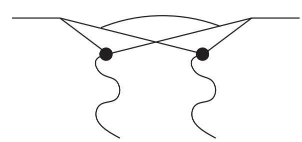
13

Interacting reggeons

13.1 Constructing effective field theory of interacting reggeons

Now we are in the position to address an important question, namely: what can be said about the vertex functions N for other, more complicated, processes.

13.1.1 Stuffing up the vertex: $R \to RR$ transition



Let us draw a more complicated diagram for N. Again, I will carry out the factorization and obtain the same structure expressing N as the energy integral (12.14) of some new amplitude (which now has a three-particle threshold in s_1), etc.

Continuing the process I will face the following problem. As the diagram becomes more complicated, it may start to decrease slower and slower with the growth of energy. For a concrete diagram, N is a number. Summing up a set of diagrams may rend, however, a *diverging answer* for N (if faraway multi-particle thresholds happen to dominate the internal integrals in A_1). Stuffing the particles into N, we effectively increase s_1 and may arrive, e.g. at the graph shown in Fig. 13.1(a). This one has all the reasons to contradict my initial expectation that it was profitable to ascribe a *finite energy* $\langle s_1 \rangle$ to the particle–reggeon scattering blocks, and the bulk of the total energy – to the parallel reggeons, $s_{\mathrm{I}} \sim s_{\mathrm{II}} \sim s$.

This observation exposes another flaw in our logic. In the previous lecture we have analysed a branch cut singularity in the angular momentum plane due to two reggeons. But who ever told us that the blocks

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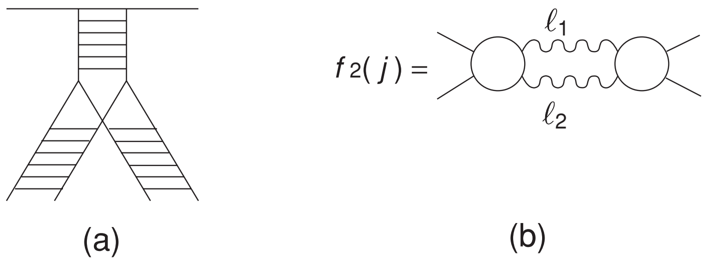
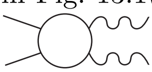


Fig. 13.1 (a) A ladder in the two-particle–two-reggeon vertex block; (b) partial wave for the two-reggeon branching.

in Fig. 13.1(b) did not contain singularities themselves? In particular, if  contains, e.g. *a pole* in the total angular momentum j (as the ladder in Fig. 13.1(a) suggests), the asymptotics of $N(s_1)$ would be bad, and the integral (12.14) would diverge. Therefore, it was a mistake to suppose that the reggeon production amplitude was a smooth function in the j-plane.

Let us write, once again, the integral for the vertex function N:

$$N = \int_{s_0}^{\infty} \frac{ds_1}{2\pi} A_1(s_1, q^2; \ldots)\,. \tag{13.1a}$$

Does this not look familiar?

Recall, how in Lecture 7 we have related the t-channel partial wave to the imaginary part of the amplitude, cf. (7.27),

$$f_j(t) \;\sim\; \int Q_j(z) A_1(z,t)\, dz \;\simeq\; \int_{s_0}^{\infty} ds \frac{A_1(s,t)}{s^{j+1}}. \tag{13.1b}$$

Comparing the two expressions (13.1) we conclude that, if the wavy lines in Fig. 13.1(b) were ordinary particles rather then reggeons, N would be just the value of the partial wave f_j for the $2 \to 2$ scattering amplitude at $j = -1$. Thus, our N is the analogue of the partial wave f_{-1} for a two particles $\to$ two reggeons transition. In this language, the problem of convergence reduces to the question where the singularities in j are: if they all lie on the left of $j = -1$ then N is finite; if the rightmost singularity is above -1 then $N = \infty$. So, the divergence of the vertex N is connected to the structure of angular-momentum singularities of the reggeon production amplitude.

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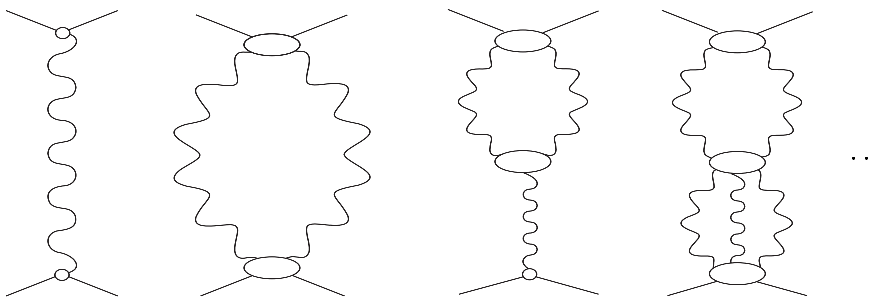
13.1.2 Reggeon field theory: construction logic

So, what is the basic idea? We have found the branchings, and realized that all of them are relevant for the asymptotic behaviour. To construct an effective field theory describing high-energy scattering phenomena, I have to have, I would say, an initial (bare) reggeon, reggeon emission amplitudes, amplitudes of inter-reggeon interactions, etc.



Dealing with a field theory we assume that once we plug in bare quantities (propagators, vertices), the true scattering amplitudes can be calculated, taking into account all renormalization corrections, repetitions, etc. From the point of view of dispersion relations, *all* the branchings are important. Obviously, if such a situation persists at the level of 'true' (renormalized) objects and interactions, we would have failed the task. In practice, we would rather have an answer that is self-consistent within a scheme that contains but a small number of bare (unknown) quantities.

Let us divide the diagrams into two classes.

(1) Diagrams with only a few lines, in configurations with not too large relative momenta and virtualities (so that the s_1 integral converges), yielding finite functions N. These we will call 'bare reggeon vertices', and will treat them as such.

(2) Diagrams with large pair energies $s_1 \gg \mu^2$. As to these diagrams, we will assume that they can be expressed, in turn, in terms of reggeons, since s_1 is large and the amplitude is in the asymptotic regime.

By so doing we make the second class of diagrams, again, subject to *calculation*. This fits well the idea of constructing a self-consistent field theory. There is, of course, a problem: how to actually separate diagrams into the two classes (when few becomes a good few?).

In the diagrammatic language, the main hypothesis can be formulated as follows. I *suppose* that after I reformulate the theory in terms of *exact* (renormalized) reggeons, the diagrams will naturally fall into two classes.

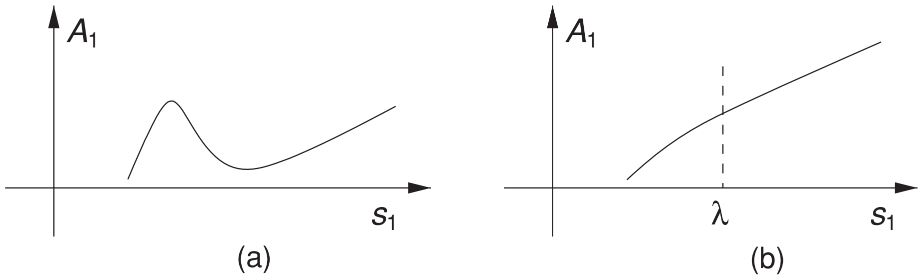


Fig. 13.2 Dividing vertex diagrams into two classes: (a) a clear-cut case; (b) setting an arbitrary separation parameter $s_1 = \lambda$.

The ideal case is when the two energy regions separate naturally as in Fig. 13.2(a). But even if the discontinuity A_1 does not have a prominent structure as shown in Fig. 13.2(b), we can introduce an arbitrary finite cut-off λ to formally divide the two regions.

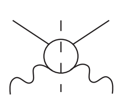
We hope that in the *high energy* region, $s_1 \gg \lambda = \mathcal{O}(1)$, the picture will get simpler in the asymptotics, and we will be able to construct a self-consistent calculation scheme. At the same time, we will treat particle–reggeon and reggeon–reggeon interaction blocks at *low energies*, $s_1 \lesssim \lambda$, as input (bare) vertices. The introduction of bare vertices is a way to separate the finite energy domain about which nothing definite can be said from first principles.

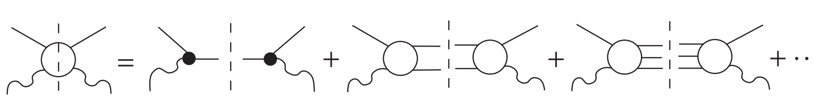
Now that we have virtually 'calculated' *all* the diagrams of the first class ($s_1 \leq \lambda$, $s_2 \leq \lambda$, $s_{\text{internal}} \sim s \to \infty$), we can make an important statement: $N \neq 0$.

Since my 'calculation' is valid only for $s_1 \leq \lambda$ anyway, I can deform the contour around the right cut in spite of the divergence of the integral, and define a 'bare' vertex

$$N_{\text{bare}} \;\equiv\; \int_{4\mu^2}^{\lambda} \frac{ds_1}{\pi} A_1(s_1). \tag{13.2}$$

About the integrand we can say that $A_1 > 0$, at least in a certain region: for (near to) forward scattering it is given by a sum of positive contributions

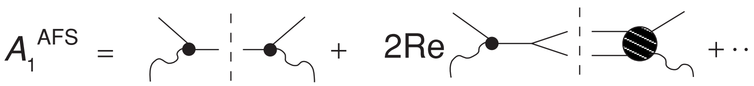
$\operatorname{Im} A = A_1 =$  $= \sum_n a_n\, a_n{*} > 0,$



(13.3)

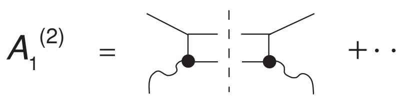
This allows us to state that the particle–reggeon scattering does exist: $N_{\text{bare}} \neq 0$.

At this point you may wonder: how did we manage to get $N = 0$ for the AFS amplitude? The total contribution of each multi-particle state is positive since the same (full) amplitude stands there on the left and on the right from the discontinuity in (13.3). To observe the AFS cancellation, we have picked *specific pieces* from the multi-particle cuts through (corrections to) the *reggeon vertex*, which cuts are not positively definite:



(13.4)

The series of selected terms cancelled, upon integration over s_1, the positive contribution $a_1 a_1^*$ of the one-particle state. Such a selection procedure eliminates not only the pole but some other positive terms on the r.h.s. of the unitarity relation (13.3) as well. For example,



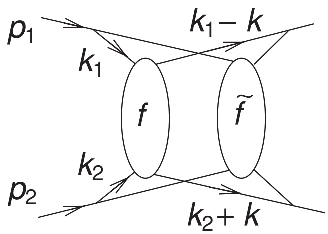
which diagram, being *planar*, has no third spectral function either. At the same time, the diagrams with $\rho_{su} \neq 0$ do survive and contribute to N. For them an artificial procedure of extracting negative pieces from the farther terms of the unitarity relation makes no sense: the series of potentially compensating contributions diverges.

13.2 Feynman diagrams for reggeon branchings

Now we have to examine different diagrams and learn to calculate them.

How can one describe the contribution of branchings in a transparent way? We start from a two-reggeon branching,

$$F_2(s, q^2) =$$



.

At high energies this diagram can be expressed in terms of the asymptotics of the blocks f and $\tilde{f}$. For one of the blocks, introducing the complex angular momentum variable ℓ_1 and the reggeon Green function G_{ℓ_1}, we

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can write

$$f = g(k_1,k)g(k_2,k)\cdot\underbrace{(-1)\int\frac{d\ell_1}{2\pi i}\xi_{\ell_1}[2k_1k_2]^{\ell_1}G_{\ell_1}(\mathbf{k})}_{\text{block}}.$$

If G has a simple pole, $G = 1/(\ell_1 - \alpha(\mathbf{k}^2))$, the integration can be carried out producing $(2k_1k_2)^{\alpha}\xi_{\alpha}$. Similarly, for the second block we have

$$\begin{aligned}\tilde{f} &= -g(p_1-k_1,q-k)g(p_2-k_2,q-k)\\ &\quad\times\int\frac{d\ell_2}{2\pi i}\xi_{\ell_2}[2(p_1-k_1)(p_2-k_2)]^{\ell_2}G_{\ell_2}(\mathbf{q}-\mathbf{k}).\end{aligned}$$

Taking into account that

$$\begin{aligned}\alpha_1\sim 1,\ \beta_1\sim s^{-1}; &\qquad \alpha_2\sim s^{-1},\ \beta_2\sim 1;\\ \alpha_k\sim\beta_k\sim s^{-1}, &\qquad k^2\simeq -\mathbf{k}_\perp^2,\end{aligned}$$

$$2k_1k_2 = \alpha_1\beta_2 s,\quad 2(p_1-k_1)(p_2-k_2) = (1-\alpha_1)(1-\beta_2)s,$$

everything becomes factorized, and we obtain

$$\begin{aligned}F_2(s,q^2) &= \frac{1}{2!}\frac{i\pi}{2}\int\frac{d\ell_1}{2\pi i}\int\frac{d\ell_2}{2\pi i}\int\frac{d^2\mathbf{k}}{(2\pi)^2}\xi_{\ell_1}G_{\ell_1}(\mathbf{k})\xi_{\ell_2}G_{\ell_2}(\mathbf{q}-\mathbf{k})\\ &\quad\cdot N_{\ell_1\ell_2}(\mathbf{k},\mathbf{q})N^*_{\ell_1\ell_2}(\mathbf{k},\mathbf{q})\,s^{\ell_1+\ell_2-1},\end{aligned}\tag{13.5}$$

where

$$N_{\ell_1\ell_2} = \frac{1}{\sqrt{2}}\int\frac{d^4k_1}{(2\pi)^4 i}\int\frac{sd\beta_k}{2\pi i}\frac{g(k_1,k)g(p_1-k_1,q-k)\alpha_1^{\ell_1}(1-\alpha_1)^{\ell_2}}{(\quad)(\quad)(\quad)(\quad)}.$$

We have discussed that this expression is analogous to a Feynman diagram for particles with spins; $\alpha_1^{\ell_1}(1-\alpha_1)^{\ell_2}$ are 'cosines of angles'. Since α_1 changes in the interval $0<\alpha_1<1$ (cf. calculation of the box diagram in Section 9.2.3 of Lecture 9), this factor does not pose any problem.

13.2.1 Two-reggeon branching as a Feynman integral

How to find the t-channel partial wave corresponding to the amplitude (13.5)? This is done using the 'relativistic projection' (7.19):

$$f_j^{(2)}(q^2) = \frac{2}{\pi}\int_{s_0}^{\infty}\frac{ds}{s^{j+1}}\,\mathrm{Im}\,F_2(s,q^2);\tag{13.6a}$$

$$F_2(s,q^2) = -\frac{1}{4i}\int dj\,\xi_j s^j f_j^{(2)}(q^2).\tag{13.6b}$$

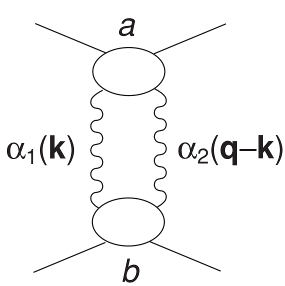
The integral is very simple:

$$f_j(q^2) = \frac{1}{2!}\int \frac{d^2\mathbf{k}}{(2\pi)^2}\int \frac{d\ell_1 d\ell_2}{(2\pi i)^2}\gamma_{\ell_1\ell_2}N^2_{\ell_1\ell_2}G_{\ell_1}(\mathbf{k})G_{\ell_2}(\mathbf{q}-\mathbf{k})\frac{s_0^{\ell_1+\ell_2-j-1}}{j+1-\ell_1-\ell_2}.$$

Here $\gamma_{\ell_1\ell_2} \equiv \mathrm{Im}(i\xi_{\ell_1}\xi_{\ell_2})$.

One may always choose units such that $s_0 = 1$. More importantly, I will always be interested in the region j close to the singularity so that $j + 1 - \ell_1 - \ell_2 \approx 0$ and the factor $s_0^{\ell_1+\ell_2-j-1}$ can be dropped, independently of the value of s_0.

The expression we have obtained reminds us very much of the old perturbation theory. Indeed, we took two 'particles' with propagators



$$G_{\ell_1} = \frac{1}{\ell_1-\alpha_1}, \qquad G_{\ell_2} = \frac{1}{\ell_2-\alpha_2},$$

and derived

$$f_j(q^2) = \frac{1}{2!}\int \frac{d^2\mathbf{k}}{(2\pi)^2}\frac{\gamma_{\alpha_1\alpha_2}N^2_{\alpha_1\alpha_2}}{j+1-\alpha_1-\alpha_2}. \tag{13.7}$$

This is a typical expression for the second-order correction to the $a \to b$ transition amplitude in non-relativistic perturbation theory,

$$\delta f_{a\to b}(E) = \sum_n \frac{V_{an}V_{nb}}{E_n - E}. \tag{13.8}$$

We get a direct correspondence if we treat

$$\alpha_1(\mathbf{k}) - 1 \equiv \varepsilon_1, \quad \alpha_2(\mathbf{q}-\mathbf{k}) - 1 \equiv \varepsilon_2$$

as energies of the two particles, $E_n = \varepsilon_1 + \varepsilon_2$, and $j - 1 \equiv \omega$ as the total energy (E):

$$E - E_n \equiv \omega - (\varepsilon_1+\varepsilon_2) = (j-1)-(\alpha_1-1)-(\alpha_2-1) = j+1-\alpha_1-\alpha_2.$$

The two-dimensional momentum $\mathbf{k}$ in (13.7) plays the rôle of the index n of the intermediate state in (13.8).

This result can be rewritten using Feynman's techniques (covariant perturbation theory). To this end we return to the original representation containing integration over ℓ_1 and ℓ_2. The contours in ℓ_i run parallel to the imaginary axis, *on the right* of the singularities of the Green functions

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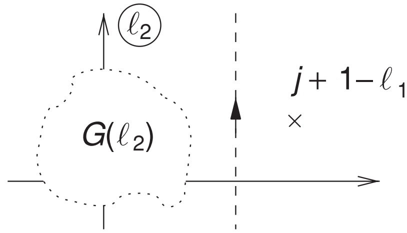
G_{ℓ_i}. Consider, e.g., the ℓ_2-plane:

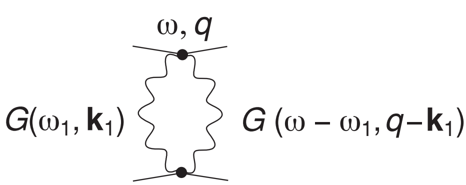


For the energy integral in (13.6a) to converge, j has to be sufficiently large: $j+1-\ell_1-\ell_2>0$. Then, in the ℓ_2-plane the contour lies *on the left* of the pole: $\ell_2<\ell_2^{\text{pole}}=j+1-\ell_1$. Therefore we may evaluate the integral by closing the contour on the right half-plane, around the pole. Introducing $\omega_1=\ell_1-1$ and $\omega_2=\ell_2-1$ as new variables, $G_{\ell_i}\to G_{\omega_i}$, the relation $j+1-\ell_1-\ell_2=0$ translates into $\omega_2=\omega-\omega_1$ and we have

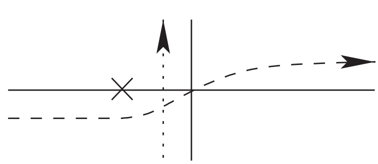
$$f_\omega(q^2)=\frac{1}{2!}\int\frac{d^2\mathbf{k}}{(2\pi)^2}\int\frac{d\omega_1}{2\pi i}\gamma_{\omega_1\omega_2}N^2_{\omega_1\omega_2}G_{\omega_1}(\mathbf{k})G_{\omega-\omega_1}(\mathbf{q}-\mathbf{k}). \tag{13.9a}$$

This expression can be cast in a more symmetric form by introducing two sets of integrations,

$$\begin{aligned} f_\omega(q^2) &= \frac{1}{2!}\int\frac{d\omega_1\,d^2\mathbf{k}_1}{(2\pi)^3 i}\int\frac{d\omega_2\,d^2\mathbf{k}_2}{(2\pi)^3 i}\gamma_{\omega_1\omega_2}N^2_{\omega_1\omega_2}G_{\omega_1}(\mathbf{k}_1)G_{\omega_2}(\mathbf{k}_2) \\ &\quad\cdot(2\pi i)\,\delta(\omega-\omega_1-\omega_2)\cdot(2\pi)^2\delta(\mathbf{q}-\mathbf{k}_1-\mathbf{k}_2). \end{aligned} \tag{13.9b}$$



This is a Feynman diagram in the exact sense of the word; a 'particle' with 'energy–momentum' $(\omega_1,\mathbf{k}_1)$ is described by the propagator $G(\omega_1,\mathbf{k}_1)$. The only unfamiliar feature is the form of the integration in 'energy' ω: we have now the integral running along the *imaginary axis* instead of the usual Feynman integration contour,



This does not make much of a difference, it is just more convenient. It is important that the singularities of the propagator $G(\omega,\mathbf{k})$ lie on the one side of the contour, i.e. our 'particles' are *non-relativistic.*

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Recall the reggeon signature factor (8.8c),

$$\xi_\ell = -\frac{\mathrm{e}^{-i\pi\ell} \pm 1}{\sin \pi\ell},$$

which can be represented as

$$\xi_\ell = -\zeta_\ell{}^{-1} \cdot \exp\left\{-i\frac{\pi}{2}\left(\ell + \frac{1-P_\ell}{2}\right)\right\}, \quad \zeta_\ell = \left\{ \begin{array}{ll} \sin\frac{\pi\ell}{2}, & P_\ell = +1, \\ \cos\frac{\pi\ell}{2}, & P_\ell = -1, \end{array}\right.$$

where $P_\ell = \pm 1$ corresponds to positive (negative) signature. The factors ζ_ℓ may be included into reggeon propagators, G_ℓ; this way we would have poles in ℓ both for reggeons and *particles*: $\ell = 2n$ for positive, and $\ell = 2n+1$ for negative signature trajectories.

Let us look at the numerator of the γ factor in (13.9),

$$\mathrm{Im}\big(i\xi_1\xi_2\big) \propto \mathrm{Re}\ \exp\left\{-i\frac{\pi}{2}\big(\ell_1 + \ell_2 + \tfrac{1}{2}(1-P_1) + \tfrac{1}{2}(1-P_2)\big)\right\},$$

giving

$$\gamma_{\ell_1\ell_2} = \zeta_{\ell_1}^{-1}\zeta_{\ell_2}^{-1}\ \cos\frac{\pi}{2}\big(j + 1 + \tfrac{1}{2}(1-P_1) + \tfrac{1}{2}(1-P_2)\big), \qquad (13.10)$$

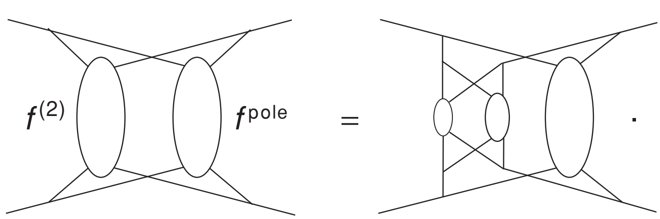
where we have taken into account that $j+1 = \ell_1 + \ell_2$. The cos factor can either be absorbed into vertices, $\sqrt{\cos}\cdot\sqrt{\cos}$, or ascribed to the intermediate state of the two reggeons.

The signature of the reggeon branching is given by the product of signatures of participating reggeons, $P^{(2)} = P_1P_2$. Indeed, the factor $s^{\ell_1+\ell_2-1}$ in the expression (13.5) contains two reggeon amplitudes, $\xi_\ell s^\ell$, each of which transforms under the $s \to -s$ reflection according to its proper signature, while s^{-1} originates from the phase space volume and is in fact $|s|^{-1}$.

13.2.2 Multi-reggeon exchange: conservation of signature

Let us take into account more complicated processes. We have considered a diagram with two blocks with Regge pole asymptotics, and arrived at the two-reggeon branching contribution. What if the asymptotic behaviour of the block amplitude itself corresponds to the *branching* rather than to the pole?

Let us insert $f^{(2)}$ that we have just calculated in one of the blocks,



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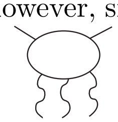
The vertex will now contain $N \cdot g$ instead of $g_1 \cdot g_2$. However, since everything remains factorized in the integrand, denoting  as $N_{\ell_1\ell_2\ell_3}$, for the amplitude we obtain

$$F_3(s,q^2) = \frac{i^2}{3!}\int \frac{d^2\mathbf{k}' d^2\mathbf{k}}{(2\pi)^4}\int \frac{d\ell_1\, d\ell_2\, d\ell_3}{(2\pi i)^3}\xi_{\ell_1}\xi_{\ell_2}\xi_{\ell_3} \cdot G_{\ell_1}G_{\ell_2}G_{\ell_3}s^{\ell_1+\ell_2+\ell_3-2}(N_{\ell_1\ell_2\ell_3})^2. \tag{13.11}$$

Here 3! accounts for different equivalent insertions, and s^{-2} originates from the phase space volume.

It is clear now how to repeat the procedure an arbitrary number of times. For the n-reggeon branching we have

$$F_n(s,q^2) = \frac{\pi}{2}\frac{(-1)^n i^{n-1}}{n!}\int \prod_1^{n-1}\frac{d^2k_i}{(2\pi)^2}\prod_1^{n}\frac{d\ell_1}{2\pi i}\xi_{\ell_i}G_{\ell_i}\cdot s^{\sum_i \ell_i - n+1}N^2_{\ell_1\ldots\ell_n}. \tag{13.12}$$

Let us look into the origin of the phase in (13.12).

In the case of two reggeons we had $i\xi_1\xi_2$; a three-reggeon amplitude contains the factor $i^2\xi_1\xi_2\xi_3$. Each additional transverse momentum integration brings in a factor i:

$$\frac{d^4k_i}{(2\pi)^4 i} = \frac{i}{2\,|s|}\cdot\frac{d\alpha_i s}{2\pi i}\frac{d\beta_i s}{2\pi i}\frac{d^2\mathbf{k}}{(2\pi)^2},$$

since integrals over α_i and β_i, as we have learned before, reduce to integrations of discontinuities of reggeon creation amplitudes and produce real-valued vertex functions. Evaluating the corresponding partial wave using (13.6a) and introducing an integral over k_n in order to symmetrize the expression, we get

$$f_j^{(n)}(q^2) = \frac{1}{n!}\int\prod_{i=1}^{n}\frac{d^2\mathbf{k}_i d\ell_i}{(2\pi)^3 i}G_{\ell_i}(k_i)\cdot(2\pi)^2\delta\left(\sum \mathbf{k}_i - \mathbf{q}\right) \cdot \operatorname{Im}\left(i^{n-1}\xi_{\ell_1}\ldots\xi_{\ell_n}\right)\frac{N^2}{j+n-1-\sum \ell_i}. \tag{13.13}$$

Finally, we may introduce ω_i and one more integration,

$$f_j^{(n)}(q^2) = \frac{1}{n!}\int\prod_{i=1}^{n}\frac{d^2\mathbf{k}_i\, d\omega_i}{(2\pi)^3 i}\cdot(2\pi)^3\, i\delta\left(\omega - \sum\omega_i\right)\,\delta\left(\sum\mathbf{k}_i - \mathbf{q}\right) \cdot\gamma_{\omega_1\ldots\omega_n}\cdot N^2_{\omega_1\ldots\omega_n}\prod_{i=1}^{n}G_{\omega_i}(\mathbf{k}_i), \tag{13.14}$$

to obtain, again, a standard expression for the Feynman diagram. The only difference is in the signature factor γ:

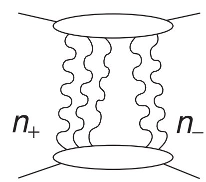
$$\gamma \equiv \mathrm{Im}\left[(-1)^n i^{n-1}\,\mathrm{e}^{-\frac{i\pi}{2}\left(\sum \ell_i+\sum_i \frac{1-P_i}{2}\right)}\right]\cdot\prod_{i=1}^{n}\zeta_{\ell_i}^{-1}. \tag{13.15}$$

Recalling that we have taken the residue $j+n-1-\sum_i \ell_i = 0$, for (13.15) it is straightforward to derive

$$\gamma = (-1)^{n-1}\sin\frac{\pi}{2}\left[j+\sum_{i=1}^{n}\frac{1-P_i}{2}\right]\cdot\prod_{i=1}^{n}\zeta_i^{-1}. \tag{13.16}$$

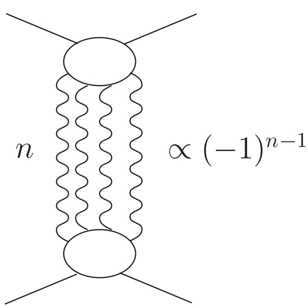
What is the meaning of this factor? Depending on the sign of the product $P^{(n)} = \prod_{i=1}^n P_i$, the amplitude is proportional to $\sin\frac{\pi}{2}j$ ($P^{(n)}=+1$) or $\cos\frac{\pi}{2}j$ ($P^{(n)}=-1$). This is the manifestation of the *conservation of signature*: the symmetry of the n-reggeon amplitude is determined by the 'product' of symmetries of all the poles.

Because of this factor, the contribution of the n-reggeon branching *vanishes* in the points of its proper signature, $\gamma_+ \propto \sin\frac{\pi}{2}j = 0$ for $j=2k$, and $\gamma_- \propto \cos\frac{\pi}{2}j = 0$ for $j=2k+1$.



The signature of a branching of n_+ positive and n_- negative signature reggeons is

$$P^{(n)} \equiv P^{(n_+ + n_-)} = (-1)^{n_-}.$$



Let us examine the most important case when all the poles are pomerons $\mathbf{P}$. Then, as we know, $j_n(t) = n\alpha(t/n^2) - n + 1$. At small t-values the branching $j_n(t) \simeq 1 + (\alpha' t/n)$ is positioned near 1, as well as all the poles ℓ_i, and (13.16) gives

$$\gamma_{\ell_1\ldots\ell_n} \equiv \gamma_{n\mathbf{P}} \simeq (-1)^{n-1}. \tag{13.17}$$

In another interesting case when we have n pomerons and one non-vacuum pole with some trajectory $\beta(t)$,

$$j_{n+1}(t) \simeq \beta(0) + \frac{\alpha'}{\alpha'+n\beta'}t \simeq \beta(0),$$

the branching signature factor reads

$$\gamma_{n\mathbf{P}+\beta} = (-1)^n \zeta_{\beta(0)}^{-1}\sin\frac{\pi}{2}\left[\beta(0)+\frac{1-P_\beta}{2}\right] = (-1)^n P_\beta. \tag{13.18}$$

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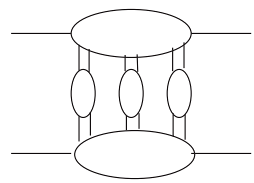
Thus, in both cases when all (or all but one) poles are **P**, adding one pomeron changes the sign of the partial wave amplitude of the branching.

We conclude that the contributions of the (simplest) branchings are very similar to Feynman diagrams, except for the alternating signs. In terms of the field theory, the fact that contributions to the unitarity condition have alternating signs means that the Hamiltonian corresponding to our theory is *anti-Hermitian*.

Where does this oscillation come from? Recall that the characteristic feature of the vacuum pole was that the corresponding scattering amplitude A was *purely imaginary* at small t values, $A \propto i$. When we iterate n vacuum amplitudes, $n-1$ loops each produce the factor i,

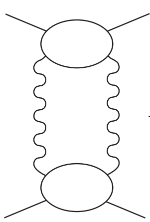


$$F^{(n)} \sim A^n \frac{d^4k_1 \dots d^4k_{n-1}}{[(2\pi)^4 i]^{n-1}} \sim A^n \left(\frac{i}{i^2}\right)^{n-1}$$

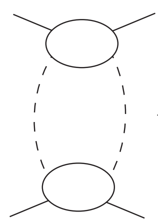
(i^2 in the denominator participates in forming the real multi-reggeon production vertices, as we have just discussed), and we get

$$F^{(n)} \sim (iA)^{n-1} A \sim (-1)^{n-1} A.$$

What if I considered a photon?



$A_{\mathbf{PP}} = -i$



$A_{\gamma\gamma} = i$

Both processes are diffraction scatterings, but in the case of the photon the basic amplitude is real. Thus the alternating sign takes its origin from the complexity of the vacuum pole amplitude.

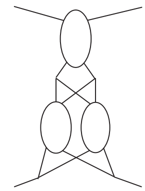
As we have discussed in the previous lecture, from the s-channel point of view the opposite sign of the two-pomeron branching is nothing but the *screening* phenomenon.

13.3 Enhanced branchings

Till now we supposed that the reggeon creation vertices N contained no singularities in ℓ and treated them as constants. What we have obtained this way is known as 'non-enhanced' reggeon branchings.

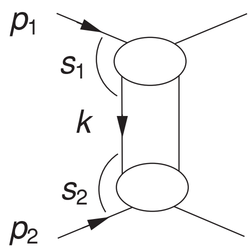
Now it is time to move further and try to combine poles with branch cut singularities.

For example, what will be the contribution to the asymptotics of a diagram like this one? This is an example of the so-called *enhanced branchings*.



To learn how to deal with enhanced branchings, it is sufficient to consider a general graph (13.19), without specifying the details of the blocks:

$$F(q^2, s) \;=\; \qquad (13.19)$$



For the time being we assume the simplest case, with two particles in the intermediate t-channel state; more complicated configurations will be considered later.

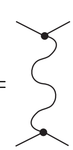
$$\begin{aligned} F \;&=\; \int \frac{d^2\mathbf{k}_\perp\, d\alpha\, d\beta\, s}{2(2\pi)^4 i} f(p_1, q, k) \frac{1}{m^2 - \alpha\beta s + \mathbf{k}_\perp^2 - i\varepsilon} \\ &\quad \frac{1}{m^2 - (\alpha - \alpha_q)(\beta - \beta_q)s + (\mathbf{k} - \mathbf{q})_\perp^2 - i\varepsilon} f'(p_2, q, k). \end{aligned} \qquad (13.20)$$

The key question is, which invariant energies s_1 and s_2 are relevant? If one of them is small, we arrive at the situation we have already considered. So, we will suppose that both energies are in the asymptotic regime, $s_1, s_2 \gg \mu^2$.

13.3.1 Renormalization of the Regge pole

First, we write for the blocks f and f' in (13.20) just the pole expressions:

$$f = g(q, p_1) \int \frac{d\ell_1}{2\pi i} \xi_{\ell_1} G_{\ell_1}(q) (2p_1 k)^{\ell_1} g(q, k) = \qquad (13.21)$$



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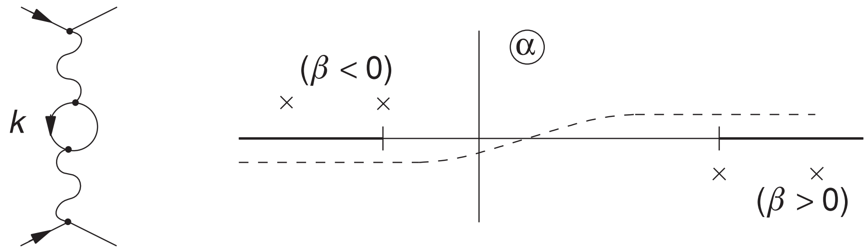
This leads to the picture of two reggeons connected by a particle loop,



Let us investigate the singularities in α. The poles are

$$\alpha_1 = \frac{m_\perp^2 - i\varepsilon}{\beta s}\,, \qquad \alpha_2 = \frac{m_\perp'^2 - i\varepsilon}{(\beta - \beta_q)s} + \alpha_q\,.$$

The cuts in α come from the lower block amplitude $f'(s_2)$:

$$\begin{aligned} s_1 &= (p_1 - k)^2 = (1-\alpha)(\gamma - \beta)s - \mathbf{k}_\perp^2 \simeq -\beta s\,, \\ s_2 &= (p_2 + k)^2 = (\gamma + \alpha)(1+\beta)s - \mathbf{k}_\perp^2 \simeq \alpha s\,. \end{aligned}$$

Depending on the sign of β, I close the contour around the left ($\beta > 0$) or the right ($\beta < 0$) cut. As always, to determine the asymptotics, we have to consider the s- and u-cuts independently (see Lecture 11):

$$F(q^2, s) = \int_{\beta<0} \frac{d\beta s\, d\alpha\, d^2\mathbf{k}}{2(2\pi)^3\pi} \frac{f(p_1, q, k)}{(\quad)(\quad)} \operatorname{Im}_s f'(p_2, q, k) + \int_{\beta>0} \operatorname{Im}_u f'(p_2, q, k). \tag{13.22}$$

Consider the first integral. It includes

$$\xi_{\ell_1}(-\beta s)^{\ell_1}(\alpha s)^{\ell_2};$$

the second signature factor, ξ_{ℓ_2}, is absent since we took the s-channel imaginary part of the lower amplitude. Everywhere in the propagators enters $\alpha\beta s = \mathcal{O}(1)$, thus it is reasonable to introduce $x = -\alpha\beta s$ as an integration variable, instead of β:

$$\begin{aligned} F(q^2, s) &= g(q, p_1) g(q, p_2) \int \frac{d\ell_1}{2\pi i} \xi_{\ell_1} G_{\ell_1}(q) \int \frac{d\ell_2}{2\pi i} G_{\ell_2}(q) \\ &\quad \cdot \int \frac{d^2\mathbf{k}\, dx}{(2\pi)^4} \frac{g(q,k) g(q,k)}{(\quad)(\quad)} \int \frac{d\alpha}{\alpha} \left(\frac{x}{\alpha}\right)^{\ell_1} (\alpha s)^{\ell_2}. \end{aligned} \tag{13.23}$$

Since the propagators depend only on x, the integral in α can be easily taken:

$$\int \frac{d\alpha}{\alpha} \alpha^{\ell_2 - \ell_1}\,, \quad \frac{1}{s} < \alpha < 1.$$

The result depends on the magnitude of the difference $\ell_2 - \ell_1$.

If the angular momenta are significantly different, for example, $\ell_2 > \ell_1$, then the lower amplitude grows with energy faster than the upper one. In this situation we have

$$\alpha \sim 1\,; (\alpha s)^{\ell_2} \quad \text{is large, while} \qquad (\beta s)^{\ell_1} \sim \left(\frac{x}{\alpha s} s\right)^{\ell_1} \sim 1 \quad \text{is small,}$$

so that the whole energy turns out to be assigned to the lower block, $s \sim s_2 \gg s_1$. In the opposite case, $\ell_2 < \ell_1$, a large energy will be assigned to the upper block, while the lower one will contain only a few particle interactions, away from the asymptotic regime.

An interesting case is $\ell_2 \approx \ell_1$. This gives a possibility to get an enhanced contribution by playing on the redistribution of energy between the two blocks:

$$\int_{1/s}^{1} \frac{d\alpha}{\alpha} = \frac{1}{\ell_2 - \ell_1}(1 - s^{(\ell_1 - \ell_2)}) \sim \ln s. \tag{13.24}$$

This is just the phenomenon which prompted me to carry out the calculation. According to (13.24), the singularity in j of our simplest enhanced diagram is *not identical* to that of the initial reggeon amplitude. Let us rewrite the expression so that this can be clearly seen:

$$F(q^2, s) = g^2 \int \frac{d\ell_1}{2\pi i} \xi_{\ell_1} G_{\ell_1}(q) \int \frac{d\ell_2}{2\pi i}\, r_{\ell_1 \ell_2} G_{\ell_2}(q) \frac{s^{\ell_1} - s^{\ell_2}}{\ell_1 - \ell_2}. \tag{13.25a}$$

Here

$$r_{\ell_1 \ell_2} = 2 \int \frac{d^2\mathbf{k}\, dx}{(2\pi)^4} \frac{g^2(q,k)}{(\quad)(\quad)} x^{\ell_1}. \tag{13.25b}$$

The partial wave is given by the integral

$$f_j(q^2) = g^2 \int \frac{d\ell_1}{2\pi i} G_{\ell_1}(q) \int \frac{d\ell_2}{2\pi i} G_{\ell_2}(q)\, r_{\ell_1 \ell_2} \int_{s_0}^{\infty} \frac{ds}{s^{j+1}} \frac{s^{\ell_1} - s^{\ell_2}}{\ell_1 - \ell_2}\,. \tag{13.26}$$

Let us calculate it at $\ell_1 \approx \ell_2$:

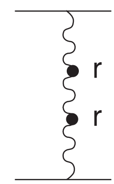
$$\int_{s_0}^{\infty} ds = \frac{1}{\ell_1 - \ell_2} \left[\frac{s_0^{j-\ell_1}}{j - \ell_1} - \frac{s_0^{j-\ell_2}}{j - \ell_2} \right] \simeq \frac{1}{(j - \ell_1)(j - \ell_2)}\,.$$

Finally, closing the ℓ-contours around the poles, we obtain

$$f_j(q^2) = g(q^2) G_j(q)\, r_{jj} G_j(q)\, g(q^2). \tag{13.27}$$

Depending on the signatures of the two reggeons, the second term of (13.22) due to the u-channel cut either cancels the s-channel contribution (opposite signatures, $P_1 = -P_2$) or doubles it ($P_1 = P_2$; the corresponding factor 2 is already included in the formula (13.25b) for $r_{\ell_1 \ell_2}$).

The expression (13.27) is rather transparent from the point of view of Feynman diagrams. The two reggeons belonging to the same scattering amplitude, having the same quantum numbers and the same signature, can transform one into another; they 'mix'.



If we insert the calculated expression in the upper block of our general graph (13.19) and repeat the analysis, we obtain a graph with two successive reggeon–reggeon transitions.

If initially the reggeon Green function has the form

$$G_0 = \frac{1}{j-\alpha_0}\,,$$

then after taking into account all iterations of this sort we arrive at

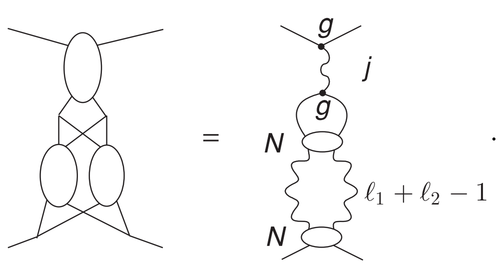
$$G = \frac{1}{j-\alpha_0} + \frac{r}{(j-\alpha_0)^2} + \frac{r^2}{(j-\alpha_0)^3} + \cdots = \frac{1}{j-\alpha_0-r}\,.$$

The trajectory has changed.

By the way, this means that if the pole is a genuine one, there cannot be such diagrams ($r=0$). (This is just an aside.)

13.3.2 Basic reggeon interactions

What truly interests us is diagrams like



Using the results of our previous calculations, we obtain

$$f_j(q) = g\,G_j(q)\int \frac{d\ell_1\,d^2\mathbf{k}}{(2\pi)^3 i} r_{j\ell_1\ell_2} G_{\ell_1}(k) G_{\ell_2}(q-k)\gamma_{\ell_1\ell_2} N_{\ell_1\ell_2}(q,k), \quad (13.28)$$

where $\ell_2 = j+1-\ell_1$. The new vertex $r_{j\ell_1\ell_2}$ is analogous to $r_{\ell_1\ell_2}$, only with the two-reggeon creation function N replacing one of the $g(q,k)$ factors in (13.25b).

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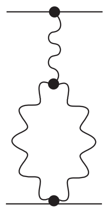
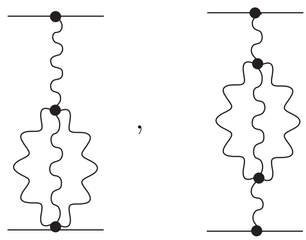


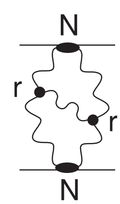
We come to the conclusion that one reggeon can transform into two: there is a mixing. Such a transition can be important only when the upper and the lower blocks have the same degree of the s behaviour, i.e. $\alpha_{\text{pole}} \approx \alpha_{\text{branching}}$. But as we already know this is just the situation that we have, at small t-values, for the vacuum pole $\mathbf{P}$ with $\alpha_{\mathbf{P}}(0) = 1$. Similarly, one arrives at more complicated diagrams. For example, taking a three-reggeon branching for one of the blocks in (13.19), we will obtain diagrams



etc.

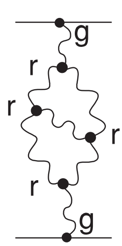
Thus, having started from a pole, we obtain all diagrams with different groups of reggeons following one another in the t-channel. These diagrams modify particle–reggeon vertex functions and the reggeon propagator.

Will there be graphs of another sort, which look like corrections to reggeon–reggeon interaction vertices?

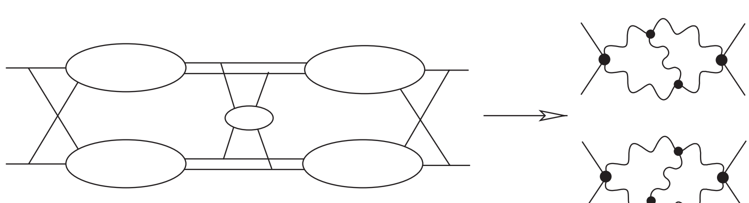


, (13.29)

Should a reggeon diagram like this one, (13.29), exist this would generate, via (13.19), a whole new family of graphs containing 'reggeon corrections' to r as well as to other reggeon interaction vertices.



The reggeon graphs of the type of (13.29) do appear from the analysis of the diagram shown in the l.h.s. of (13.30). To show this is a rather cumbersome task because of many regions in the α_i, β_i variables in an amplitude with five scattering blocks.



(13.30)

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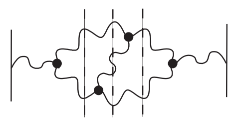
The answer, however, turns out to be quite simple: just the sum of two terms shown on the r.h.s. of (13.30). If the outer reggeons forming the basic loop are different, these contributions are not identical, exactly as it would have been the case in a non-relativistic quantum mechanical problem.

Essentially, the answer reduces to a simple statement that using reggeons we can draw all diagrams that can be imagined in a field theory. There is one important additional rule: every reggeon vertex r has its own signature factor γ, which is characterized by the set of reggeon complex angular momenta ℓ_i in each intermediate state,



We have investigated only the structures of the diagrams. What are the coefficients these diagrams should be added up with? Is there any correspondence with a (non-relativistic) field theory? The answer is simple: the coefficients should be such that the *unitarity conditions* are satisfied, since the series of Feynman diagrams is a formal solution of the unitarity.

The asymptotic behaviour of the amplitudes at $s \to \infty$ can be expressed in terms of the asymptotics of the internal blocks. If we assume that the basic scattering block is just a Regge pole, then, applying the diagrammatical rules, everything can be constructed from this reggeon.

13.4 Feynman diagrams and reggeon unitarity conditions

In Lecture 11 we derived the reggeon unitarity condition in the t-channel, at $t > 0$, which contained unknown reggeon creation amplitudes. Now we know that the branchings can be obtained directly from the physical region of the s-channel ($t < 0$) where all the ingredients are well defined, diagrammatically. It is the series of these diagrams that allows us to simplify the unitarity conditions.

Recall the two-reggeon unitarity condition that we have written in the t-channel,

$$\delta f_j = -\pi \int dt_1\, dt_2\, \tau(t, t_1, t_2) N_j^+ \cdot \delta(j + 1 - \alpha_1 - \alpha_2) \cdot N_j^-, \qquad (13.31)$$

where τ is the two particle phase space volume, and $N_j^\pm = N_j^\pm(t, t_1, t_2)$ are the values of the full two-particle–two-reggeon transition amplitude above and below the cut in j, cf. (11.37).

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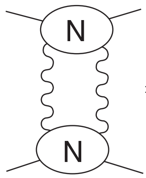
We have calculated above a similar expression in terms of Feynman diagrams, using the simplest explicit vertex functions N:

$f_j =$ 

$$= -\int \frac{d^2\mathbf{k}\, d\ell_1}{(2\pi)^3 i} G_{\ell_1}(k) G_{j+1-\ell_1}(q-k) N\, N\,. \qquad (13.32)$$

Substituting $G = 1/\ell - \alpha$,

$$f_j = -\int \frac{d^2\mathbf{k}}{(2\pi)^2} \frac{NN}{j+1-\alpha_1-\alpha_2}\,.$$

The imaginary part of the partial wave,

$$\delta f_j = -\pi \int \frac{d^2\mathbf{k}}{(2\pi)^2} N\delta(j+1-\alpha_1-\alpha_2)N, \qquad (13.33)$$

has a structure resembling the t-channel unitarity condition. The only difference is that now we have $d^2\mathbf{k}$ instead of $dt_1\, dt_2\, \tau$ in (13.31).

So, is there a correspondence between the two formulae? Let us trade the two-dimensional transverse momentum integration,

$$d^2k = k\, dk\, d\varphi = \tfrac{1}{2} d(k^2)\, d\varphi,$$

for the arguments of the reggeon Green functions,

$$-t_1 = \mathbf{k}^2, \;\; -t_2 = \mathbf{k}'^2 \equiv (\mathbf{q}-\mathbf{k})^2; \qquad (-t = \mathbf{q}^2)$$
$$k'^2 = q^2 + k^2 - 2kq\cos\varphi\,, \quad dk'^2 = 2kq\sin\varphi\, d\varphi\,.$$

We obtain

$$\sin^2\varphi = 1 - \left(\frac{k'^2 - q^2 - k^2}{2kq}\right)^2, \quad d^2\mathbf{k} = \frac{dt_1\, dt_2}{2kq|\sin\varphi|} = \frac{dt_1\, dt_2}{2\sqrt{-t}\cdot p_c},$$

where

$$p_c = p_c(t, t_1, t_2) = \left[\frac{t^2 - 2t(t_1+t_2) + (t_1-t_2)^2}{4t}\right]^{\frac{1}{2}}.$$

The factor p_c appearing in the *denominator* of (13.33) is nothing but the relative momentum of the two reggeons in the centre-of-mass of the t-channel. Curiously, the same factor p_c is present in the unitarity condition (13.31) but in the *numerator*, $\tau \propto p_c$.

The two expressions would match if we could extract $1/p_c$ from the full vertex $N^\pm$ by introducing $N^\pm = \tilde{N}/p_c(t, t_1, t_2)$, and claim that $\tilde{N}$ behaves similarly to N from (13.33).

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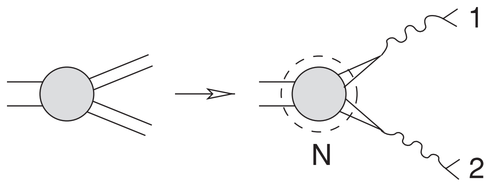
Such an operation, however, looks intrinsically dangerous, since the singularity we are studying emerges just near $p_c = 0$.

Let us recall the origin of the two-reggeon creation amplitude N:



The production amplitude of *usual particles* with the orbital momentum L behaves at small p_c as $N \sim p_c^L$. Our branch point corresponds to $j = \sigma_1 + \sigma_2 - 1$. Comparing with

$$\mathbf{j} = \boldsymbol{\sigma}_1 + \boldsymbol{\sigma}_2 + \mathbf{L}\ , \quad |\mathbf{j}|_{\max} = \sigma_1 + \sigma_2 + L,$$

we observe that the *two-reggeon* singularity appears when the orbital momentum assumes the first unphysical value $L = -1$. Consequently, the vertex $N^{\pm}$ is *obliged* to behave like $1/p_c$, and this is just what we need: $\tilde{N}$ turns to a constant in the $p_c \to 0$ limit.

As for multi-reggeon diagrams, using in the partial wave (13.14) the Regge-pole expression for $G_{\ell_i}(k_i)$ and evaluating the imaginary part gives

$$\delta f_j^{(n)} = \frac{(-1)^{n-1}\pi}{n!} \int \prod_{i=1}^{n-1} \frac{d^2\mathbf{k}_i}{(2\pi)^2} N\delta\big(j + n - 1 - \sum_{m=1}^{n} \alpha(k_m)\big)N. \quad (13.34)$$

This expression describes the simplest contribution to the multi-reggeon unitarity condition, corresponding to $N = \text{const}$. To satisfy the genuine unitarity condition (11.40) one has to draw and analyse the full set of diagrams for the creation of n reggeons in a two-particle interaction, which determines the exact vertex $N^{\pm}$.

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