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A Lower Bound for the Length of Closed Geodesics on a Finsler Manifold

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Abstract. In this paper, we obtain a lower bound for the length of closed geodesics on an arbitrary closed Finsler manifold.

1 Introduction

The study of closed geodesics is a classical and important problem in differential geometry. There are many important results, which in turn to lead to a better understanding of the global geometry of differential manifolds. In Riemannian geometry, following Klingenberg[K], Cheeger [Ch] gives a lower bound for the length of simple closed geodesics in terms of an upper bound for the diameter and lower bounds for the volume and the sectional curvature. Finsler geometry is a natural generalization of Riemannian geometry. The analogue of sectional curvature in Finsler geometry is the so-called the flag curvature. It is a natural question whether Cheeger's theorem still holds in the Finslerian case. However, even to most trivial Finsler metrics, such as Berwald–Randers metrics, the answer is negative.

Example 1.1 ([BCS]) Let α be the canonical Riemannian product metric on $\mathbb{S}^2 \times \mathbb{S}$ and let β be a parallel 1-form. Denote by (r, θ) and t the spherical coordinates of $\mathbb{S}^2$ and $\mathbb{S}$, respectively. Then $\beta = dt$. For each $\epsilon \in [0, 1)$, $F_\epsilon := \alpha + \epsilon\beta$ is a Berwald–Randers metric with the flag curvature $\mathbf{K}_\epsilon \geq 0$, $\mathrm{diam}_\epsilon(M) \leq 3\pi$ and the Holmes–Thompson volume $\mu_\epsilon(M) = 8\pi^2$. However, $\sigma(t) = (0, 0, -t)$ is a geodesic of F_ϵ with the length $2\pi(1-\epsilon) \to 0$ (as $\epsilon \to 1$).

The purpose of this paper is to study the length of simple geodesics on a closed Finsler manifold. Given a Finsler manifold (M, F), let $\mathbf{T}$ and Λ_F be the T-curvature and the uniformity constant, respectively (see [E, S] or Section 2). These quantities are non-Riemannian quantities. In fact, $\mathbf{T} = 0$ if only if F is Berwaldian, while $\Lambda_F = 1$ if and only if F is Riemannian. Our main result is the following theorem.

Theorem 1.2 *Let (M, F) be a closed Finsler m-manifold with $\mathbf{K} \geq \delta$, $\mathbf{T} \leq \varsigma$, $\Lambda_F \geq \Lambda$,*

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and diameter $\leq d$. *Then for any simple closed geodesic* γ,

$$L_F(\gamma) \geq \frac{\mu(M)}{c_{m-2}\Lambda^{\frac{3m}{2}}\left[\frac{\mathfrak{s}_\delta^{m-1}(\min\{d,\pi/(2\sqrt{\delta})\})}{m-1} + \max\{0,\varsigma\}\int_0^d \mathfrak{s}_\delta^{m-1}(t)dt\right]},$$

where $\mu(M)$ *is either the Busemann–Hausdorff volume or the Holmes–Thompson volume of* M, $L_F(\gamma)$ *is the length of* γ, *and* $c_{m-2} := \mathrm{Vol}(\mathbb{S}^{m-2})$.

According to Theorem 1.2, a lower bound for the length of the simple closed geodesics in Example 1.1 is $8/(9\pi\Lambda_{F_\epsilon}^{9/2})$. Note that $\Lambda_{F_\epsilon} \geq (1+\epsilon)^2(1-\epsilon)^{-2}$. Hence,

$$L_{F_\epsilon}(\sigma) \geq 8/(9\pi\Lambda_{F_\epsilon}^{9/2}) \longrightarrow 0$$

(as $\epsilon \to 1$). In fact, by a better estimate for Randers manifolds (see Theorem 6.3), we have $L_{F_\epsilon}(\gamma) \geq 8(1-\epsilon)^4/(9\pi(1+\epsilon))$ for any simple closed geodesic γ in Example 1.1.

We remark that Cheeger's argument in [Ch] was carried out using Toponogov's comparison theorem. But Toponogov's comparison theorem does not hold in a non-Riemannian Finsler manifold. In [HK], Heintze and Karcher gave a more direct proof of Cheeger's theorem by studying the normal bundle of a simple closed geodesic. However, in the general case, the normal bundle of a Finsler submanifold is not a vector bundle but a cone-bundle [Ru, S]. Apparently, it is rather hard to handle this cone-bundle due to nonlinearity. The principal idea in the proof of Theorem 1.2 is to investigate the conormal bundle, which is the homeomorphic image of the normal bundle under the Legendre transformation. In fact, our method works for Finsler submanifolds with arbitrary codimensions. This will be discussed elsewhere.

2 Preliminaries

In this section, we recall some definitions and properties about Finsler manifolds. See [BCS, S] for more details.

Let (M, F) be a (connected) Finsler m-manifold with Finsler metric $F\colon TM \to [0,\infty)$. Define $S_xM := \{y \in T_xM : F(x,y) = 1\}$ and $SM := \cup_{x\in M} S_xM$. Let $(x,y) = (x^i, y^i)$ be local coordinates on TM. Define

$$\ell^i := \frac{y^i}{F},\ g_{ij}(x,y) := \frac{1}{2}\frac{\partial^2 F^2(x,y)}{\partial y^i \partial y^j}, \qquad A_{ijk}(x,y) := \frac{F}{4}\frac{\partial^3 F^2(x,y)}{\partial y^i\partial y^j\partial y^k},$$

$$\gamma^i_{jk} := \frac{1}{2}g^{il}\left(\frac{\partial g_{jl}}{\partial x^k} + \frac{\partial g_{kl}}{\partial x^j} - \frac{\partial g_{jk}}{\partial x^l}\right), \qquad N^i_j := \left(\gamma^i_{jk}\ell^j - A^i_{jk}\gamma^k_{rs}\ell^r\ell^s\right)\cdot F.$$

The Chern connection ∇ is defined on the pulled-back bundle π^*TM and its forms are characterized by the following structure equations:

(1) Torsion freeness: $dx^j \wedge \omega^i_j = 0$;

(2) Almost g-compatibility: $dg_{ij} - g_{kj}\omega^k_i - g_{ik}\omega^k_j = 2\frac{A_{ijk}}{F}(dy^k + N^k_l dx^l)$.

From the above, it is easy to obtain $\omega^i_j = \Gamma^i_{jk}dx^k$, and $\Gamma^i_{jk} = \Gamma^i_{kj}$.

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The curvature form of the Chern connection is defined as

$$\Omega^i_j := d\omega^i_j - \omega^k_j \wedge \omega^i_k =: \frac{1}{2} R^i_{j\,kl} dx^k \wedge dx^l + P^i_{j\,kl} dx^k \wedge \frac{dy^l + N^l_s dx^s}{F}.$$

Given a non-zero vector $V \in T_xM$, the flag curvature $K(y, V)$ on $(x, y) \in TM\backslash 0$ is defined as

$$\mathbf{K}(y, V) := \frac{V^i y^j R_{jikl} y^l V^k}{g_y(y, y) g_y(V, V) - [g_y(y, V)]^2},$$

where $R_{jikl} := g_{is} R^s_{j\,kl}$.

Given $y, v \in T_xM$ with $y \neq 0$, define the T-curvature $\mathbf{T}$ as

$$\mathbf{T}_y(v) := g_y(\nabla^V_v V, y) - g_y(\nabla^Y_v V, y),$$

where V (resp. Y) is a vector field with $V_x = v$ (resp. $Y_x = y$). And we say $\mathbf{T} \leq \varsigma$ if

$$\mathbf{T}_y(v) \leq \varsigma \left[\sqrt{g_y(v, v)} - g_y\left(v, \frac{y}{F(y)}\right) \right]^2 F(y),$$

for all $y, v \in TM\backslash 0$.

Remark 2.1 We modify the definition of $\mathbf{T} \leq \varsigma$ here, because the original one in [S] is not well defined when F is a Randers metric and $y = -v$.

The uniformity constant Λ_F of (M, F) is defined by ([E])

$$\Lambda_F := \sup_{X,Y,Z \in SM} \frac{g_X(Y, Y)}{g_Z(Y, Y)}.$$

Clearly, $\Lambda_F \geq 1$; $\Lambda_F = 1$ if and only if F is Riemannian.

Given any volume form $d\mu$ on M, in a local coordinate system (x^i), express $d\mu = \sigma(x) dx^1 \wedge \cdots \wedge dx^n$. For $y \in T_xM\backslash 0$, define the distortion of $(M, F, d\mu)$ as

$$\tau(y) := \log \frac{\sqrt{\det(g_{ij}(x, y))}}{\sigma(x)}.$$

The Legendre transformation $\mathcal{L} \colon TM \to T^*M$ is defined by

$$\mathcal{L}(Y) = \begin{cases} 0, & Y = 0, \\ g_Y(Y, \cdot), & Y \neq 0. \end{cases}$$

For each $x \in M$, the Legendre transformation is a smooth diffeomorphism from $T_xM\backslash\{0\}$ onto $T^*_xM\backslash\{0\}$.

Define the dual Finsler metric $F^* \colon T^*M \to [0, \infty)$ of F by $F^*(\xi) := \sup_{y \in SM} \xi(y)$. By [BCS, S], $F^*(\mathcal{L}(y)) = F(y)$ and $g^{*ij}(\xi) := \frac{1}{2}[F^{*2}]_{\xi_i \xi_j}(\xi) = g^{ij}(y)$, where $\xi = \mathcal{L}(y)$.

3 Conormal Bundle and Exponential Map

Throughout this paper, we assume that (M, F) is a forward complete Finsler m-manifold and that $\gamma(s)$, $0 \le s \le \ell = L_F(\gamma)$, is a unit speed simple closed geodesic on M. And we always identify γ with its image $\gamma([0,\ell])$. Denote by $c_y(t)$ a constant speed geodesic with $\dot{c}_y(0) = y$. The rules that govern our index gymnastics are as follows:

- i, j run from 1 to m.
- A, B run from 2 to m. $\mathfrak{g}, \mathfrak{h}$ run from 2 to $m-1$.
- $\mathbf{a}, \mathbf{b}$ run from 1 to $m-1$.

According to [Ru, S], the normal bundle $\mathcal{V}\gamma$ of γ is defined as

$$\mathcal{V}\gamma := \{n \in TM : n = 0 \text{ or } g_n(n, \dot{\gamma}) = 0\}.$$

In general, $\mathcal{V}\gamma$ is not a vector bundle even F is reversible. Consider the following subbundle of T^*M:

$$\mathcal{V}^*\gamma := \{\omega \in T^*M : \omega(\dot{\gamma}) = 0\}.$$

It is easy to see that $\mathcal{V}^*\gamma = \mathcal{L}(\mathcal{V}\gamma)$, where $\mathcal{L}\colon TM \to T^*M$ is the Legendre transformation. Note that $\mathcal{L}$ is a homeomorphism from TM to T^*M and a diffeomorphism from $TM\backslash 0$ to $T^*M\backslash 0$. Hence, $\mathcal{V}^*\gamma$ is called the *conormal bundle* over γ in M.

Let $\pi\colon \mathcal{V}^*\gamma \to \gamma$ denote the bundle projection. For each s_0, there exists a local coordinate system $(U; x^i)$ at $\gamma(s_0)$ such that $x^1 \circ \gamma(s) = s$ and $x^A \circ \gamma(s) = 0$. Hence, for each $\xi \in \pi^{-1}(U \cap \gamma)$, $\xi = \xi_A dx^A$ and $\pi^{-1}(U \cap \gamma) \approx (U \cap \gamma) \times \mathbb{R}^{m-1}$. We call (x^i) (resp. (s, ξ_A)) an *adapted coordinate system* for γ (resp. $\mathcal{V}^*\gamma$).

Define the *conormal exponential map* $\mathrm{Exp}^c\colon \mathcal{V}^*\gamma \to M$ by

$$\mathrm{Exp}^c(\xi) := \exp_{\pi(\xi)}(\mathcal{L}^{-1}(\xi)).$$

Let $S^*M := \{\omega \in T^*M : F^*(\omega) = 1\}$ and $\mathcal{V}^*S\gamma := S^*M \cap \mathcal{V}^*\gamma$. Now we have the following theorem.

Theorem 3.1 *For each $\eta \in \mathcal{V}^*S\gamma$, there exists a small $\epsilon(\eta) > 0$ and an open neighborhood $\mathcal{W}$ of η in $\mathcal{V}^*S\gamma$ such that ${\mathrm{Exp}^c}_{*t\xi}$ is nonsingular for all $\xi \in \mathcal{W}$ and $t \in (0, \epsilon(\eta))$.*

Proof For the sake of clarity, we use (x, ξ) to denote a point $\xi \in \mathcal{V}^*\gamma$. Given $(x_0, \eta_0) \in \mathcal{V}^*S\gamma \subset \mathcal{V}^*\gamma$, let $(U; x^i)$ be an adapted coordinate system at x_0 for γ and $V := U \cap \gamma$. We can choose a small $\delta > 0$ such that $\mathrm{Exp}^c(\mathcal{D}) \subset U$, where $\mathcal{D} = \{(x, t\eta) : t \in (0,\delta), (x,\eta) \in \mathcal{V}^*SV\}$. Let (x^i, y^i) and (s, ξ_A) be the local (adapted) coordinates for TM and $\mathcal{V}^*\gamma$, respectively. Thus, for each $(x, t\eta) \in \mathcal{D}$, we have

$${\mathrm{Exp}^c}_{*(x,t\eta)} \frac{\partial}{\partial s} = \frac{\partial \exp(x, \mathcal{L}^{-1}(\xi))}{\partial s}\bigg|_{x,\,\xi = t\eta} = \left[\delta_1^i + H(t,x,\eta)_1^i\right] \frac{\partial}{\partial x^i},$$

where

$$H(t,x,\eta)_1^i := \left[\frac{\partial \exp^i}{\partial x^1}(x, t\mathcal{L}^{-1}(\eta)) - \delta_1^i\right] + \frac{\partial \exp^i}{\partial y^k}(x, t\mathcal{L}^{-1}(\eta)) \cdot \frac{\partial g^{*Ak}}{\partial x^1}(x,\eta) \cdot t\eta_A.$$

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Likewise,

$$\text{Exp}^{c}{}_{*(x,t\eta)}\frac{\partial}{\partial\xi_A}=g^{*Ak}_{(\eta)}\big[\,\delta^i_k+L(t,x,\eta)^i_k\big]\,\frac{\partial}{\partial x^i},\tag{3.1}$$

where

$$L(t,x,\eta)^i_k:=\frac{\partial\exp^i}{\partial y^k}\big(x,t\mathcal{L}^{-1}(\eta)\big)\,-\,\delta^i_k.$$

Clearly, $\lim_{t\to0^+}H(t,x,\eta)^i_1=\lim_{t\to0^+}L(t,x,\eta)^i_k=0$. The matrix of $\text{Exp}^{c}{}_{*(x,t\eta)}$ is

$$S(t,x,\eta)=\begin{pmatrix}1+H(t,x,\eta)^1_1 & H(t,x,\eta)^B_1\\ g^{*A1}_{(\eta)}+g^{*Ak}_{(\eta)}L(t,x,\eta)^1_k & g^{*AB}_{(\eta)}+g^{*Ak}_{(\eta)}L(t,x,\eta)^B_k\end{pmatrix}.$$

Since $\det S(0,x_0,\eta_0)>0$, there exists a small $\epsilon(x_0,\eta_0)>0$ and an open neighborhood $\mathcal{W}$ of (x_0,η_0) in $\mathcal{V}^*S\gamma$ such that $\text{Exp}^{c}{}_{*t\xi}$ is nonsingular for all $\xi\in\mathcal{W}$ and $t\in(0,\epsilon(x_0,\eta_0))$. ■

Remark 3.2 In general, Exp^c is not C^1 at all the zero sections of $\mathcal{V}^*\gamma$. Otherwise, it follows from (3.1) that $\mathcal{L}^{-1}|_{\mathcal{V}^*\gamma}\colon\mathcal{V}^*\gamma\to\mathcal{V}\gamma$ is an isomorphism, which implies that $\mathcal{V}\gamma$ is a vector bundle.

Definition 3.3 Given $\xi\in\mathcal{V}^*_s\gamma\backslash0$, the co-(second fundamental form) of γ along ξ in M is defined as

$$h_\xi(X,Y):=\langle\xi,\nabla^{\overline{n}}_X\overline{Y}\rangle,\quad\forall X,Y\in T_s\gamma,$$

And co-Weingarten map $\mathfrak{A}^\xi\colon T_s\gamma\to T_s\gamma$ is defined as

$$\mathfrak{A}^\xi(X):=-(\nabla^{\overline{n}}_X\overline{n})^{\top_\xi},$$

where $\overline{n}:=\mathcal{L}^{-1}(\overline{\xi})$, $\overline{\xi}$ (resp. $\overline{Y}$) is an extension of ξ (resp. Y) to a co-normal (resp. tangent) vector field along γ, and the superscript $\top_\xi$ denotes projection to $T_s\gamma$ by $g_{\mathcal{L}^{-1}(\xi)}$.

By the definition of Legendre transformation and [S, (3.10), p. 39], it is easy to check that h and $\mathfrak{A}^\xi$ are well defined. A direct calculation yields

$$g_n\big(\mathfrak{A}^\xi(X),Y\big)\,=h_\xi(X,Y)=-g_{\dot\gamma}(\dot\gamma,X)g_{\dot\gamma}(\dot\gamma,Y)\mathbf{T}_n(\dot\gamma),\ \forall X,Y\in T_s\gamma.\tag{3.2}$$

Definition 3.4 Given $\xi\in\mathcal{V}^*_sS\gamma$, a vector field X along the geodesic $c_{\mathcal{L}^{-1}(\xi)}(t)$, $t\in[0,a]$, is called a transverse vector field if

$$g_T(T,X)=0,\ X(0)\in T_s\gamma,\ (\nabla^T_TX)(0)+\mathfrak{A}^\xi(X(0))\in T^\perp_s\gamma,$$

where $T=\dot c_{\mathcal{L}^{-1}(\xi)}(t)$ and $T^\perp_s\gamma=\{Y\in T_{\gamma(s)}M:g_{\mathcal{L}^{-1}(\xi)}(Y,\dot\gamma(s))=0\}$.

Let $\mathfrak{T}$ denote the collection of transverse Jacobi fields along the geodesic $c_{\mathcal{L}^{-1}(\xi)}(t)$, $t \in [0, a]$. Then $\mathfrak{T}$ is a vector space. A similar argument to the one given in [C, p. 141] shows that $\dim(\mathfrak{T}) = m - 1$.

Let $\pi_1 \colon \mathcal{V}^* S\gamma \to \gamma$ be the natural projection. Clearly, $\mathcal{V}^*_s S\gamma := \pi_1^{-1}(\gamma(s))$ is a $(m-2)$-dimensional unit Minkowski sphere in $T^*_{\gamma(s)}M$. Let $(s, \theta_{\mathfrak{g}})$ be a local coordinate system on $\mathcal{V}^* S\gamma$, where $(\theta_{\mathfrak{g}})$ are the local coordinates for $\mathcal{V}^*_s S\gamma$. Thus, we obtain a local *conic coordinate system* $(t, s, \theta_{\mathfrak{g}})$ on $\mathcal{V}^*\gamma\backslash 0$, that is, for $\xi \in \mathcal{V}^*\gamma\backslash 0$, $t = F^*(\xi)$ and $\xi / F^*(\xi) = (s, \theta_{\mathfrak{g}})$.

Define a map $\mathrm{E} \colon [0, +\infty) \times \mathcal{V}^* S\gamma \to M$ by $\mathrm{E}(t, \xi) = \mathrm{Exp}^c(t\xi)$. Then we have the following lemma.

Lemma 3.5 *$J_1(t) = \mathrm{E}_{*(t,\xi)} \frac{\partial}{\partial s}$ and $J_{\mathfrak{g}}(t) = \mathrm{E}_{*(t,\xi)} \frac{\partial}{\partial \theta_{\mathfrak{g}}}$ are $m-1$ transverse Jacobi fields along the geodesic $c_{\mathcal{L}^{-1}(\xi)}(t)$ with initial data*

$$J_1(0) = \dot{\gamma}(s_0), \ J_{\mathfrak{g}}(0) = 0, \ (\nabla^T_T J_{\mathfrak{g}})(0) = \mathcal{L}^{-1}_{*\xi}\Big(\frac{\partial}{\partial \theta_{\mathfrak{g}}}\Big),$$

where $\gamma(s_0) := \pi(\xi)$, $T := \dot{c}_{\mathcal{L}^{-1}(\xi)}(t)$ and $\mathcal{L}^{-1}_{*\xi} \colon T_\xi(T^*_{\gamma(s_0)}M) \to T_{\mathcal{L}^{-1}(\xi)}(T_{\gamma(s_0)}M)$ is the tangent map.

Proof Suppose $\xi = (s_0, \theta^0_{\mathfrak{g}})$.

(1) Set $\xi(s) = (s, \theta^0_{\mathfrak{g}})$, where $s \in (-\epsilon + s_0, \epsilon + s_0)$. Consider the variation $\sigma(t, s) = \mathrm{E}(t, \xi(s)) = \exp_{\gamma(s)} t\mathcal{L}^{-1}(\xi(s))$. Thus, $J_1(t) = \frac{\partial}{\partial s}|_{s=s_0}\sigma(t, s)$, which implies

$$J_1(0) = \dot{\gamma}(s_0) \quad \text{and} \quad (\nabla^T_T J_1)(0) = \nabla^{\mathcal{L}^{-1}(\xi(s))}_{J_1(0)} \mathcal{L}^{-1}(\xi(s)).$$

Hence, $(\nabla^T_T J_1)(0) + \mathfrak{A}^\xi(J_1(0)) \in T^\perp_{s_0}\gamma$. Since $F^*(\xi(s)) = 1$, $g_T(T, (\nabla^T_T J_1)(0)) = 0 = g_T(T, J_1)$. Therefore, J_1 is a transverse Jacobi field along $c_{\mathcal{L}^{-1}(\xi)}(t)$.

(2) Set $\xi(u) = (s_0, \theta_{\mathfrak{g}}(u))$, $u \in (-\epsilon, \epsilon)$ with $\theta_{\mathfrak{g}}(0) = \theta^0_{\mathfrak{g}}$ and $\frac{d}{du}|_{u=0}\xi(u) = \frac{\partial}{\partial \theta_{\mathfrak{g}}}$. Consider the variation $\sigma(t, u) = \mathrm{E}(t, \xi(u)) = \exp_{\gamma(s_0)} t\mathcal{L}^{-1}(\xi(u))$. Clearly,

$$J_{\mathfrak{g}}(t) = \mathrm{E}_{*(t,\xi)} \frac{\partial}{\partial \theta_{\mathfrak{g}}} = \frac{\partial}{\partial u}\Big|_{u=0} \sigma(t, u) = (\exp_{\gamma(s_0)})_{*t\mathcal{L}^{-1}(\xi)} t\mathcal{L}^{-1}_{*\xi}\Big(\frac{\partial}{\partial \theta_{\mathfrak{g}}}\Big).$$

Since $F(\mathcal{L}^{-1}(\xi(u))) = 1$,

$$0 = \frac{d}{du}\Big|_{u=0} F^2(\mathcal{L}^{-1}(\xi(u))) = 2g_{\mathcal{L}^{-1}(\xi)}\Big(\mathcal{L}^{-1}(\xi), \mathcal{L}^{-1}_{*\xi}\Big(\frac{\partial}{\partial \theta_{\mathfrak{g}}}\Big)\Big).$$

The Gauss lemma[BCS, p. 140] then yields $g_T(T, J_{\mathfrak{g}}) = 0$. ∎

Now we extend focal points to Finsler manifolds.

Definition 3.6 Given $\xi \in \mathcal{V}^* S\gamma$, a point $c_{\mathcal{L}^{-1}(\xi)}(t_0)$ $(t_0 > 0)$ is said to be focal to γ along $c_{\mathcal{L}^{-1}(\xi)}(t)$ if there exists a nontrivial transverse Jacobi field J such that $J(t_0) = 0$.

Given $\xi \in \mathcal{V}_s^* S\gamma$, let $\mathfrak{X}$ denote the collection of all vector fields X along $c_{\mathcal{L}^{-1}(\xi)}(t)$, $t \in [0, a]$, such that $g_T(T, X) = 0$ and $X(0) \in T_s\gamma$ and let $\mathfrak{X}_0$ consist of those elements of $\mathfrak{X}$ that vanish at $t = a$. On $\mathfrak{X}$, *the index form* is defined by

$$I(X, Y) := -h_\xi\big(0), Y(0)\big) + \int_0^a g_T(\nabla_T^T X, \nabla_T^T Y) + R_T(T, X, T, Y)\, dt.$$

By Lemma 3.5 and the arguments given in [BCS, pp. 180–185], one can easily show the following theorem.

Theorem 3.7 *Suppose that $c_{\mathcal{L}^{-1}(\xi)}(t)$ has not focal points on $(0, a]$ to γ. Given $X \in \mathfrak{X}$, let J denote the (unique) transverse Jacobi field along $c_{\mathcal{L}^{-1}(\xi)}$ with $J(a) = X(a)$. Then $I(X, X) \geq I(J, J)$ with equality if and only if $X = J$.*

Suppose that some point $c_{\mathcal{L}^{-1}(\xi)}(t_0)$, $0 < t_0 < a$ is focal to γ along $c_{\mathcal{L}^{-1}(\xi)}$. Then there is $U \in \mathfrak{X}_0$ with $I(U, U) < 0$.

Lemma 3.5 together with Theorem 3.1 furnishes the following proposition.

Proposition 3.8 *Given $\xi \in \mathcal{V}^* S\gamma$, the following statements are mutually equivalent:*

(i) *$c_{\mathcal{L}^{-1}(\xi)}(t_0)$, $0 < t_0 < \infty$ is a focal point of γ along $c_{\mathcal{L}^{-1}(\xi)}(t)$;*
(ii) *$\mathrm{Exp}^{c}{}_{*t_0\xi}$ is singular;*
(iii) *$\mathrm{E}_{*(t_0,\xi)}$ is singular.*

Proof Define a diffeomorphism $\mathscr{F} \colon (0, +\infty) \times \mathcal{V}^* S\gamma \to \mathcal{V}^*\gamma \backslash 0$ by $\mathscr{F}(t, \xi) = t\xi$. Clearly, $\mathrm{Exp}^{c}{}_{*} \circ \mathscr{F}_* = \mathrm{E}_*$, which implies (ii) $\Leftrightarrow$ (iii). It follows Lemma 3.5 that (iii) $\Rightarrow$ (i). Now we show (i) $\Rightarrow$ (iii).

Let $J_{\mathbf{a}}(t)$, $\mathbf{a} = 1, \mathfrak{g}$, be as in Lemma 3.5. By Theorem 3.1, there exists $\epsilon(\xi) > 0$ such that $\mathrm{E}_{*(t,\xi)}$ is nonsingular for $0 < t \leq \epsilon(\xi)$. Thus, $\{J_{\mathbf{a}}(t)\}$ form a basis for the space of the transverse Jacobi fields along $c_n(t)$, $0 \leq t \leq \epsilon(\xi)$, where $n = \mathcal{L}^{-1}(\xi)$.

Suppose $c_n(t_0)$ is a focal point. Then there exists a nontrivial transverse Jacobi field J along c_n such that $J(t_0) = 0$. We can suppose $J(t) = C^{\mathbf{a}} J_{\mathbf{a}}(t)$ for $t \geq 0$. Here, $C^{\mathbf{a}}$'s are constants not all zero. Then $J(t_0) = 0$ implies (i) $\Rightarrow$ (iii). ■

Definition 3.9 Given $\xi \in \mathcal{V}^* S\gamma$, the focal value $c_f(\xi)$ is defined by

$$c_f(\xi) := \sup\{r > 0 : \text{ no point } c_{\mathcal{L}^{-1}(\xi)}(t),\ 0 < t < r \text{ is focal point }\}$$

By Theorem 3.1 and Proposition 3.8, we have the following lemma.

Lemma 3.10 *The function $c_f \colon \mathcal{V}^* S\gamma \to (0, +\infty]$ is lower semicontinuous.*

Proof Given $\xi_0 \in \mathcal{V}^* S\gamma$ and $0 < r < c_f(\xi_0)$, let $\varepsilon(\xi_0)$ and $\mathcal{W}$ be as in Theorem 3.1. If $r \leq \varepsilon(\xi_0)/2$, then we take $\epsilon_r = \varepsilon(\xi_0)/2$ and $\mathcal{U} = \mathcal{W}$. Suppose $r > \varepsilon(\xi_0)/2$. For each $t \in [\varepsilon(\xi_0)/2, r]$, there exist a neighborhood U_t of ξ and a interval $I_t = (t - \epsilon_t, t + \epsilon_t)$ such that $\mathrm{Exp}^{c}{}_{*}$ is nonsingular at $s\eta$ for all $\eta \in U_t$ and $s \in I_t$. Then one can find finitely many $\{I_{t_s}\}_{s=1}^k$ such that $\cup_s I_{t_s} \supset [\varepsilon(\xi_0)/2, r]$. Without loss of generality, we suppose that $t_1 < \cdots < t_k$ and $t_k = r$ (so $\epsilon_{t_k} = \epsilon_r$). Set $\mathcal{U} := \cap_s U_{t_s} \cap \mathcal{W}$. Thus, $\mathrm{Exp}^{c}{}_{*(x,t\xi)}$ is not singular for all $t \in (0, r + \epsilon_r)$ and $(x, \xi) \in \mathcal{U}$, *i.e.*, $c_f(\xi) > r + \epsilon_r$. From above, $\liminf_{\xi \to \xi_0} c_f(\xi) \geq r + \epsilon_r$ and $\lim_{r \to c_f(\xi_0)} \epsilon_r = 0$. We complete the proof by letting $r \to c_f(\xi_0)$. ■

Given $\xi \in \mathcal{V}_s^* S\gamma$, let $n = \mathcal{L}^{-1}(\xi)$ and $n^\perp := \{X \in T_{\gamma(s)}M : g_n(n, X) = 0\}$. The proof of Lemma 3.5 furnishes the following decomposition

$$(T_{\gamma(s)}M, g_n) = \mathbb{R} \cdot n \oplus \mathbb{R} \cdot \dot{\gamma}(s) \oplus \mathrm{Span}_{\mathbb{R}} \Big\{ \mathcal{L}_{*\xi}^{-1} \Big(\frac{\partial}{\partial \theta_{\mathfrak{g}}} \Big) \Big\}.$$

For convenience, set $e_1 := \dot{\gamma}(s)$ and $e_{\mathfrak{g}} := \mathcal{L}_{*\xi}^{-1}(\partial / \partial \theta_{\mathfrak{g}})$. Denote by $P_{t;n}$ the parallel translation along c_n from $T_{c_n(0)}M$ to $T_{c_n(t)}M$ (with respect to the Chern connection) for all $t \geq 0$. Set $T = \dot{c}_n(t)$, $R_T := R_T(\cdot, T)T$ and

$$\mathcal{R}(t, n) := P_{t;n}^{-1} \circ R_T \circ P_{t;n} \colon n^\perp \to n^\perp.$$

Let $\mathcal{A}(t, n)$ denote the solution of the matrix differential equation on $n^\perp$:

$$\begin{cases} \mathcal{A}'' + \mathcal{R}(t, y)\mathcal{A} = 0, \\ \mathcal{A}(0, n)e_1 = e_1, \ \mathcal{A}'(0, n)e_1 = (\nabla_T^T J_1)(0), \\ \mathcal{A}(0, n)e_{\mathfrak{g}} = 0, \ \mathcal{A}'(0, n)e_{\mathfrak{g}} = e_{\mathfrak{g}}, \end{cases}$$

where $\mathcal{A}' = \frac{d}{dt}\mathcal{A}$. Note that $c_n(t) = P_{t;n}n$. Thus, for each $X \in n^\perp$,

$$g_{P_{t;n}n}\big(P_{t;n}n, P_{t;n}\mathcal{A}(t, n)X \big) = g_n\big(n, \mathcal{A}(t, n)X \big) = 0.$$

Hence, $P_{t;n}\mathcal{A}(t, n)X$ is a transverse Jacobi filed along $c_n(t)$. Let $J_{\mathbf{a}}(t)$, $\mathbf{a} = 1, \mathfrak{g}$, be as in Lemma 3.5. Thus, $J_{\mathbf{a}}(t) = P_{t;n}\mathcal{A}(t, n)e_{\mathbf{a}}$. Set $\mathcal{A}e_{\mathbf{a}} =: \mathcal{A}_{\mathbf{a}}^{\mathbf{b}} e_{\mathbf{b}}$ and $\det \mathcal{A} := \det \mathcal{A}_{\mathbf{a}}^{\mathbf{b}}$. Clearly, $\det \mathcal{A}(t_0, n) = 0$ $(t_0 > 0)$ if and only if $c_n(t_0)$ is a focal point of γ along $c_n(t)$. Moreover, we have the following lemma.

Lemma 3.11

$$\lim_{t \to 0^+} \frac{\det \mathcal{A}(t, n)}{t^{m-2}} = 1.$$

Proof The Lagrange identity [BCS, p. 135] together with (3.2) implies that

$$g_T\big(\nabla_T^T J_1(t), J_{\mathfrak{g}}(t) \big) - g_T\big(J_1(t), \nabla_T^T J_{\mathfrak{g}}(t) \big) = \\ - g_n\big(\mathfrak{A}^\xi\big(J_1(0) \big), J_{\mathfrak{g}}(0) \big) + g_n\big(J_1(0), \mathfrak{A}^\xi\big(J_{\mathfrak{g}}(0) \big) \big) = 0.$$

By L'Hôspital's rule, we have

$$\lim_{t \to 0^+} \frac{g_T(J_1(t), J_{\mathfrak{g}}(t))}{t^2} = \lim_{t \to 0^+} \frac{g_T(\nabla_T^T J_1(t), J_{\mathfrak{g}}(t))}{t} = g_n\big((\nabla_T^T J_1)(0), e_{\mathfrak{g}} \big).$$

And it is easy to see that $\lim_{t \to 0^+} \frac{g_T(J_{\mathfrak{h}}, J_{\mathfrak{g}})}{t^2} = g_n(e_{\mathfrak{h}}, e_{\mathfrak{g}})$. Hence,

$$\lim_{t \to 0^+} \frac{\det g_T(J_{\mathbf{a}}(t), J_{\mathbf{b}}(t))}{t^{2(m-2)}} = \det g_n(e_1, e_1) \det g_n(e_{\mathfrak{h}}, e_{\mathfrak{g}}).$$

Now the conclusion follows from

$$\det\big[g_T(J_{\mathbf{a}}(t), J_{\mathbf{b}}(t)) \big] = (\det \mathcal{A})^2 \cdot \det g_n(e_1, e_1) \cdot \det g_n(e_{\mathfrak{h}}, e_{\mathfrak{g}}). \qquad \blacksquare$$

By the arguments above and Lemma 3.11, we have the following Heintze–Karcher type inequality.

Theorem 3.12 *Given $\xi \in \mathcal{V}_s^* S\gamma$, let $n = \mathcal{L}^{-1}(\xi)$. If the flag curvature $\mathbf{K}(\dot{c}_n(t); \cdot) \geq \delta$, then $c_f(\xi) \leq \min\{\zeta, \pi/\sqrt{\delta}\}$ and*

$$\det \mathcal{A}(t,n) \leq \Big(\mathfrak{s}_\delta' + \frac{\mathbf{T}_n(\dot\gamma)}{g_n(\dot\gamma,\dot\gamma)} \mathfrak{s}_\delta \Big)(t) \cdot \mathfrak{s}_\delta^{m-2}(t), \text{ for } t \in [0, c_f(\xi)],$$

where ζ is the first positive zero of

$$\Big(\mathfrak{s}_\delta' + \frac{\mathbf{T}_n(\dot\gamma)}{g_n(\dot\gamma,\dot\gamma)} \mathfrak{s}_\delta \Big)(t)$$

(should such a zero exist; otherwise, set $\zeta = +\infty$).

Proof Fix some positive number $r < c_f(\xi)$. Recall $J_{\mathbf{a}}(t) = P_{t;n}\mathcal{A}e_{\mathbf{a}}$, for $\mathbf{a} = 1, \mathfrak{g}$. For $l \in (0,r)$, we have

$$\frac{(\det \mathcal{A})'}{\det \mathcal{A}}(l) = \frac{1}{2} \frac{(\det g_T(J_{\mathbf{a}}, J_{\mathbf{b}}))'}{\det g_T(J_{\mathbf{a}}, J_{\mathbf{b}})}(l).$$

Note that $\{J_{\mathbf{a}}(t)\}$ is a basis for the space $\mathfrak{T}$ of transverse Jacobi fields along $c_n(t)$, $0 \leq t \leq l$. Let $\{\overline{J}_{\mathbf{a}}(t)\}$ be another $m-1$ transverse Jacobi fields such that $\{T(l), \overline{J}_{\mathbf{a}}(l)\}$ is a g_T-orthonormal basis. Then $\{\overline{J}_{\mathbf{a}}(t)\}$ is also a basis for $\mathfrak{T}$. Hence,

$$\frac{(\det \mathcal{A})'}{\det \mathcal{A}}(l) = \frac{1}{2} \frac{(\det g_T(J_{\mathbf{a}}, J_{\mathbf{b}}))'}{\det g_T(J_{\mathbf{a}}, J_{\mathbf{b}})}(l) = \frac{1}{2} \frac{(\det g_T(\overline{J}_{\mathbf{a}}, \overline{J}_{\mathbf{b}}))'}{\det g_T(\overline{J}_{\mathbf{a}}, \overline{J}_{\mathbf{b}})}(l). \tag{3.3}$$

A direct calculation yields

$$\frac{1}{2} \frac{(\det g_T(\overline{J}_{\mathbf{a}}, \overline{J}_{\mathbf{b}}))'}{\det g_T(\overline{J}_{\mathbf{a}}, \overline{J}_{\mathbf{b}})}(l) = \sum_{\mathbf{a}} (g_T(\nabla_T^T \overline{J}_{\mathbf{a}}, \overline{J}_{\mathbf{a}}))'(l) = \sum_{\mathbf{a}} I_{[0,l]}(\overline{J}_{\mathbf{a}}, \overline{J}_{\mathbf{a}}), \tag{3.4}$$

where $I_{[0,l]}$ is the index form restricted to $c_n(t)$, $0 \leq t \leq l$.

Consider the solution $\mathcal{A}_\delta(t)$ to the matrix differential equation in $n^\perp$:

$$\mathcal{A}_\delta'' + k\mathcal{A}_\delta = 0,$$

with the same initial conditions as $\mathcal{A}(t)$. Let f_1 be a g_n-unit eigenvector of $\mathfrak{A}^\xi$ with the eigenvalue λ. It follows from (3.2) that $\lambda = -\mathbf{T}_n(\dot\gamma)/g_n(\dot\gamma,\dot\gamma)$.

Let $\{f_{\mathfrak{g}}\}$ be a g_n-orthonormal basis for $n^\perp \cap T_s^\perp \gamma$. Then we have

$$\mathcal{A}_\delta(t) f_1 = (\mathfrak{s}_\delta' - \lambda \mathfrak{s}_\delta)(t) \cdot f_1 + \mathfrak{s}_\delta(t) \cdot (C^{\mathfrak{g}} f_{\mathfrak{g}}), \quad \mathcal{A}_\delta(t) f_{\mathfrak{g}} = \mathfrak{s}_\delta(t) \cdot f_{\mathfrak{g}},$$

where $C^{\mathfrak{g}}$'s are constants determined by the initial data of $\mathcal{A}_k(t)$. Clearly, $\det \mathcal{A}_\delta(t) = \mathfrak{s}_\delta^{m-2}(t) \cdot (\mathfrak{s}_\delta' - \lambda \mathfrak{s}_\delta)(t)$.

Let $r < \zeta_0$, where ζ_0 is the first positive zero of $\det \mathcal{A}_\delta(t)$. Then $\{T, P_{t;n}\mathcal{A}_\delta(t) f_{\mathbf{a}}\}$ is a frame field along $c_n(t)$, $0 < t \leq r$. Now consider the vector fields $Y_{\mathbf{a}}(t) :=$

$C_{\mathbf{a}}^{\mathbf{b}} P_{t;n} \mathcal{A}_\delta(t) f_{\mathbf{b}}$, where $C_{\mathbf{a}}^{\mathbf{b}}$'s are constants such that $Y_{\mathbf{a}}(l) = \overline{J}_{\mathbf{a}}(l)$. Clearly, $\nabla_T^T \nabla_T^T Y_{\mathbf{a}} + \delta Y_{\mathbf{a}} = 0$ and $g_T(T, Y_{\mathbf{a}}) = 0$. Theorem 3.7 then yields

$$\sum_{\mathbf{a}} I_{[0,l]}(\overline{J}_{\mathbf{a}}, \overline{J}_{\mathbf{a}}) \leq \sum_{\mathbf{a}} I_{[0,l]}(Y_{\mathbf{a}}, Y_{\mathbf{a}}) \leq \sum_{\mathbf{a}} g_T(\nabla_T^T Y_{\mathbf{a}}, Y_{\mathbf{a}})(l). \tag{3.5}$$

Since $g_T(Y_{\mathbf{a}}, Y_{\mathbf{b}})(l) = \delta_{\mathbf{ab}}$,

$$\sum_{\mathbf{a}} g_T(\nabla_T^T Y_{\mathbf{a}}, Y_{\mathbf{a}})(l) = \operatorname{tr}(\mathcal{A}_\delta' \cdot \mathcal{A}_\delta^{-1})(l) = \frac{(\det \mathcal{A}_\delta)'}{\det \mathcal{A}_\delta}(l). \tag{3.6}$$

Equation (3.3) together with (3.4), (3.5), (3.6), and Lemma 3.11 furnishes $\det \mathcal{A}(t) \leq \det \mathcal{A}_\delta(t)$ for all $t \in [0, r]$, which implies that $c_f(\xi) \leq \zeta_0$. ■

4 Proof of Theorem 1.2

4.1 Volume of a Unit Conormal Sphere

Note that $\mathcal{L}^{-1}$ is an isometry from $(T_x^* M \backslash 0, g_x^*)$ to $(T_x M \backslash 0, g_x)$. Denote by $d\nu_s$ the Riemannian volume form on $\mathcal{V}_s^* S\gamma$ induced by $g_{\gamma(s)}^*$. Given $\xi \in \mathcal{V}_s^* S\gamma$, let n and $e_{\mathbf{a}}$, $\mathbf{a} = 1, \mathfrak{g}$ be defined as before. Then we have

$$g_\xi^*\Big(\frac{\partial}{\partial \theta_{\mathfrak{g}}}, \frac{\partial}{\partial \theta_{\mathfrak{h}}}\Big) = ((\mathcal{L}^{-1})^* g_n)\Big(\frac{\partial}{\partial \theta_{\mathfrak{g}}}, \frac{\partial}{\partial \theta_{\mathfrak{h}}}\Big) = g_n(e_{\mathfrak{g}}, e_{\mathfrak{h}}), \tag{4.1}$$

which implies $d\nu_s(\xi) = \sqrt{\det g_n(e_{\mathfrak{g}}, e_{\mathfrak{h}})}\, d\Theta$, where $d\Theta = \wedge_{\mathfrak{g}} d\theta_{\mathfrak{g}}$. Using the technique in [W, Proposition 3.1], one can easily show that the uniformity constant Λ_{F^*} of F^* coincides with Λ_F. Then we have the following estimate.

Lemma 4.1 $\nu_s(\mathcal{V}_s^* S\gamma) \leq c_{m-2} \cdot \Lambda_F^{(m-1)/2}$.

Proof Let (s, ξ_A) be an adapted coordinate system for $\mathcal{V}^*\gamma$. Thus,

$$\mathcal{V}_s^* S\gamma = \big\{\, \xi = \xi_A dx^A : F^*(\gamma(s), \xi) = 1 \big\}.$$

Hence,

$$d\nu_s(\xi) = \sqrt{\det g_\xi^{*AB}} \Big(\sum_A (-1)^{A+1} \xi_A d\xi_2 \wedge \cdots d\hat{\xi}_A \wedge \cdots d\xi_m \Big).$$

Set $\mathcal{V}_s^* B\gamma := \{\xi = \xi_A dx^A : F^*(\gamma(s), \xi) < 1\}$. Stokes's theorem then yields

$$\frac{\nu_s(\mathcal{V}_s^* S\gamma)}{m-1} = \int_{\mathcal{V}_s^* B\gamma} \sqrt{\det g_\xi^{*AB}} \wedge_A d\xi_A \leq \int_{\mathcal{V}_s^* B\gamma} \Big(\max_{\eta \in \mathcal{V}_s^* B\gamma} \sqrt{\det g_\eta^{*AB}} \Big) \wedge_A d\xi_A.$$

Now the conclusion follows from $\Lambda_{F^*} = \Lambda_F$. ■

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4.2 A m-form on $\mathcal{V}^*\gamma\backslash 0$

Given a volume form $d\mu$ on M, one can define a global m-form ϖ on $\mathcal{V}^*\gamma\backslash 0$. In a conic coordinate system,

$$\varpi_{(t,\xi)} = e^{-\tau(\dot{c}_{\mathcal{L}^{-1}(\xi)}(t))} \det \mathcal{A}(t, \mathcal{L}^{-1}(\xi))dt \wedge \sqrt{g_{\mathcal{L}^{-1}(\xi)}(\dot{\gamma}(s), \dot{\gamma}(s))}ds \wedge d\nu_s(\xi),$$

where τ is the distortion of $d\mu$. It is easy to check that ϖ is well defined.

4.3 Conic Coordinate Systems on M

Given an arbitrary point $p \in M\backslash\gamma$, there exists a unit speed minimizing geodesic c_v from γ to p. A simple first variation argument yields $\eta := \mathcal{L}(v) \in \mathcal{V}^*S\gamma$. If $c_v(t_0)$ is a focal point to γ along c_v, then the second variation of arc length formula together with Theorem 3.7 furnishes $d(\gamma, p) \leq t_0$. Hence, $\mathrm{E}(D) = M$, where

$$D := \left\{ (t, \xi) : \xi \in \mathcal{V}^*S\gamma,\ 0 \leq t \leq c_f(\xi) \right\}.$$

Moreover, for each $x_0 = \mathrm{E}(t_0, \xi_0) \in M$ with $0 < t_0 < c_f(\xi_0)$, by Lemma 3.10, there exists an open set $\mathcal{Q}(t_0, \xi_0) = (t_0 - \varepsilon, t_0 + \varepsilon) \times \mathcal{W}(\xi_0)$ such that $\mathrm{E}\,|_{\mathcal{Q}(t_0,\xi_0)} \colon \mathcal{Q}(t_0, \xi_0) \to \mathrm{E}(\mathcal{Q}(t_0, \xi_0))$ is a diffeomorphism. Thus,

$$(t \circ \mathrm{E}\,|^{-1}_{\mathcal{Q}(t_0,\xi_0)}, s \circ \mathrm{E}\,|^{-1}_{\mathcal{Q}(t_0,\xi_0)}, \theta_{\mathfrak{g}} \circ \mathrm{E}\,|^{-1}_{\mathcal{Q}(t_0,\xi_0)})$$

is a conic coordinate system on $\mathrm{E}(\mathcal{Q}(t_0, \xi_0))$. In particular, it follows from (4.1) that $\mathrm{E}\,|^*_{\mathcal{Q}(t_0,\xi_0)}\, d\mu = \varpi$.

Proof of Theorem 1.2 Sard's theorem implies that $\mu(M) = \mu(\mathrm{E}(D_d))$, where

$$D_d := \{(t, \xi) : \xi \in \mathcal{V}^*S\gamma,\ 0 < t < \min\{d, c_f(\xi)\}\}.$$

By the argument above and the proof of Lemma 3.10, there is a countable open covering $\{\mathcal{Q}(t_i, \xi_i)\}$ of D_d such that $\mathcal{Q}(t_i, \xi_i) \subset D_d$ and

$$\mathrm{E}\,|_{\mathcal{Q}(t_i,\xi_i)} \colon \mathcal{Q}(t_i, \xi_i) \to \mathrm{E}(\mathcal{Q}(t_i, \xi_i))$$

is a diffeomorphism. For simplicity, set $\mathcal{Q}_i := \mathcal{Q}(t_i, \xi_i)$ and $\mathrm{E}_i := \mathrm{E}\,|_{\mathcal{Q}_i}$. Note that $\{\mathrm{E}(\mathcal{Q}_i)\}$ is an open covering of $\mathrm{E}(D_d)$. Let $\{\rho_i\}$ be a partition of unity subordinate to $\{\mathrm{E}(\mathcal{Q}_i)\}$. Define a sequence of nonnegative continuous functions $\varrho_i \colon D_d \to \mathbb{R}$ by

$$\varrho_i(t, \xi) := \begin{cases} \rho_i \circ \mathrm{E}_i, & (t, \xi) \in \mathcal{Q}_i, \\ 0, & \text{otherwise.} \end{cases}$$

A simple argument based on [W, Proposition 3.1, Proposition 4.1] shows that $e^{-\tau(y)} \leq \Lambda_F^m$. Then Theorem 3.12 together with Lemma 4.1 furnishes

$$\begin{aligned}
\mu(M) &= \sum_i \int_{\mathrm{E}_i(\Omega_i)} \rho_i \cdot d\mu = \sum_i \int_{\Omega_i} \varrho_i \cdot \varpi \leq \int_{D_d} \varpi \\
&\leq \int_0^\ell \sqrt{g_{\mathcal{L}^{-1}(\xi)}(\dot\gamma(s), \dot\gamma(s))}ds \int_{\mathcal{V}_s^* S\gamma} e^{-\tau(\dot c_{\mathcal{L}^{-1}(\xi)}(t))} d\nu_s(\xi) \\
&\quad \times \int_0^{\min\{d, c_f(\xi)\}} \left(\mathfrak{s}_\delta' + \frac{\mathbf{T}_{\mathcal{L}^{-1}(\xi)}(\dot\gamma)}{g_{\mathcal{L}^{-1}(\xi)}(\dot\gamma,\dot\gamma)} \mathfrak{s}_\delta \right)(t) \cdot \mathfrak{s}_\delta^{m-2}(t) dt \\
&\leq c_{m-2} \Lambda^{\frac{3m}{2}} \ell \left[\frac{\mathfrak{s}_\delta^{m-1}\big(\min\{d, \frac{\pi}{2\sqrt{\delta}}\}\big)}{m-1} + \max\{0,\varsigma\} \int_0^d \mathfrak{s}_\delta^{m-1}(t) dt \right]. \qquad \blacksquare
\end{aligned}$$

It follows from [S, Lemma 12.2.5] that Kingenberg's lemma can be extended to the case of a reversible Finsler manifold. Hence, we have a generalization of Cheeger's injectivity radius estimate.

Corollary 4.2 *Let (M,F) be a closed reversible Finsler m-manifold with $|\mathbf{K}| \leq \delta$, $\mathbf{T} \leq \varsigma$, $\Lambda_F \geq \Lambda$, diameter $\leq d$, and $\mu(M) \geq V$, where $\mu(M)$ is either the Busemann–Hausdorff volume or the Holmes–Thompson volume of M. Then*

$$\mathfrak{i}_M \geq \min\left\{ \frac{\pi}{\sqrt{\delta}}, \frac{V}{2c_{m-2}\Lambda^{\frac{3m}{2}} \left[\mathfrak{s}_{-\delta}^{m-1}(d)/(m-1) + \max\{0,\varsigma\} \int_0^d \mathfrak{s}_{-\delta}^{m-1}(t)dt \right]} \right\}.$$

5 Non-Riemannian Examples

In [Ch], Cheeger gives the existence of the lower bound for the length of simple closed geodesics in a closed Riemannian manifold in terms of an upper bound for the diameter and lower bounds for the volume and the curvature. However, this is false for general Finsler manifolds. Before giving more examples, we first introduce the notations used in this section.

We say a function $\phi\colon (-1,1) \to \mathbb{R}$ satisfies Condition (Δ) if one of the following conditions is true:

(1) there exists a positive constant C such that $\mathscr{T}_{m,t}(s) \geq C$ for $|s| \leq t < 1$;
(2) $\varphi(s) := \mathscr{T}_{m,t}(s) - 1$ is an odd function.

Here $\mathscr{T}_{m,t}(s) := \phi(s)\cdot(\phi(s) - s\phi'(s))^{m-2}[\phi(s) - s\phi'(s) + (t^2 - s^2)\phi''(s)]$. Let $\wp$ denote the collection of smooth positive functions ϕ defined on $(-1,1)$ such that ϕ satisfies Condition (Δ), $\sup_{s\in(-1,1)}\phi(s) < +\infty$, $\lim_{s\to -1}\phi(s) = 0$, and

$$\phi(s) - s\phi'(s) + (t^2 - s^2)\phi''(s) > 0, \ |s| \leq t < 1.$$

Let (M,α) be a closed Riemannian m-manifold with nonnegative curvature and let β be a 1-form on M with $\sup_{x\in M} \|\beta\|_\alpha = 1$. Given $\phi \in \wp$ and $\epsilon \in [0,1)$, define

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a function F_ϵ on TM by $F_\epsilon := \alpha\phi(\epsilon\beta/\alpha)$. It follows from [CS] that F_ϵ is a Finsler metric on M. Let $\mathbf{K}_\epsilon$, diam_ϵ and μ_ϵ denote the flag curvature, the diameter, and the Holmes–Thompson volume of (M, F_ϵ), respectively.

A simple argument based on [BC, (5), Proposition 4.1, Corollary 4.2] shows the following lemma.

Lemma 5.1 *If β is parallel corresponding to α, then F_ϵ is a Berwald metric with $\mathbf{K}_\epsilon \geq 0$, $\mathrm{diam}_\epsilon(M) \leq \mathrm{diam}_\alpha(M) \cdot \mathcal{M}$, and $\mu_\epsilon(M) \geq D$, where $\mathcal{M} := \sup_{s\in(-1,1)}\phi(s)$, and D is a positive constant independent of ϵ. In particular, a geodesic of α is also a geodesic of F_ϵ and vice versa.*

Let $M = \mathbb{S}^n \times \mathbb{S}$, $n \geq 2$ and let α be the canonical Riemannian product metric on M. There exists a parallel 1-form β on (M, α). Denote by (r, θ) and t the spherical coordinates of $\mathbb{S}^n$ and $\mathbb{S}$, respectively. Then $\beta = dt$. It should be noted that β is global defined on M, even though the coordinate t is not. Given r_0 and θ_0, $\gamma(t) = (r_0, \theta_0, -t)$ is a (closed) geodesic on (M, α). Thus, for each $\phi \in \wp$, the Finsler metric F_ϵ, $\epsilon \in (0, 1)$ has the properties stated in Lemma 5.1 and $L_{F_\epsilon}(\gamma) \to 0$ as $\epsilon \to 1$. In particular, $\phi(s) = 1 + s \in \wp$. Hence, we have the following example.

Example 5.2 There always exist a sequence of Randers metrics $\{F_\epsilon\}$ on $M = \mathbb{S}^n \times \mathbb{S}$ $(n \geq 2)$ with $\mathbf{K}_\epsilon \geq 0$, $\mathrm{diam}_\epsilon(M) \leq (\sqrt{2} + 1)\pi$, and $\mu_\epsilon(M) = 2\pi c_n$. In particular, there exists a closed geodesic γ of all (M, F_ϵ) such that $L_{F_\epsilon}(\gamma) \to 0$ as $\epsilon \to 1$. Hence, the injective radius of $F_\epsilon \to 0$ as $\epsilon \to 1$.

Let $\mathbb{T}^k = \mathbb{S} \times \cdots \times \mathbb{S}$ denote the flat torus. From the construction above, one can easily show the following example.

Example 5.3 There always exists a sequence of Randers metrics $\{F_\epsilon\}$ on $M = \mathbb{S}^n \times \mathbb{T}^k$ $(n \geq 2, k \geq 1)$ with $\mathbf{K}_\epsilon \geq 0$, $\mathrm{diam}_\epsilon(M) \leq (\sqrt{1 + k} + 1)\pi$ and $\mu_\epsilon(M) = c_n(2\pi)^k$. In particular, there exists a closed geodesic γ of all (M, F_ϵ) such that $L_{F_\epsilon}(\gamma) \to 0$ as $\epsilon \to 1$. Hence, the injective radius of $F_\epsilon \to 0$ as $\epsilon \to 1$.

6 Randers Metric

In general, it is very difficult to compute the uniformity constant and the T-curvature of a Finsler metric. However, for a Randers metric $F = \alpha + \beta$, instead of the uniformity constant and the T-curvature, one can use $\|\beta\|_\alpha$ and $\|\nabla^\alpha\beta\|_\alpha$ to estimate the lower bound for the length of closed geodesics, where ∇^α is the Levi-Civita connection of α. Before stating our result, we need the following estimate.

Lemma 6.1 *If $F = \alpha + \beta$ is a Randers metric, then $\nu_s(\mathcal{V}_s^* S\gamma) \leq c_{m-2} \cdot (1 - b(s))^{-\frac{m}{2}}$, where $b(s) := \|\beta\|_\alpha(\gamma(s))$.*

Proof By [S, Example 3.1.1], $F^* = \alpha^* + \beta^*$ is also a Randers metric. Let (x^i) be an adapted coordinate system for γ. Denote by Σ_s the subspace $\{\xi = \xi_i dx^i : \xi_1 = 0\}$ of $T^*_{\gamma(s)}M$. Example 3.1.1 of [S] also furnishes $\sup_{\xi\in\Sigma_s\backslash 0}\big(\beta^*(\xi)/\alpha^*(\xi)\big) \leq b(s)$, which implies that $\det g_\xi^{*AB} \leq (\det \alpha^{*AB})(1 + b(s))^m$, for all $\xi \in \Sigma_s\backslash 0$. Now the conclusion follows from the proof of [S, Example 2.2.2]. ∎

A direct calculation yields the following lemma.

Lemma 6.2 *Let $F = \alpha + \beta$ be a Randers metric, where $\alpha(y) = \sqrt{a_{ij}y^i y^j}$ and $\beta(y) = b_i y^i$. Let $b_{i|j}$ denote the covariant derivative corresponding with α. Set*

$$r_{ij} = \frac{1}{2}(b_{i|j}+b_{j|i}),\ s_{ij} := \frac{1}{2}(b_{i|j}-b_{j|i}),\ s^i{}_j := a^{ik}s_{kj},\ s_j := b_i s^i{}_j,\ e_{ij} := r_{ij}+b_i s_j+b_j s_i.$$

Then we have

$$\mathbf{T}_y(v) = \left[-2\Big(\frac{e_{11}}{2F(v)} - s_1\Big) + 2\frac{s_{01}}{\alpha(y)} + \frac{1}{\alpha(y)}\Big(\frac{e_{00}}{2F(y)} - s_0\Big)\Big(\alpha(v) + \frac{\langle v, y\rangle}{\alpha(y)}\Big)\right]F(y) \cdot \Big(\alpha(v) - \frac{\langle v, y\rangle}{\alpha(y)}\Big),$$

where the index "0" *(resp.* "1"*) means the contraction with y^i (resp. v^i).*

Theorem 6.3 *Let (M, F) be a compact Randers manifold with $\mathbf{K} \geq \delta$, $\|\beta\|_\alpha \leq b$ and $\|\nabla^\alpha \beta\|_\alpha \leq b_1$. For each simple closed geodesic γ, we have*

$$L_F(\gamma) \geq \frac{(1-b)^{\frac{m+1}{2}}\mu_{BH}(M)}{c_{m-2}(1+b)^{\frac{m+3}{2}}\mathfrak{S}(b, b_1, \delta, d, m)},$$

where

$$\mathfrak{S}(b, b_1, \delta, d, m) = \frac{\mathfrak{s}_\delta^{m-1}\big(\min\{d, \frac{\pi}{2\sqrt{\delta}}\}\big)}{m-1} + \frac{b_1(7+13b+3b^2-13b^3+2b^4-4b^5)}{2(1-b)^5}\int_0^d \mathfrak{s}_\delta^{m-1}(t)dt.$$

Proof For each $n \in \mathcal{L}^{-1}(\mathcal{V}^* S\gamma)$, we have $\alpha(n)\beta(\dot\gamma) = -\langle\dot\gamma, n\rangle$, where $\langle\cdot,\cdot\rangle$ is the inner product induced by α. Hence,

$$\frac{(1-b)^2}{(1+b)} \leq g_n(\dot\gamma, \dot\gamma) = \frac{F(-\dot\gamma)}{\alpha(n)} \leq \frac{(1+b)^2}{(1-b)}.$$

And Lemma 6.2 yields

$$\mathbf{T}_n(\dot\gamma) \leq \frac{b_1(7+13b+3b^2-13b^3+2b^4-4b^5)}{2(1+b)(1-b)^3}.$$

By [BC], one can easily check that $e^{-\tau_{BH}(y)} \leq (1+b)^{(m+1)/2}$. Now the conclusion follows from Lemma 6.1, the proof of Theorem 1.2, and the inequalities above. ■

Remark 6.4 Note that $\mu_{HT}(M) = \mathrm{Vol}_\alpha(M)$ and $\mu_{BH}(M) \geq (1-b^2)^{\frac{m+1}{2}}\mathrm{Vol}_\alpha(M)$. By this, one can obtain a weak version of Theorem 6.3.

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