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COMPACT NON-ORIENTABLE SURFACES OF GENUS 5 WITH EXTREMAL METRIC DISCS

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Abstract. A compact hyperbolic surface of genus g is called an extremal surface if it admits an extremal disc, a disc of the largest radius determined by g. Our problem is to find how many extremal discs are embedded in non-orientable extremal surfaces. It is known that non-orientable extremal surfaces of genus $g > 6$ contain exactly one extremal disc and that of genus 3 or 4 contain at most two. In the present paper we shall give all the non-orientable extremal surfaces of genus 5, and find the locations of all extremal discs in those surfaces. As a consequence, non-orientable extremal surfaces of genus 5 contain at most two extremal discs.

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1. Introduction. A compact hyperbolic surface S of genus g has the unit disc $\mathbb{D}$ as its universal covering surface, where g denotes the number of handles ($g \geqq 2$) if S is orientable, or the number of cross caps ($g \geqq 3$) if S is non-orientable. The hyperbolic metric on S is the one induced by the hyperbolic metric on $\mathbb{D}$. Bavard showed in [**1**] that the radius r of a disc embedded in S satisfies the inequality

$$\cosh r \leqq \frac{1}{2\sin\frac{\pi}{6-6\chi_g}}, \tag{1}$$

where χ_g denotes the Euler characteristic, that is $\chi_g = 2 - 2g$ in the orientable case and $\chi_g = 2 - g$ in the non-orientable case. For each case we denote by R_g the radius satisfying equality in (1). A compact surface S of genus g is called an extremal surface if a disc of radius R_g, called an extremal disc, is isometrically embedded in S. A natural problem arising here is to find how many extremal discs are embedded in extremal surfaces. If the surfaces are orientable, then the problem is completely solved ([**2, 3, 6, 7**]). If the surfaces are non-orientable, previous research has revealed that extremal surfaces of genus $g > 6$ contain a unique extremal disc ([**4**]) and that those of genus 3 or 4 contain at most two ([**4, 8**]), where the surfaces containing two extremal discs are obtained. In the present paper we shall consider the case of non-orientable extremal surfaces of genus 5. Our problem is still open for the case of genus 6.

THEOREM 1.1. *There exist 3,627 non-orientable extremal surfaces of genus 5. They contain at most two extremal discs, and 17 of them contain exactly two extremal discs.*

Table 1 shows the 17 surfaces, the location of the centres of extremal discs and the group of automorphisms $\mathrm{Aut}^{\pm}$. To describe the centres we assume that the fundamental

Table 1. Extremal surfaces with two extremal discs

Surface	Centres of extremal discs	Aut$^{\pm}$
S_1	$\pi(0), \pi\left(\frac{1-\sqrt{2}+(\sqrt{2}-\sqrt{3}-2)i}{\sqrt{6(\sqrt{2}+\sqrt{6}-2)}}\right)$	$\mathbb{Z}_2$
S_2, S_3, S_4, S_5	$\pi(0), \pi\left(\frac{\sqrt{\sqrt{2}+\sqrt{6}-2}}{2}\, i\right)$	$\mathbb{Z}_2 \times \mathbb{Z}_2$
S_6, S_7, S_8, S_9	$\pi(0), \pi\left(\frac{\sqrt{2\sqrt{2}-\sqrt{3}-1}+i\sqrt{2\sqrt{2}+\sqrt{3}+1}}{2\sqrt{\sqrt{6}+\sqrt{3}+1}}\right)$	$\mathbb{Z}_2 \times \mathbb{Z}_2$
$S_{10}, S_{11}, S_{12}, S_{13}$	$\pi(0), \pi\left(\frac{\sqrt{\sqrt{2}+\sqrt{6}-2}(\sqrt{2\sqrt{2}-\sqrt{3}-1}+i\sqrt{2\sqrt{2}+\sqrt{3}+1})}{4\sqrt{(2+\sqrt{3})(\sqrt{2}-1)}}\right)$	$\mathbb{Z}_2 \times \mathbb{Z}_2$
$S_{14}, S_{15}, S_{16}, S_{17}$	$\pi(0), \pi\left(\frac{1+(\sqrt{2}-\sqrt{3})i}{\sqrt{\sqrt{2}+\sqrt{6}-2}}\right)$	$\mathbb{Z}_2 \times \mathbb{Z}_2$

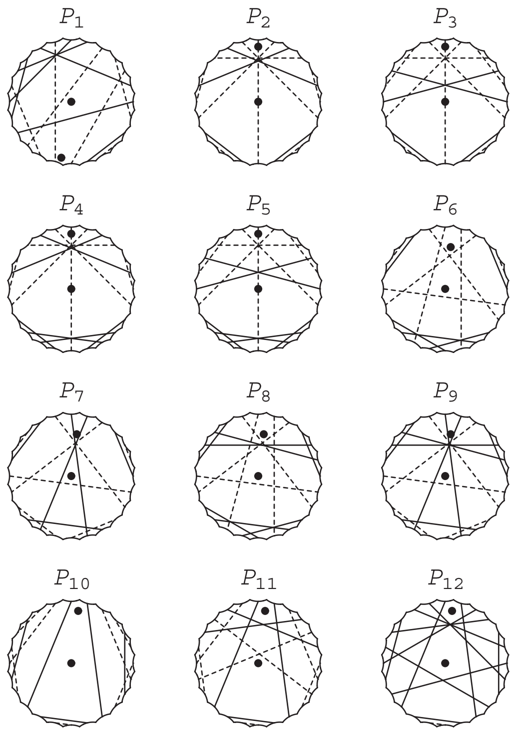

Figure 1. Side-pairing patterns and the centres of extremal discs (•).

region is centrally located in $\mathbb{D}$ such that the vertices $v_n, n = 1, 2, \ldots, 24$, satisfy $\arg v_n = (2n-1)\pi/24$, and π denotes the projection from $\mathbb{D}$ onto each surface S_j.

Surfaces S_j are derived from the hyperbolic polygons P_j in Figures 1. and 2., where lines and dotted lines indicate pairs of sides pasted by the different and the same direction, respectively; bullets correspond to the centres of extremal discs.

2. Side-pairing patterns of 24-gon. It is known that a non-orientable extremal surface S of genus $g \geqq 3$ has a regular $(6g-6)$-gon in $\mathbb{D}$ as its fundamental region ([**1**]). It is proved by facts that the hyperbolic area of S is $2\pi(g-2)$ and the density of a disc of radius r in a fundamental region (the Dirichlet–Volonoi cell determined by the points that project to the same centre of a disc of radius r in S) is bounded by the

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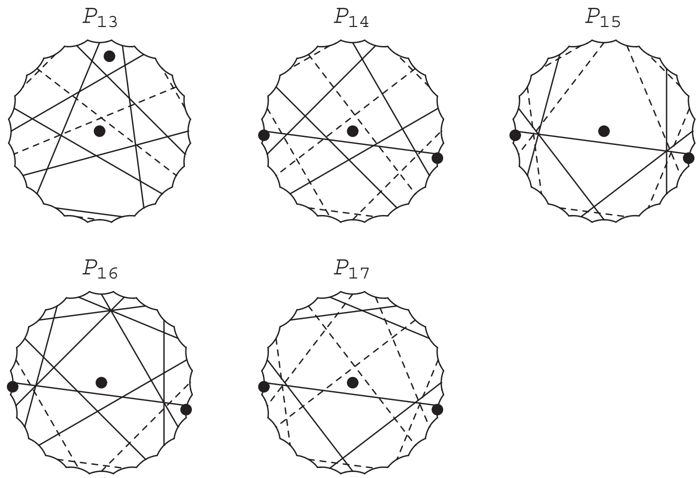

Figure 2. Side-pairing patterns and the centres of extremal discs (•).

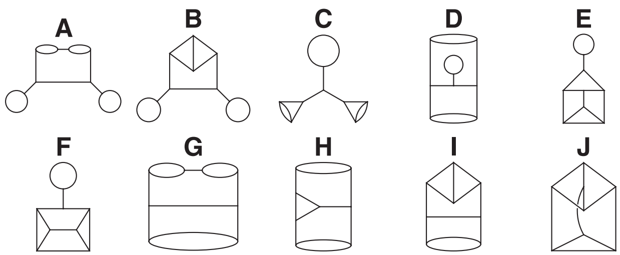

Figure 3. Trivalent graphs for 17 surfaces.

density of three discs of radius r mutually tangential to one another with respect to the triangle whose vertices are the centres of the three discs. We shall therefore consider a regular 24-gon for $g = 5$. The polygon has a generic property of having three edges in every vertex of the underling graph of the surface. Considering all the trivalent graphs with eight vertices and 12 edges, we see that there are 3,627 side-pairing patterns for the 24-gon to be a non-orientable surface of genus 5 [**9**].

Figure 3. shows 10 trivalent graphs obtained from 17 surfaces. Graph **A** is from P_2; **B** is from P_3, P_6 and P_7; **C** is from P_{10}; **D** is from P_{13}; **E** is from P_1; **F** is from P_{11} and P_{12}; **G** is from P_4; **H** is from P_{14} and P_{15}; **I** is from P_5, P_8 and P_9 and graph **J** is from P_{16} and P_{17} (cf. [**5**]).

3. Extremal discs for 3,627 surfaces. In order to describe the centres of extremal discs, we shall normalise the hyperbolic regular 24-gon P such that the centre is the origin and that the vertices v_n satisfy $\arg v_n = (2n-1)\beta$ $(n = 1, \ldots, 24)$, where $\beta = \pi/24$. We denote by C_n the sides between v_n and v_{n+1} and by w_n the middle point of C_n, where subscripts are regarded as modulo 24. The two hyperbolic distances $l = d(0, v_1)$ and $s = d(v_1, v_2)$ are calculated by $l = \sinh^{-1}\left(\frac{2}{\sqrt{3}} \sinh R\right) \approx 2.158$ and $s = 2\sinh^{-1}\left(\frac{2}{\sqrt{3}} \sin\beta \sinh R\right) \approx 1.064$, where $R = R_5$ denotes the maximal radius satisfying (1) in Section 1.

LEMMA 3.1. *Let S be a non-orientable extremal surface of genus 5 and $\pi : \mathbb{D} \to S$ the natural projection. If $p \in S$ is the centre of an extremal disc, then the list of hyperbolic distances between two points in the set $\pi^{-1}(p)$ consists of $d_1 = 2R \approx 4.037$, $d_2 \approx 5.380$, $d_3 \approx 6.157$ and so on.*

Proof. The elements of the list are calculated by considering the tessellation of $\mathbb{D}$ by hyperbolic regular 24-gons. For example, $d_2 = 2\sinh^{-1}(\sinh 2R \sin 2\beta)$ and $d_3 = 2\sinh^{-1}(\sinh 2R \sin 3\beta)$. □

Let $K_n \subset P$ $(n = 1, \dots, 24)$ be the pentagon with vertices at w_{n-1}, v_n, v_{n+1}, w_{n+1} and the origin.

LEMMA 3.2. *Let n be fixed. If $z \in K_n$ projects to the centre of an extremal disc, then $d(z, t_n(z)) = d_1$, where $t_n = t_{n,m}$ denotes an orientation preserving or reversing side-pairing mapping from C_n onto the other side C_m.*

Proof. Our proof is the same as the case of $g = 4$ (Lemma 3.2 in **[8]**) because it is independent of genus g. The two hyperbolic lengths l and s are used here. □

Suppose $z \in K_n$ projects to the centre of an extremal disc. The equation $d(z, t_{n,m}(z)) = 2R$ implies that z is on the curves $L_{n,m}$ or $M_{n,m}$ ($L'_{n,m}$ or $M'_{n,m}$, respectively) if $t_{n,m}$ is orientation preserving (or reversing) (see Lemma 5.5 in **[4]**):

$$L_n = L_{n,m} : \left| z - \frac{\tanh R\, e^{i(n+m)\beta}}{2\cos(n-m)\beta} \right| = \frac{\tanh R}{2|\cos(n-m)\beta|} \ (n - m \not\equiv 12 \pmod{24}),$$
$$M_n = M_{n,m} : z = \coth R\, e^{2in\beta} - te^{i(n+m+12)\beta} \ (t \in \mathbb{R}),$$
$$L'_n = L'_{n,m} : \left| z - \frac{\coth R\, e^{i(n+m)\beta}}{2\cos(n-m)\beta} \right| = \frac{\coth R}{2|\cos(n-m)\beta|} \ (n - m \not\equiv 12 \pmod{24}),$$
$$M'_n = M'_{n,m} : z = \tanh R\, e^{2in\beta} - te^{i(n+m+12)\beta} \ (t \in \mathbb{R}).$$

Figure 4. shows examples of these curves.

Though our process to find the centres of extremal discs is the same as that of the cases $g = 3$ and $g = 4$, we shall describe it for the sake of completeness.

(1) According to the side-pairing mappings $t_n = t_{n,m}$ of P_j, draw L_n, M_n (or L'_n, M'_n) on K_n for every $n = 1, \dots, 24$.
(2) Find intersections of these curves on $K_n \cap K_{n+1}$ for every n.
(3) Select every point ζ in the intersections such that the hyperbolic distance $d(\zeta, t_k(\zeta))$ is in the list of Lemma 3.1 for every side-pairing mapping t_k of P_j.

Applying this process to P_j $(j = 1, \dots, 3, 627)$ by a computer, we see that only 17 side-pairing patterns yield two points (the origin and $\zeta \neq 0$) and that the others yield a unique (the origin). For each of the 17 patterns, we can show that the two points are transitive by a certain isometry f of $\mathbb{D}$, which is compatible with the side-pairing mappings. Since the origin projects to the centre of an extremal disc, so does the other point.

EXAMPLE. We shall apply the process to P_1. Then we get two points $\zeta = (1 - \sqrt{2} + (\sqrt{2} - \sqrt{3} - 2)i)/\sqrt{6(\sqrt{2} + \sqrt{6} - 2)}$ and the origin. Let $\alpha_{n,m}$ (or $\gamma_{n,m}$) denote the orientation preserving (or reversing) side-pairing mapping from C_n onto C_m. Put

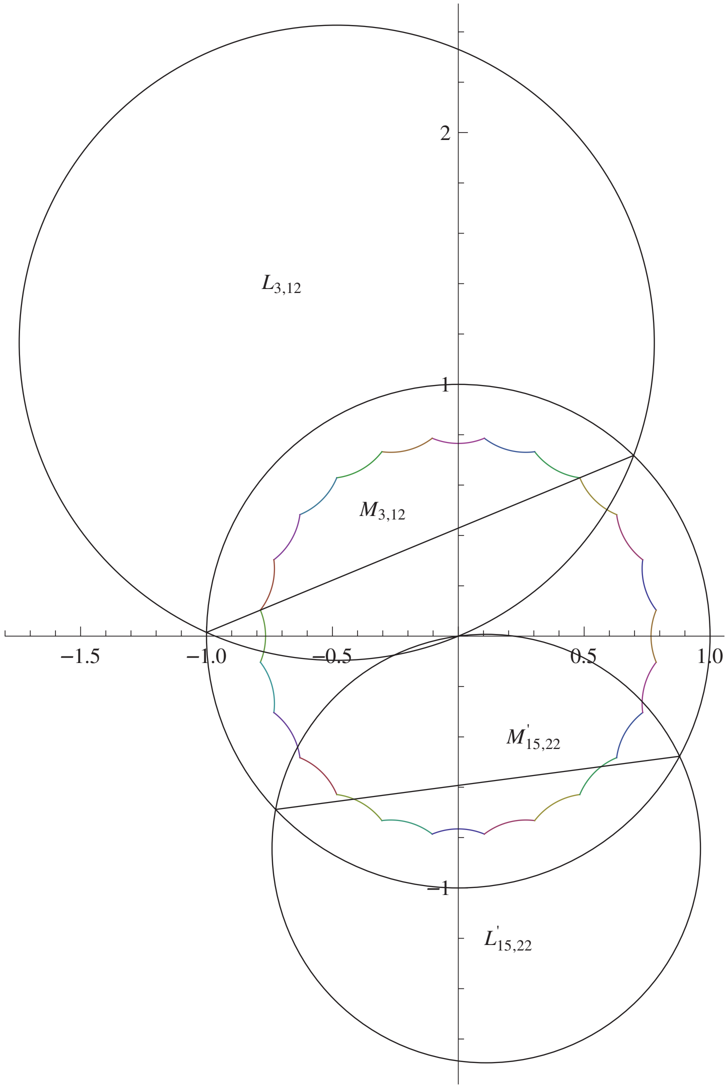

Figure 4. $L_{3,12}$, $M_{3,12}$, $L'_{15,22}$ and $M'_{15,22}$.

$f(z) := (|\zeta|^2 - \zeta\bar{z})/(\bar{\zeta} - |\zeta|^2\bar{z})$, then we can verify that f is compatible with the side-pairing mappings of P_1:

$$\begin{aligned}
&f\alpha_{1,8}f^{-1} = \gamma_{7,17}\gamma_{18,2}, && f\gamma_{2,18}f^{-1} = \gamma_{18,2},\\
&f\gamma_{3,23}f^{-1} = \alpha_{22,19}\gamma_{18,2}, && f\gamma_{4,15}f^{-1} = \alpha_{0,14}\gamma_{18,2},\\
&f\alpha_{5,10}f^{-1} = \gamma_{7,17}\alpha_{10,5}\gamma_{17,7}, && f\alpha_{6,12}f^{-1} = \gamma_{11,16}\gamma_{17,7},\\
&f\gamma_{7,17}f^{-1} = \gamma_{17,7}, && f\alpha_{9,13}f^{-1} = \gamma_{4,15}\gamma_{17,7},\\
&f\gamma_{11,16}f^{-1} = \gamma_{16,11}, && f\alpha_{14,24}f^{-1} = \alpha_{22,19}\alpha_{14,0},\\
&f\alpha_{19,22}f^{-1} = \alpha_{22,19}, && f\gamma_{20,21}f^{-1} = \gamma_{21,20}.
\end{aligned}$$

Consequently, the projection $\pi(\zeta)$ of ζ is the centre of an extremal disc.

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It is considered in [**9**] that the surfaces S_j derived from P_j ($j = 1, \dots, 3, 627$) are not isomorphic to each other. If there exists an isomorphism T between two extremal surfaces S_j and S_k ($j \neq k$), then it maps the centre of an extremal disc of S_j to that of S_k. Since we can assume that a lift $\tilde{T} : \mathbb{D} \to \mathbb{D}$ of T fixes the origin, $\tilde{T}$ must be a rotation around the origin or a reflection in a line passing through the origin. It is clear that such a mapping is incompatible with the side-pairing patterns P_j and P_k. Furthermore, it is not difficult to determine the full group of automorphisms by the fact that the centres of extremal discs are fixed or interchanged by automorphisms.

For example, $\mathrm{Aut}^{\pm}(S_1) = \mathbb{Z}_2 = \langle T \rangle$, where T denotes the automorphism induced by $z \mapsto f(z)$.

REMARK. For the other side-pairing patterns $P_2, \dots, P_{17}$, we shall give a required isometry f of $\mathbb{D}$ and the relations between f and the side-pairing mappings. In the following, ζ denotes the point representing the centre of an extremal disc that appeared in Table 1. In the case of P_j, $j = 14, 15, 16, 17$, we set

$$\zeta_1 = \frac{1 + (\sqrt{2} - \sqrt{3})i}{\sqrt{\sqrt{2} + \sqrt{6} - 2}} \quad \text{and} \quad \zeta_2 = \frac{-1 - 2\sqrt{2} + \sqrt{3} + (1 - 2\sqrt{2} + \sqrt{3})i}{2\sqrt{\sqrt{2} + \sqrt{6} - 2}},$$

which represent the same centre of an extremal disc and are located in the boundary of P_j in Figure 2.

P_2: $f(z) = (\zeta - z)/(1 - \bar{\zeta} z)$.

$$\begin{aligned}
&f\alpha_{1,8}f^{-1} = \alpha_{8,1}, && f\gamma_{2,24}f^{-1} = \alpha_{1,8}\,\gamma_{9,3},\\
&f\gamma_{3,9}f^{-1} = \gamma_{9,3}, && f\alpha_{4,11}f^{-1} = \alpha_{11,4},\\
&f\gamma_{5,13}f^{-1} = \gamma_{13,5}, && f\gamma_{6,18}f^{-1} = \gamma_{18,6},\\
&f\gamma_{7,23}f^{-1} = \gamma_{23,7}, && f\gamma_{10,12}f^{-1} = \alpha_{11,4}\,\gamma_{3,9},\\
&f\alpha_{14,17}f^{-1} = \gamma_{18,6}\,\alpha_{17,14}\,\gamma_{6,18}, && f\gamma_{15,16}f^{-1} = \gamma_{18,6}\,\gamma_{16,15}\,\gamma_{6,18},\\
&f\alpha_{19,22}f^{-1} = \gamma_{18,6}\,\alpha_{22,19}\,\gamma_{6,18}, && f\gamma_{20,21}f^{-1} = \gamma_{18,6}\,\gamma_{21,20}\,\gamma_{6,18}.
\end{aligned}$$

P_3: $f(z) = (\zeta - z)/(1 - \bar{\zeta} z)$.

$$\begin{aligned}
&f\gamma_{1,4}f^{-1} = \gamma_{8,11}, && f\alpha_{2,12}f^{-1} = \gamma_{1,4}\,\gamma_{9,3},\\
&f\gamma_{3,9}f^{-1} = \gamma_{9,3}, && f\gamma_{5,13}f^{-1} = \gamma_{13,5},\\
&f\gamma_{6,18}f^{-1} = \gamma_{18,6}, && f\gamma_{7,23}f^{-1} = \gamma_{23,7},\\
&f\gamma_{8,11}f^{-1} = \gamma_{1,4}, && f\alpha_{10,24}f^{-1} = \gamma_{11,8}\,\gamma_{3,9},\\
&f\alpha_{14,17}f^{-1} = \gamma_{18,6}\,\gamma_{5,13}, && f\gamma_{15,16}f^{-1} = \gamma_{18,6}\,\gamma_{15,16}\,\gamma_{5,13},\\
&f\alpha_{19,22}f^{-1} = \gamma_{23,7}\,\gamma_{6,18}, && f\gamma_{20,21}f^{-1} = \gamma_{23,7}\,\gamma_{20,21}\,\gamma_{6,18}.
\end{aligned}$$

P_4: $f(z) = (\zeta - z)/(1 - \bar{\zeta} z)$.

$$\begin{aligned}
&f\alpha_{1,8}f^{-1} = \alpha_{8,1}, && f\gamma_{2,24}f^{-1} = \gamma_{23,7}\,\alpha_{8,1},\\
&f\gamma_{3,9}f^{-1} = \gamma_{9,3}, && f\alpha_{4,11}f^{-1} = \alpha_{11,4},\\
&f\gamma_{5,13}f^{-1} = \gamma_{13,5}, && f\gamma_{6,18}f^{-1} = \gamma_{18,6},\\
&f\gamma_{7,23}f^{-1} = \gamma_{23,7}, && f\gamma_{10,12}f^{-1} = \gamma_{13,5}\,\alpha_{4,11},\\
&f\alpha_{14,17}f^{-1} = \gamma_{18,6}\,\gamma_{5,13}, && f\alpha_{15,20}f^{-1} = \gamma_{18,6}\,\alpha_{15,20}\,\gamma_{5,13},\\
&f\alpha_{16,21}f^{-1} = \gamma_{23,7}\,\alpha_{16,21}\,\gamma_{6,18}, && f\alpha_{19,22}f^{-1} = \gamma_{23,7}\,\gamma_{6,18}.
\end{aligned}$$

P_5: $f(z) = (\zeta - z)/(1 - \overline{\zeta} z)$.

$$\begin{aligned}
&f\gamma_{1,4}f^{-1} = \gamma_{8,11}, && f\alpha_{2,12}f^{-1} = \gamma_{1,4}\gamma_{9,3},\\
&f\gamma_{3,9}f^{-1} = \gamma_{9,3}, && f\gamma_{5,13}f^{-1} = \gamma_{13,5},\\
&f\gamma_{6,18}f^{-1} = \gamma_{18,6}, && f\gamma_{7,23}f^{-1} = \gamma_{23,7},\\
&f\gamma_{8,11}f^{-1} = \gamma_{1,4}, && f\alpha_{10,24}f^{-1} = \gamma_{11,8}\gamma_{3,9},\\
&f\alpha_{14,17}f^{-1} = \gamma_{18,6}\gamma_{5,13}, && f\alpha_{15,20}f^{-1} = \gamma_{18,6}\alpha_{15,20}\gamma_{5,13},\\
&f\alpha_{16,21}f^{-1} = \gamma_{23,7}\alpha_{16,21}\gamma_{6,18}, && f\alpha_{19,22}f^{-1} = \gamma_{23,7}\gamma_{6,18}.
\end{aligned}$$

P_6: $f(z) = (\zeta - z)/(1 - \overline{\zeta} z)$.

$$\begin{aligned}
&f\gamma_{1,2}f^{-1} = \gamma_{9,10}, && f\alpha_{3,24}f^{-1} = \alpha_{11,8},\\
&f\gamma_{4,13}f^{-1} = \gamma_{13,4}, && f\gamma_{5,19}f^{-1} = \gamma_{16,6},\\
&f\gamma_{6,16}f^{-1} = \gamma_{19,5}, && f\gamma_{7,22}f^{-1} = \gamma_{22,7},\\
&f\alpha_{8,11}f^{-1} = \alpha_{24,3}, && f\gamma_{9,10}f^{-1} = \gamma_{1,2},\\
&f\gamma_{12,23}f^{-1} = \gamma_{22,7}\alpha_{3,24}, && f\alpha_{14,18}f^{-1} = \gamma_{16,6}\gamma_{4,13},\\
&f\alpha_{15,20}f^{-1} = \gamma_{16,6}\gamma_{5,19}, && f\alpha_{17,21}f^{-1} = \gamma_{22,7}\gamma_{5,19}.
\end{aligned}$$

P_7: $f(z) = (|\zeta|^2 - \zeta\overline{z})/(\overline{\zeta} - |\zeta|^2\overline{z})$.

$$\begin{aligned}
&f\gamma_{1,2}f^{-1} = \gamma_{9,10}, && f\alpha_{3,24}f^{-1} = \alpha_{11,8},\\
&f\gamma_{4,13}f^{-1} = \gamma_{13,4}, && f\alpha_{5,16}f^{-1} = \alpha_{16,5},\\
&f\alpha_{6,19}f^{-1} = \alpha_{19,6}, && f\gamma_{7,22}f^{-1} = \gamma_{22,7},\\
&f\alpha_{8,11}f^{-1} = \alpha_{24,3}, && f\gamma_{9,10}f^{-1} = \gamma_{1,2},\\
&f\gamma_{12,23}f^{-1} = \gamma_{22,7}\alpha_{3,24}, && f\gamma_{14,17}f^{-1} = \alpha_{16,5}\gamma_{4,13},\\
&f\alpha_{15,20}f^{-1} = \gamma_{16,6}\gamma_{5,19}, && f\gamma_{18,21}f^{-1} = \gamma_{22,7}\alpha_{6,19}.
\end{aligned}$$

P_8: $f(z) = (\zeta - z)/(1 - \overline{\zeta} z)$.

$$\begin{aligned}
&f\alpha_{1,9}f^{-1} = \alpha_{9,1}, && f\alpha_{2,10}f^{-1} = \alpha_{10,2},\\
&f\alpha_{3,24}f^{-1} = \alpha_{11,8}, && f\gamma_{4,13}f^{-1} = \gamma_{13,4},\\
&f\gamma_{5,19}f^{-1} = \gamma_{16,6}, && f\gamma_{6,16}f^{-1} = \gamma_{19,5},\\
&f\gamma_{7,22}f^{-1} = \gamma_{22,7}, && f\alpha_{8,11}f^{-1} = \alpha_{24,3},\\
&f\gamma_{12,23}f^{-1} = \gamma_{22,7}\alpha_{3,24}, && f\alpha_{14,18}f^{-1} = \gamma_{16,6}\gamma_{4,13},\\
&f\alpha_{15,20}f^{-1} = \gamma_{16,6}\gamma_{5,19}, && f\alpha_{17,21}f^{-1} = \gamma_{22,7}\gamma_{5,19}.
\end{aligned}$$

P_9: $f(z) = (\zeta - z)/(1 - \overline{\zeta} z)$.

$$\begin{aligned}
&f\alpha_{1,9}f^{-1} = \alpha_{9,1}, && f\alpha_{2,10}f^{-1} = \alpha_{10,2},\\
&f\alpha_{3,24}f^{-1} = \alpha_{11,8}, && f\gamma_{4,13}f^{-1} = \gamma_{13,4},\\
&f\alpha_{5,16}f^{-1} = \alpha_{16,5}, && f\alpha_{6,19}f^{-1} = \alpha_{19,6},\\
&f\gamma_{7,22}f^{-1} = \gamma_{22,7}, && f\alpha_{8,11}f^{-1} = \alpha_{24,3},\\
&f\gamma_{12,23}f^{-1} = \gamma_{22,7}\alpha_{3,24}, && f\gamma_{14,17}f^{-1} = \alpha_{16,5}\gamma_{4,13},\\
&f\alpha_{15,20}f^{-1} = \alpha_{19,6}\alpha_{5,16}, && f\gamma_{18,21}f^{-1} = \gamma_{22,7}\alpha_{6,19}.
\end{aligned}$$

P_{10}: $f(z) = (\zeta - z)/(1 - \overline{\zeta} z)$.

$$\begin{aligned}
&f\gamma_{1,3}f^{-1} = \gamma_{8,10}, && f\alpha_{2,22}f^{-1} = \alpha_{15,6}\,\alpha_{9,13},\\
&f\gamma_{4,23}f^{-1} = \gamma_{12,7}, && f\alpha_{5,20}f^{-1} = \alpha_{15,6},\\
&f\alpha_{6,15}f^{-1} = \alpha_{20,5}, && f\gamma_{7,12}f^{-1} = \gamma_{23,4},\\
&f\gamma_{8,10}f^{-1} = \gamma_{1,3}, && f\alpha_{9,13}f^{-1} = \gamma_{23,4}\,\gamma_{3,1},\\
&f\gamma_{11,14}f^{-1} = \alpha_{20,5}\,\gamma_{4,23}, && f\alpha_{16,19}f^{-1} = \alpha_{15,6}\,\alpha_{5,20},\\
&f\gamma_{17,18}f^{-1} = \alpha_{15,6}\,\gamma_{18,17}\,\alpha_{5,20}, && f\gamma_{21,24}f^{-1} = \gamma_{12,7}\,\alpha_{6,15}.
\end{aligned}$$

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P_{11}: $f(z) = (\zeta - z)/(1 - \overline{\zeta} z)$.

$$f\,\alpha_{1,8}f^{-1} = \alpha_{8,1}, \qquad f\,\gamma_{2,13}f^{-1} = \gamma_{23,4}\,\alpha_{8,1},$$
$$f\,\alpha_{3,10}f^{-1} = \alpha_{10,3}, \qquad f\,\gamma_{4,23}f^{-1} = \gamma_{12,7},$$
$$f\,\alpha_{5,20}f^{-1} = \alpha_{15,6}, \qquad f\,\alpha_{6,15}f^{-1} = \alpha_{20,5},$$
$$f\,\gamma_{7,12}f^{-1} = \gamma_{23,4}, \qquad f\,\gamma_{9,22}f^{-1} = \alpha_{15,6}\,\gamma_{2,13},$$
$$f\,\gamma_{11,14}f^{-1} = \alpha_{20,5}\,\gamma_{4,23}, \qquad f\,\alpha_{16,19}f^{-1} = \alpha_{15,6}\,\alpha_{5,20},$$
$$f\,\gamma_{17,18}f^{-1} = \alpha_{15,6}\,\gamma_{18,17}\alpha_{5,20}, \qquad f\,\gamma_{21,24}f^{-1} = \gamma_{12,7}\,\alpha_{6,15}.$$

P_{12}: $f(z) = (\zeta - z)/(1 - \overline{\zeta} z)$.

$$f\,\alpha_{1,8}f^{-1} = \alpha_{8,1}, \qquad f\,\alpha_{2,22}f^{-1} = \alpha_{15,6}\,\alpha_{9,13},$$
$$f\,\alpha_{3,10}f^{-1} = \alpha_{10,3}, \qquad f\,\alpha_{4,12}f^{-1} = \alpha_{12,4},$$
$$f\,\alpha_{5,20}f^{-1} = \alpha_{15,6}, \qquad f\,\alpha_{6,15}f^{-1} = \alpha_{20,5},$$
$$f\,\alpha_{7,23}f^{-1} = \alpha_{23,7}, \qquad f\,\alpha_{9,13}f^{-1} = \alpha_{12,4}\,\alpha_{3,10},$$
$$f\,\alpha_{11,21}f^{-1} = \alpha_{15,6}\,\alpha_{4,12}, \qquad f\,\alpha_{14,24}f^{-1} = \alpha_{23,7}\,\alpha_{5,20},$$
$$f\,\alpha_{16,19}f^{-1} = \alpha_{15,6}\,\alpha_{5,20}, \qquad f\,\gamma_{17,18}f^{-1} = \alpha_{15,6}\,\gamma_{18,17}\,\alpha_{5,20}.$$

P_{13}: $f(z) = (\zeta - z)/(1 - \overline{\zeta} z)$.

$$f\,\gamma_{1,3}f^{-1} = \gamma_{8,10}, \qquad f\,\gamma_{2,13}f^{-1} = \alpha_{12,4}\,\gamma_{8,10},$$
$$f\,\alpha_{4,12}f^{-1} = \alpha_{12,4}, \qquad f\,\alpha_{5,20}f^{-1} = \alpha_{15,6},$$
$$f\,\alpha_{6,15}f^{-1} = \alpha_{20,5}, \qquad f\,\alpha_{7,23}f^{-1} = \alpha_{23,7},$$
$$f\,\gamma_{8,10}f^{-1} = \gamma_{1,3}, \qquad f\,\gamma_{9,22}f^{-1} = \alpha_{23,7}\,\gamma_{3,1},$$
$$f\,\alpha_{11,21}f^{-1} = \alpha_{15,6}\,\alpha_{4,12}, \qquad f\,\alpha_{14,24}f^{-1} = \alpha_{23,7}\,\alpha_{5,20},$$
$$f\,\alpha_{16,19}f^{-1} = \alpha_{15,6}\,\alpha_{5,20}, \qquad f\,\gamma_{17,18}f^{-1} = \alpha_{15,6}\,\gamma_{18,17}\,\alpha_{5,20}.$$

P_{14}: $f(z) = (\zeta_1\overline{z} - \overline{\zeta_2}\zeta_1)/(|\zeta_1|^2\overline{z} - \overline{\zeta_2})$.

$$f\,\alpha_{1,15}f^{-1} = \alpha_{15,1}\,\alpha_{23,12}, \qquad f\,\gamma_{2,4}f^{-1} = \alpha_{6,22}\,\gamma_{4,2}\,\alpha_{22,6},$$
$$f\,\gamma_{3,14}f^{-1} = \gamma_{4,2}\,\alpha_{22,6}, \qquad f\,\alpha_{5,13}f^{-1} = \alpha_{22,6},$$
$$f\,\alpha_{6,22}f^{-1} = \alpha_{12,23}\,\alpha_{22,6}, \qquad f\,\gamma_{7,9}f^{-1} = \gamma_{8,21}\,\alpha_{22,6},$$
$$f\,\gamma_{8,21}f^{-1} = \alpha_{12,23}\,\gamma_{21,8}, \qquad f\,\alpha_{10,20}f^{-1} = \alpha_{12,23}\,\alpha_{20,10},$$
$$f\,\gamma_{11,17}f^{-1} = \gamma_{18,24}, \qquad f\,\alpha_{12,23}f^{-1} = \alpha_{12,23},$$
$$f\,\gamma_{16,19}f^{-1} = \gamma_{18,24}\,\alpha_{1,15}, \qquad f\,\gamma_{18,24}f^{-1} = \alpha_{12,23}\,\gamma_{24,18}.$$

P_{15}: $f(z) = (\zeta_1\overline{z} - \overline{\zeta_2}\zeta_1)/(|\zeta_1|^2\overline{z} - \overline{\zeta_2})$.

$$f\,\gamma_{1,20}f^{-1} = \alpha_{17,24}\,\gamma_{16,19}\,\alpha_{23,12}, \qquad f\,\gamma_{2,4}f^{-1} = \gamma_{5,22}\,\alpha_{14,8}\,\alpha_{23,12},$$
$$f\,\alpha_{3,21}f^{-1} = \alpha_{12,23}\,\alpha_{14,8}\,\alpha_{23,12}, \qquad f\,\gamma_{5,22}f^{-1} = \alpha_{12,23}\,\gamma_{22,5},$$
$$f\,\gamma_{6,13}f^{-1} = \gamma_{22,5}, \qquad f\,\gamma_{7,9}f^{-1} = \alpha_{3,21}\,\gamma_{22,5},$$
$$f\,\alpha_{8,14}f^{-1} = \alpha_{21,3}, \qquad f\,\gamma_{10,15}f^{-1} = \gamma_{20,1},$$
$$f\,\alpha_{11,18}f^{-1} = \alpha_{17,24}, \qquad f\,\alpha_{12,23}f^{-1} = \alpha_{12,23},$$
$$f\,\gamma_{16,19}f^{-1} = \alpha_{17,24}\,\gamma_{1,20}, \qquad f\,\alpha_{17,24}f^{-1} = \alpha_{12,23}\,\alpha_{24,17}.$$

P_{16}: $f(z) = (\zeta_1\overline{z} - \overline{\zeta_2}\zeta_1)/(|\zeta_1|^2\overline{z} - \overline{\zeta_2})$.

$$f\,\alpha_{1,15}f^{-1} = \alpha_{15,1}\,\alpha_{23,12}, \qquad f\,\alpha_{2,7}f^{-1} = \alpha_{6,22}\,\alpha_{14,8}\,\alpha_{23,12},$$
$$f\,\alpha_{3,21}f^{-1} = \alpha_{12,23}\,\alpha_{14,8}\,\alpha_{23,12}, \qquad f\,\alpha_{4,9}f^{-1} = \alpha_{3,21}\,\alpha_{22,6},$$
$$f\,\alpha_{5,13}f^{-1} = \alpha_{22,6}, \qquad f\,\alpha_{6,22}f^{-1} = \alpha_{12,23}\,\alpha_{22,6},$$
$$f\,\alpha_{8,14}f^{-1} = \alpha_{21,3}, \qquad f\,\alpha_{10,20}f^{-1} = \gamma_{18,24}\,\gamma_{19,16},$$
$$f\,\gamma_{11,17}f^{-1} = \gamma_{18,24}, \qquad f\,\alpha_{12,23}f^{-1} = \alpha_{12,23},$$
$$f\,\gamma_{16,19}f^{-1} = \alpha_{12,23}\,\alpha_{20,10}\,\gamma_{24,18}, \qquad f\,\gamma_{18,24}f^{-1} = \alpha_{12,23}\,\gamma_{24,18}.$$

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P_{17}: $f(z) = (\zeta_1\overline{z} - \overline{\zeta_2}\zeta_1)/(|\zeta_1|^2\overline{z} - \overline{\zeta_2})$.

$$\begin{array}{ll}
f\,\gamma_{1,20}f^{-1} = \alpha_{17,24}\,\gamma_{16,19}\,\alpha_{23,12}, & f\,\alpha_{2,7}f^{-1} = \gamma_{5,22}\,\gamma_{14,3}\,\alpha_{23,12}, \\
f\,\gamma_{3,14}f^{-1} = \gamma_{14,3}\,\alpha_{23,12}, & f\,\alpha_{4,9}f^{-1} = \gamma_{8,21}\,\gamma_{22,5}, \\
f\,\gamma_{5,22}f^{-1} = \alpha_{12,23}\,\gamma_{22,5}, & f\,\gamma_{6,13}f^{-1} = \gamma_{22,5}, \\
f\,\gamma_{8,21}f^{-1} = \alpha_{12,23}\,\gamma_{21,8}, & f\,\gamma_{10,15}f^{-1} = \gamma_{20,1}, \\
f\,\alpha_{11,18}f^{-1} = \alpha_{17,24}, & f\,\alpha_{12,23}f^{-1} = \alpha_{12,23}, \\
f\,\gamma_{16,19}f^{-1} = \alpha_{17,24}\,\gamma_{1,20}, & f\,\alpha_{17,24}f^{-1} = \alpha_{12,23}\,\alpha_{24,17}.
\end{array}$$

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