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RECURSIVE COLORINGS OF GRAPHS

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A *graph* G is an ordered pair $G = (V, E)$ where E is a set of 2-element subsets of V. The set V is the set of vertices, and E is the set of edges. The vertices x and y are *joined* by an edge if $\{x, y\}$ is an edge. If X is a set (of *colors*) and $\chi : V \rightarrow X$, then we say that χ is an *X-coloring* of G if whenever two vertices x and y are joined by an edge, then $\chi(x) \neq \chi(y)$. A graph is *X-colorable* if there is an X-coloring of it. We will identify the natural number n with the set $\{0, 1, \ldots, n - 1\}$, and often refer to n-colorings and to graphs being n-colorable.

A graph $G = (V, E)$ is *recursive* if both V and E are recursive sets. An X-coloring χ of the graph G is recursive if χ is a recursive function; and G is said to be *recursively X-colorable* if such a recursive X-coloring exists. Unfortunately, as shown by Bean [**1**], counter-examples abound unless we make additional effectiveness assumptions about G.

The *degree* of a vertex x is the number of vertices to which it is joined; it is denoted by $\deg(x)$. The graph $G = (V, E)$ is *locally finite* if $\deg(x)$ is finite for each vertex x. It is *highly recursive* if it is recursive and locally finite and the function deg is recursive. (Bean also required that G be connected, but this seems not to be important here.)

A study of the relationship between effectiveness and colorability was undertaken by Bean [**1**]. He proved that for every integer $n \geqq 3$ and every connected, highly recursive graph G, if G is n-colorable then it is recursively $2n$-colorable. We improve upon this result in Theorem 1 below.

In the other direction, Bean produced an example of a connected, highly recursive, n-colorable graph which is not recursively n-colorable. (Such an example was anticipated by Manaster and Rosenstein [**3**].) This left as unsettled what the possibilities are with regard to a highly recursive, n-colorable graph being recursively m-colorable for $n + 1 \leqq m \leqq 2n - 1$. This gap is eliminated in this paper. Specifically, we prove the following two theorems.

THEOREM 1. *If $n \geqq 2$ and G is a highly recursive, n-colorable graph, then G is recursively $(2n - 1)$-colorable.*

THEOREM 2. *If $n \geqq 2$, then there is a highly recursive, n-colorable graph which is not recursively $(2n - 2)$-colorable.*

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Theorems 1 and 2 are proved in Sections 1 and 2, respectively. Some additional comments are contained in § 3. In § 4 we make an improvement of Theorem 2 by obtaining a graph which has even no $(2n - 2)$-coloring in the Boolean algebra generated by the r.e. relations.

1. Proof of theorem 1. Let $G = (V, E)$ be a highly recursive, n-colorable graph. Let $V = \{a_s : s < \omega\}$ be a recursive enumeration of V. For each $s < \omega$ define V_s and $\tilde{V}_s$ so that:

$$V_0 = \tilde{V}_0 = \{a_0\};$$
$$V_{s+1} = \{x \in V : \exists y(y \in V_s \wedge \{x, y\} \in E\} \cup \{a_{s+1}\} \cup V_s;$$
$$\tilde{V}_{s+1} = V_{s+1} - V_s.$$

Clearly $V = \bigcup_s V_s$. Since G is highly recursive, it follows that each V_s is finite and that $\langle V_s : s < \omega\rangle$ is recursive. Let G_s be the (induced) subgraph of G whose vertices are just those in V_s. For each $s < \omega$ let $X_s = \{0, 1, \ldots, n - 1\}$ if s is even, and $X_s = \{0\} \cup \{n, n + 1, \ldots, 2n - 2\}$ if s is odd.

We now recursively define functions $f_s : V_{2s} \to 2n - 1$ and $F_s : V_{2s+1} \to X_s$ such that each of the following conditions is satisfied for each $s < \omega$:

(1) f_s is a $(2n - 1)$-coloring of G_{2s};

(2) $f_s = f_{s+1}| V_{2s}$;

(3) F_s is an X_s-coloring of G_{2s+1};

(4) $F_s| \tilde{V}_{2s} = f_s| \tilde{V}_{2s}$.

Having obtained such a sequence $\langle f_s : s < \omega\rangle$, we easily see from (1) and (2) that $f = \bigcup_s f_s$ is a recursive $(2n - 1)$-coloring of G. Conditions (3) and (4) are present only to maintain the recursion.

Stage $s = 0$. Let F_0 be the first n-coloring of G_1, and let $f_0 = F_0| V_0$.

Stage $s + 1$. Suppose we already have f_s and F_s satisfying (3) and (4). Let

$$F_{s+1} : V_{2s+3} \to X_{s+1}$$

be the first X_{s+1}-coloring of G_{2s+3}. (Such an F_{s+1} exists since G, and thus also G_{2s+3}, is n-colorable.) Now define $f_{s+1} : V_{2s+2} \to 2n - 1$ as follows:

$$f_{s+1}(x) = \begin{cases} f_s(x) & \text{if } x \in V_{2s}, \\ F_{s+1}(x) & \text{if } x \in \tilde{V}_{2s+2}, \\ F_{s+1}(x) & \text{if } x \in \tilde{V}_{2s+1} \quad \text{and} \quad F_{s+1}(x) \neq 0, \\ F_s(x) & \text{if } x \in \tilde{V}_{2s+1} \quad \text{and} \quad F_{s+1}(x) = 0. \end{cases}$$

It is immediate that $f_s = f_{s+1}| V_{2s}$ and that $F_{s+1}| \tilde{V}_{2s+2} = f_{s+1}| \tilde{V}_{2s+2}$. Thus, all that remains is to show that f_{s+1} is a $(2n - 1)$-coloring of G_{2s+2}.

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To see that f_{s+1} is a $(2n - 1)$-coloring of G_{2s+2}, suppose that $x, y \in V_{2s+2}$ and that x and y are joined by an edge. Several cases need to be considered, all of which are very straightforward.

Case 1. $x, y \in V_{2s}$. Then $f_{s+1}(x) = f_s(x) \neq f_s(y) = f_{s+1}(y)$ by (1).

Case 2. $x \in V_{2s}$ and $y \in \tilde{V}_{2s+1}$. If $F_{s+1}(y) \neq 0$, then

$$f_{s+1}(y) = F_{s+1}(y) \notin X_s.$$

But $f_{s+1}(x) = f_s(x) \in X_s$. Hence, $f_{s+1}(x) \neq f_{s+1}(y)$. On the other hand, if $F_{s+1}(y) = 0$, then $f_{s+1}(y) = F_s(y)$. But $x \in \tilde{V}_{2s}$ since x and y are joined by an edge. Hence,

$$f_{s+1}(x) = f_s(x) = F_s(x)$$

by (4), so that from (3) we get $F_s(x) \neq F_s(y)$. Hence, $f_{s+1}(x) \neq f_{s+1}(y)$.

Case 3. $x \in \tilde{V}_{2s+1}$ and $y \in V_{2s+1}$. Proceed as in Case 2.

Case 4. $x, y \in \tilde{V}_{2s+1}$. Since x and y are joined by an edge, $F_{s+1}(x) \neq F_{s+1}(y)$, so without loss of generality we can assume that $0 \neq F_{s+1}(x) \notin X_s$. If $F_{s+1}(y) \neq 0$, then

$$f_{s+1}(x) = F_{s+1}(x) \neq F_{s+1}(y) = f_{s+1}(y).$$

However, if $F_{s+1}(y) = 0$, then

$$f_{s+1}(y) = F_s(y) \in X_s,$$

so that $f_{s+1}(y) \neq f_{s+1}(x)$.

Case 5. $x \in \tilde{V}_{2s+2}$ and $y \in \tilde{V}_{2s+1}$. If $F_{s+1}(y) \neq 0$, then

$$f_{s+1}(x) = F_{s+1}(x) \neq F_{s+1}(y) = f_{s+1}(y).$$

If $F_{s+1}(y) = 0$, then $F_{s+1}(x) \neq 0$ since x and y are joined by an edge. Thus, $f_{s+1}(x) = F_{s+1}(x) \notin X_s$, but $f_{s+1}(y) = F_s(y) \in X_s$.

Case 6. $x \in \tilde{V}_{2s+1}$ and $y \in \tilde{V}_{2s+2}$. Proceed as in Case 5.

Case 7. $x, y \in \tilde{V}_{2s+2}$. Then $f_{s+1}(x) = F_{s+1}(x) \neq F_{s+1}(y) = f_{s+1}(y)$. This completes the proof of Theorem 1.

2. Proof of theorem 2. We will first prove the theorem for $n = 2$. Let $X, Y \subseteq \omega$ be disjoint, recursively inseparable, recursively enumerable sets. Let $Z = X \cup Y$, and let $\langle z_i : i < \omega\rangle$ be a recursive, one to one enumeration of Z arranged so that $z_i \in X$ if and only if i is even. We define a graph $G = (V, E)$ as follows. Let

$$V = \{\langle z, n, k\rangle \in \omega^3 : (\forall i < n)(z_i \neq z) \wedge k < 2\} \cup \{\langle z, i, 2\rangle \in \omega^3 : z = z_i \text{ and } i \text{ is even}\}.$$

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Let E, the set of edges, consist exactly of those pairs following which are subsets of V:

$$\{\langle z, n, k\rangle, \langle z, n+1, k\rangle\},$$
$$\{\langle z, n, k\rangle, \langle z, n, 2\rangle\}, \text{ if } z = z_n,\ n \text{ is even, and } k < 2,$$
$$\{\langle z, n, 0\rangle, \langle z, n, 1\rangle\}, \text{ if } z = z_n \text{ and } n \text{ is odd.}$$

The graph G is certainly highly recursive. It is also 2-colorable. A particular 2-coloring of G is the function $\chi : V \rightarrow 2$ where $\chi(\langle z, n, k\rangle) = 0$ if and only if one of the following holds:

$k = 0$ and n is even;

$k = 1$, $z \in X$ and n is even;

$k = 1$, $z \notin X$ and n is odd.

However, G is not recursively 2-colorable. For, suppose χ is a recursive 2-coloring of G. Then let

$$A = \{z < \omega : \chi(\langle z, 0, 0\rangle) = \chi(\langle z, 0, 1\rangle)\}.$$

Then $X \subseteq A$ and $Y \cap A = \emptyset$. But clearly A is recursive, thus contradicting the recursive inseparability of X and Y. This proves Theorem 2 in the case that $n = 2$.

From now on assume that $n \geqq 3$. We are going to define a graph $G_n = (V_n, E_n)$. The set V_n consists of those ordered pairs (i, j) where $0 \leqq i, j < n$. Two vertices (i, j) and (r, s) are joined by an edge in E_n if and only if $i \neq r$ and $j \neq s$. (The graph G_n is the complement of the line graph of the complete bipartite graph $K_{n,n}$.) The *i-th row* of G_n is the set $\{(i, j) : 0 \leqq j < n\}$, and the *j-th column* of G_n is the set $\{(i, j) : 0 \leqq i < n\}$. Thus, two vertices are joined if, and only if, they are in different rows and different columns. The graph G_n is easily seen to be n-colorable: just color the vertices in the i-th row with color i.

Now let χ be an X-coloring of G_n. We say that χ is *row-oriented* if for each $i < n$ there are two distinct vertices y and z in the i-th row such that $\chi(y) = \chi(z)$. Similarly, χ is *column-oriented* if for every $j < n$ there are two distinct vertices y and z in the j-th column such that $\chi(y) = \chi(z)$.

LEMMA 2.1. *If χ is a $(2n - 2)$-coloring of G_n, then χ is either row-oriented or column-oriented, but not both.*

Proof. Suppose that χ is both row and column-oriented. For each $i < n$ let $p_i < 2n - 2$ be such that there are two distinct vertices in the i-th row which are colored with color p_i. For each $j < n$ let $q_j < 2n - 2$ be such that there are two distinct vertices in the j-th column which are colored with color q_j. Since χ is a coloring, it follows that $p_0, \ldots, p_{n-1}$,

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$q_0, \ldots, q_{n-1}$ are pairwise distinct colors. But this contradicts there being only $2n - 2$ colors. Thus χ is not both row and column-oriented.

Now suppose χ is neither row-oriented nor column-oriented. Then there are $i, j < n$ such that no two distinct vertices in the i-th row have the same color, and no two distinct vertices in the j-th column have the same color. But then $\{\chi(r, s) : r, s < n \text{ and either } r = i \text{ or } s = j\}$ is a set of $2n - 1$ distinct colors, and this is impossible. Hence χ is either row-oriented or column-oriented.

LEMMA 2.2. *Suppose χ_1 and χ_2 are $(2n - 2)$-colorings of G_n such that if $i \neq s$ and $j \neq r$, then $\chi_1((i, j)) \neq \chi_2((r, s))$. Then χ_1 is row-oriented if, and only if, χ_2 is column-oriented.*

Proof. By symmetry it will suffice to prove that χ_2 is column-oriented if χ_1 is row-oriented. So we will assume χ_1 is row-oriented. Let χ_3 be the $(2n - 2)$-coloring of G_n such that $\chi_3((s, r)) = \chi_2((r, s))$. Thus, χ_2 is column-oriented if, and only if, χ_3 is row-oriented, so that it suffices to prove that χ_3 is row-oriented.

Suppose χ_3 is not row-oriented. Also χ_1 is not column-oriented by Lemma 2.1. Thus the same reasoning as in the second half of the proof of Lemma 2.1 will produce $2n - 1$ distinct colors, yielding a contradiction.

We are now prepared to construct the highly recursive, n-colorable graph which is not recursively $(2n - 2)$-colorable. Let $G = (V, E)$ be a highly recursive, 2-colorable graph which is not recursively 2-colorable. Such a graph was constructed at the beginning of this proof. We will define a graph (G', V'). The set V' of vertices will be $V \times V_n$. If $u, v \in V'$ and (i, j), $(r, s) \in V_n$, then there is an edge in E' which joins the vertices $(u, (i, j))$ and $(v, (r, s))$ if, and only if, one of the following holds:

(i) $u = v$, and (i, j), (r, s) are joined by an edge (of G_n);
(ii) u and v are joined by an edge of V, and $i \neq s$ and $j \neq r$.

Clearly, G' is a highly recursive graph.

We first show that G' is n-colorable. Let χ be a 2-coloring of G. Now define $\chi' : V' \to n$ so that:

(iii) if $\chi(u) = 0$, then $\chi'((u, (i, j))) = i$;
(iv) if $\chi(u) = 1$, then $\chi'((u, (i, j))) = j$.

It is easily seen that χ' is an n-coloring of G'.

Next we show that G' is not recursively $(2n - 2)$-colorable. Let $\chi' : V' \to 2n - 2$ be any $(2n - 2)$-coloring of G'. For each $u \in V$, let $\psi_u : V_n \to 2n - 2$ be such that

$$\psi_u((i, j)) = \chi'((u, (i, j))).$$

It follows from (i) that ψ_u is a $(2n - 2)$-coloring of G_n, so that by Lemma 2.1 it is either row-oriented or column-oriented. Let $\chi : V \rightarrow 2$ be such that $\chi(u) = 0$ if and only if ψ_u is row-oriented. Then χ is a 2-coloring of V. For, suppose $u, v \in V$ are joined by an edge. Let $\chi_1 = \psi_u$ and $\chi_2 = \psi_v$. It follows from (ii) that χ_1 and χ_2 satisfy the hypothesis of Lemma 2.2. Thus, it follows from that lemma that $\chi(u) \neq \chi(v)$, so that χ is a 2-coloring.

Finally, notice that χ is recursive in χ', and this implies that χ' is not recursive since there are no recursive 2-colorings of G.

This completes the proof of Theorem 2.

3. Additional comments. The example in Theorem 2 can be transformed into a connected one by a rather general procedure. Suppose $G = (V, E)$ is a highly recursive graph, and $\{a_s : s < \omega\}$ is a recursive list of V. Let $\{b_i : i < \omega\}$ be a recursive set disjoint from V. Let $G' = (V', E')$ be the graph in which

$$V' = V \cup \{b_s : s < \omega\},$$

$$E' = E \cup \{\{a_s, b_i\} : s \leqq i \leqq s + 1\}.$$

Then G' is a connected, highly recursive graph. If $n \geqq 3$ and G is n-colorable, then G' is also n-colorable. To see this just color b_s with the first color which does not color either a_s or a_{s+1}. This procedure also shows that if G is recursively n-colorable, then so is G'.

One way of guaranteeing that a recursive graph is highly recursive is to have it be k-regular for some $k < \omega$. Recall that a graph is *k-regular* if each vertex is joined to exactly k vertices. For sufficiently large k we can arrange for our examples to be k-regular.

An aspect of the graphs constructed in the proof of Theorem 2 is that they are not just locally finite, but that deg is uniformly bounded. In the case of our examples, the least such bound is $3(n - 1)^2$. It is rather easy to see that if a highly recursive graph has such a bound, say k, then G can be "fattened" to a recursive, k-regular graph G'. Furthermore, if G is n-colorable then so is G', and if G is recursively n-colorable, then so is G'.

Thus, we can "fatten" our examples to obtain $(3(n - 1)^2)$-regular ones. At the same time, it would be possible to turn them into connected graphs, being a little more careful than we were with the procedure previously described, so as to obtain connected, $(3(n - 1)^2)$-regular graphs. But we will not worry about that.

All this suggests the following question.

Question 3. If $2 \leqq n \leqq m \leqq 2n - 2$, what is the least k for which there is a recursive, k-regular, n-colorable graph which is not recursively m-colorable?

For $m = 2n - 2$ we know that $k \leqq 3(m-1)^2$. For $m = n$ we obtain from Bean's example a recursive, $(2n-2)$-regular, n-colorable graph which is not recursively n-colorable. Thus $k \leqq 2m - 2$. This same bound is obtainable from the example of Manaster and Rosenstein [**3**]. However, it is not optimal as a simple modification of Bean's example yields a recursive $\left(\left[\frac{3n-1}{2}\right]\right)$-regular, n-colorable graph which is not recursively n-colorable. This results in an improvement whenever $n \geqq 4$.

To construct this example, let $G = (V, E)$ be the recursive, 2-regular, 2-colorable graph which was constructed at the beginning of § 2. Now let $n \geqq 3$. We will define a graph $G_n = (V_n, E_n)$ as follows. Let

$$V_n = \{\langle p, i\rangle \in V \times n : \text{if } p = \langle z, n, k\rangle \text{ and } k = 2, \text{ then } i < n/2\}.$$

Suppose $p = \langle z_1, n_1, k_1\rangle$, $q = \langle z_2, n_2, k_2\rangle$ and $\langle p, i\rangle, \langle q, j\rangle \in V_n$. Then there is an edge in E_n joining $\langle p, i\rangle$ and $\langle q, j\rangle$ if and only if they are distinct and one of the following holds:

(1) $p = q$;

(2) $z_1 = z_2$, $k_1 = k_2 < 2$, $n_1 = n_2 + 1$ and $i < n/2 \leqq j$;

(2′) $z_1 = z_2$, $k_1 = k_2 < 2$, $n_2 = n_1 + 1$ and $j < n/2 \leqq i$;

(3) $z_1 = z_2$, $k_1 < k_2 = 2$, $n_1 = n_2$, and $n/2 \leqq i$;

(3′) $z_1 = z_2$, $k_2 < k_1 = 2$, $n_1 = n_2$ and $n/2 \leqq j$;

(4) $z_1 = z_2$, $\{k_1, k_2\} = \{0, 1\}$, $n_1 = n_2$, $n/2 \leqq i, j$, $\langle z_1, n_1, 2\rangle \notin V$ and $\langle z_1, n_1 + 1, k_1\rangle \notin V$.

Clearly the graph G_n is highly recursive. Also, if $n = 2m$ then each vertex is joined to at most $3m - 1$ vertices, and if $n = 2m + 1$, then each vertex is joined to at most $3m + 1$ vertices. In either case, each vertex is joined to at most $\left[\frac{3n-1}{2}\right]$ vertices.

To see that G_n is n-colorable, let $\psi : V \rightarrow 2$ be a 2-coloring of V. Let $\chi : V_n \rightarrow n$ be such that if $\langle p, i\rangle \in V_n$, where $p = \langle z, m, k\rangle$, then

$$\chi(\langle p, i\rangle) = \begin{cases} i, \text{ if } k = 0 \text{ or } k = 2, \\ i, \text{ if } k = 1 \text{ and } \psi(p) = \psi(\langle z, m, 0\rangle), \\ n - i - 1, \text{ if } k = 1 \text{ and } \psi(p) \neq \psi(\langle z, m, 0\rangle). \end{cases}$$

Then χ is an n-coloring of G_n.

Now let χ be any n-coloring of G_n. For $p = \langle z, m, k\rangle$ and $j < 2$ set

$$I_j = \{\chi(\langle\langle z, m, j\rangle, i\rangle) : i < n/2\}.$$

Then define

$$\psi(p) = \begin{cases} 0, \text{ if } I_0 = I_1, m \text{ is odd, and } k < 2; \\ 0, \text{ if } I_0 \neq I_1 \text{ and } m + k \text{ is even}; \\ 1, \text{ otherwise.} \end{cases}$$

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Then ψ is a 2-coloring of G and is recursive in χ. Since there are no recursive 2-colorings of G, then there are no recursive n-colorings of G_n.

Then, by "fattening" this example we obtain a recursive $\left(\left[\frac{3n-1}{2}\right]\right)$-regular graph which is n-colorable but not recursively n-colorable.

Is the bound $\left[\frac{3n-1}{2}\right]$ optimal? For $n \leqq 4$ we know that it is. For, in a subsequent paper [**4**] we will show if $n = m$ then the least k answering Question 3 is $> n$.

Another way of improving Theorem 2 would be to increase the recursiveness of the graph to the point of having it decidable. Bean did just that in his examples. It is not hard to verify that all the examples constructed in the proof of Theorem 2 actually are decidable, since they all have effective elimination of quantifiers.

One final way we shall improve Theorem 2 is by constructing graphs which not only do not have recursive colorings, but do not have colorings which are in the Boolean algebra generated by the r.e. relations. This we do in § 4.

4. An improvement. Let $\mathscr{B}$ be the Boolean algebra generated by the r.e. subsets of ω (or, where the context requires, by the r.e. binary relations on ω). In this section we will make an improvement of Theorem 2 by considering colorings in $\mathscr{B}$ rather than just recursive colorings.

THEOREM 4. *If $n \geqq 2$, then there is a highly recursive, n-colorable graph which has no $(2n-2)$-coloring in $\mathscr{B}$.*

Proof. We will prove the Theorem only in the case $n = 2$. The construction in the proof of Theorem 2 can then be used to extend it to arbitrary $n > 2$. The graph $G = (V, E)$ which will be constructed is 2-regular and contains no cycles. (That is, if $x_0, \ldots, x_{n+2} \in V$ are such that x_i is joined to x_{i+1} for each $i < n + 2$, then x_0 and x_{n+2} are not joined.) Such a graph is 2-colorable.

At stage s of the construction we will define a graph $G_s = (V_s, E_s)$, each of its vertices having degree 1 or 2, and each of its components having exactly 2 vertices of degree 1. If $s < t < \omega$, then G_s is a subgraph of G_t. The graph $G = \bigcup \{G_s : s < \omega\}$ will be the desired graph.

In component X of G_s let $p_s(X)$ be the smaller vertex of degree 1 and $q_s(X)$ be the larger vertex of degree 1. Each graph G_s is 2-colorable; in fact, there is a unique 2-coloring $\psi_s : V_s \rightarrow 2$ such that $\psi_s(p_s(X)) = 0$ for each component X of G_s. If $A \subseteq \omega$, then we will say that the component X of G_s *splits* A if there are $a, b \in A$ such that $\psi_s(a) \neq \psi_s(b)$.

Let W_n be the n-th r.e. set. Let $\{\langle x_r, n_r\rangle : r < \omega\}$ be a recursive enumeration arranged so that

$$W_n = \{x < \omega : \exists s < \omega(\langle x_s, n_s\rangle = \langle x, n\rangle)\}.$$

For each $r < \omega$, let

$$W_n^r = \{x < \omega : \exists s \leqq r(\langle x_s, n_s \rangle = \langle x, n \rangle)\}.$$

Stage $s = 0$. Let $V_s = \{0, 1\}$ and $E_s = \{\{0, 1\}\}$.

Stage $s + 1 = 2r + 1$. Let $X_0, \ldots, X_n$ be the components of G_s arranged so that $p_s(X_0) < \ldots < p_s(X_n)$. Let $a_0, b_0, a_1, b_1, \ldots, a_{n+1}, b_{n+1}$ be the first $2n + 4$ natural numbers not in V_s. Let

$$V_{s+1} = V_s \cup \{a_0, b_0, \ldots, a_{n+1}, b_{n+1}\},$$
$$E_{s+1} = E \cup \{\{a_i, p_s(X_i)\} : i \leqq n\} \cup \{\{b_i, q_s(X_i)\} : i \leqq n\} \cup \{\{a_{n+1}, b_{n+1}\}\}.$$

Stage $s + 1 = 2r + 2$. Let $n = n_r$. If there do not exist components X, Y of G_s such that $X \cap W_n^r \neq \emptyset \neq Y \cap W_n^r$, $n < \min(X) < \min(Y)$, and X and Y do not split W_n^r, then let $G_{s+1} = G_s$. Otherwise, select such X and Y so that $\min(X)$ and $\min(Y)$ are minimal. Let a, b be the two least natural numbers not in V_s. If $\psi_s(p_s(X)) = \psi_s(p_s(Y))$, then set

$$V_{s+1} = V_s \cup \{a, b\},$$
$$E_{s+1} = E_s \cup \{\{p_s(X), a\}, \{a, b\}, \{b, p_s(Y)\}\}.$$

If $\psi_s(p_s(X)) \neq \psi_s(p_s(Y))$, then set

$$V_{s+1} = V_s \cup \{a\},$$
$$E_{s+1} = E_s \cup \{\{p_s(X), a\}, \{a, p_s(Y)\}\}.$$

It is clear in either case that X and Y are subsets of the same component Z of G_{s+1}, and that Z splits W_n^r.

Clearly, G is recursive. Each vertex in G has degree 1 or 2, and if $x \in V_{2r}$ then x has degree 2 in G_{2r+1}. Thus G is 2-regular and, consequently, is highly recursive. Also, G has no cycles since none of the G_s has a cycle; therefore, G is 2-colorable.

Let ψ be a 2-coloring of G. We will show that ψ is not in $\mathscr{B}$.

Let $\mathscr{I}$ be the collection of sets $I \subseteq \omega$ such that whenever $E \subseteq \omega$ is r.e. and either

(1) $E \cap X \subseteq I$ for all but finitely many components X of G,

or

(2) $E \cap X \cap I = \emptyset$ for all but finitely many components X of G,

then

(3) $E \cap X = \emptyset$ for all but finitely many components X of G.

We prove two lemmas about $\mathscr{I}$ which together imply that $\psi \notin \mathscr{B}$.

LEMMA 4.1. $\psi^{-1}(0) \in \mathscr{I}$.

To prove the lemma, let $I = \psi^{-1}(0)$ and $E = W_n$, and without loss of generality suppose I and E satisfy (1). Then there are components X and Y of G such that $E \cap X$, $E \cap Y \subseteq I$ and $n < \min(X) < \min(Y)$. It follows from the construction of G (at even stages s) that either $E \cap X = \emptyset$ or $E \cap Y = \emptyset$. Thus, there is at most one component Z of G such that $\min(Z) > n$ and $E \cap Z \neq \emptyset$. Hence, (3) holds, so the lemma is proved.

LEMMA 4.2. $\mathscr{B} \cap \mathscr{I} = \emptyset$.

To prove the lemma, it suffices to show that whenever $I \notin \mathscr{I}$ and B, C are r.e. sets such that $C \subseteq B$ and $I \cap (B - C) = \emptyset$ then $I \cup (B - C) \notin \mathscr{I}$. Let $A = I \cup (B - C)$, and assume $A \in \mathscr{I}$. Since $I \notin \mathscr{I}$ but $A \in \mathscr{I}$, there is an r.e. set E such that (2) holds and (3) fails. Since $E \cap C$ is r.e. and $E \cap C \cap A \cap X = \emptyset$ for all but finitely many components X of G, it follows that $E \cap C \cap X = \emptyset$ for all but finitely many components X. Since $E \cap B$ is r.e. and $E \cap B \cap X \subseteq A$ for all but finitely many X, then $E \cap B \cap X = \emptyset$ for all but finitely many X. Therefore, $E \cap A \cap X = \emptyset$ for all but finitely many X. But since $A \in \mathscr{I}$, it follows that (3) holds, and this is a contradiction, so the lemma is proved.

To complete the proof of the Theorem, simply note that if $\psi \in \mathscr{B}$, then $\psi^{-1}(0) \in \mathscr{B}$, which by Lemma 4.2 implies $\psi^{-1}(0) \notin \mathscr{I}$, and this contradicts Lemma 4.1.

Since the class of 2-colorings of a recursive graph is a Π_1^0 class, Theorem 4 implies the existence of a non-empty Π_1^0 class which is disjoint from $\mathscr{B}$. The existence of such a class had been shown some time ago by Specker [**5**] and later, independently, by Jockusch [**2**]. Our proof of Theorem 4 makes use of ideas from Jockusch's proof.

A related question, raised by Bean and still unresolved, is this: If X is a Π_1^0 class, is there an $n < \omega$ and a recursive graph G such that the class of n-colorings of G is degree-isomorphic to X?

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