T)$ and $0 < t < 1$. Following Calderón [4, p. 29], we define $f_1(x) = f(x) - f^*(t)$ when $f(x) > f^*(t)$, $f_1(x) = f(x) + f^*(t)$ when $f(x) < -f^*(t)$, and $f_1(x) = 0$ otherwise. If, further, $f_2 = f - f_1$, then f_1 and f_2 are integrable, $f^* = f_1^* + f_2^*$, $f_1^*(s) = 0$ when $1 > s > t$, and $f_2^*(s) = f^*(t)$ when $0 < s < t$. Moreover

$$(C_bf)^*(t) \leqq (C_bf_1)^*(t/2) + (C_bf_2)^*(t/2).$$

By (2.1) and (1.5)

$$\begin{aligned}(C_bf_1)^*(t/2) &\leqq K\frac{b(1/t)}{t}\int_0^1 f_1^*(y)dy \\ &\leqq K\frac{b(1/t)}{t}\int_0^t [f^*(y) - f^*(t)]dy \leqq K\frac{b(1/t)}{t}\int_0^t f^*(y)dy.\end{aligned}$$

Again, from (2.9)

$$\begin{aligned}(C_bf_2)^*(t/2) &\leqq P(C_bf_2)^*(t/2) \leqq Kt^{-1}\int_0^{t/2}(P^b + Q^b)f_2^*(u)du \\ &\leqq Kf^*(t)t^{-1}\int_0^t\left[b(1/u) + \int_u^t b(1/y)\frac{dy}{y}\right]du + (Q^bf^*)(t) \\ &\leqq K(P^b + Q^b)f^*(t),\end{aligned}$$

since $b(1/u)$ is nondecreasing and f^* is nonincreasing.

We now consider the question of how, being given a rearrangement invariant (r.i.) space Y, one can construct another such space X so that C_b: $X \to Y$. See Boyd [2] for the definition of an r.i. space Y and its upper index, $\alpha(Y)$, and lower index, $\beta(Y)$.

THEOREM 2.2 and $\|f^+\|_Y = \|f\|_Y$ guarantee that one need only consider the general problem for $P^b + Q^b$. This is done in

THEOREM 2.3. *Suppose b is a decreasing, slowly varying function on* $[1, \infty)$. *Let Y be an r.i. space with norm $\| \ \|_Y$. Then, the set X of all Lebesgue measurable functions f on T such that*

$$\|[(P^b + Q^b)f^*]^+\|_Y < \infty \tag{2.14}$$

is a function space satisfying all the properties of an r.i. space except that it may not contain characteristic functions of measurable sets. This latter property will also hold if and only if $K^+ \in Y$, where

$$K(t) = \int_0^1 k(s, t)ds \quad \textit{with} \quad k(s, t) = \min\left[\frac{b(1/s)}{s}, \frac{b(1/t)}{t}\right].$$

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PROOF. Only the proof that $K^+ \in Y$ implies $\chi_E \in X$ for all measurable $E \subset T$ offers any problem. The method of Theorem 2.2 of [6] can be used to prove this once one observes $k(s, t)$ is nonincreasing in each of s and t in $(0, 1)$ and hence

$$\int_0^1 k(s, t)dt \text{ is nonincreasing in } s.$$

Under certain conditions there are simpler tests for membership in the space X constructed above.

THEOREM 2.4. *Assume b, X and Y are as in Theorem 2.3. Then the condition*

$$\|b(1/|t|)f^+(t)\|_Y < \infty \tag{2.15}$$

is necessary in order that $f \in X$. Moreover, if $0 < \beta(Y) \leqq \alpha(Y) < 1$, then (2.15) *is also sufficient to have $f \in X$; in particular, $\chi_E \in X$ for all measurable $E \subset T$.*

PROOF. The necessity of (2.15) follows from the simple inequality

$$b(1/t)f^*(t) \leqq (P^b f^*)(t).$$

Since b is slowly varying, we have, when $p \in (1, \infty)$,

$$(P_b f^*)(t) \leqq K_p t^{-1/p} \int_0^t f^*(s)s^{1/p-1}ds;$$

$$(Q^b f^*)(t) \leqq \int_t^1 b(1/s)f^*(s)\frac{ds}{s}$$

Thus, choosing $p < 1/\alpha(Y)$ will guarantee the sufficiency of (2.15), in view of [2], Theorem 1.

REMARKS 2.5.

1. Suppose b, X and Y are as in Theorem 2.3. The proof of Theorem 2.4 shows that $\beta(Y) > 0$ ensures $\|[P^b f^*]^+\|_Y < \infty$ is a necessary and sufficient condition for $f \in X$.

2. When Y is a Lorentz space, $\Lambda(\phi)$, the requirement $\beta(Y) > 0$ is equivalent to

$$\int_0^1 \phi(s) \log t/s ds \leqq K \int_0^t \phi(s)ds \tag{2.16}$$

by (1.8). In this case, the condition for $f \in X$ amounts to

$$\int_0^1 f^*(t)dt \int_t^1 b(1/s)\phi(s)\frac{ds}{s} < \infty$$

We note that

$$\int_t^1 b(1/s)\phi(s)\frac{ds}{s} \leqq K \int_t^1 \phi(s)\frac{ds}{s}$$

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and that the latter function is integrable on (0, 1). Thus, when b satisfies the hypotheses of Theorem 2.3 and (2.16) holds for ϕ, we have

$$C_b: \Lambda(\phi) \to \Lambda(\psi), \tag{2.17}$$

where

$$\psi(t) = \int_t^1 b(1/s)\phi(s)\frac{ds}{s}.$$

Unlike Theorem 8 of [10], (2.17) doesn't require the additional restriction $b(t^{1/2}) \leqq Kb(t)$ and so is true for, say, $b(t) = \exp(-\sqrt{\log et})$.

3. Further refinements are possible. Thus, if $Y = L^p(T)$, $1 < p < \infty$, and $0 \leqq a \leqq 1$, Theorem 1 of [3] allows one to prove

$$\|b(1/|t|)^{-a}(C_b f)^+(t)\|_p \leqq K\|b(1/|t|)^{1-a}f^+(t)\|_p.$$

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Department of Mathematics,
Brock University,
St. Catharines, Ontario, L2S 3A1