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AN ADDITION THEOREM AND SOME PRODUCT FORMULAS FOR THE HAHN-EXTON q-BESSEL FUNCTIONS

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ABSTRACT. In this paper a q-analogue of Gegenbauer's addition formula for Bessel functions is obtained by using the orthogonality relation for the q-Ultraspherical polynomials of Rogers'. Also some product formulas and an integral representation for the Hahn-Exton q-Bessel functions are obtained.

1. **Introduction.** Several possible q-analogues of the Bessel function

$$J_\nu(x) = \sum_{k=0}^{\infty} \frac{(-1)^k (x/2)^{\nu+2k}}{k!\,\Gamma(\nu+k+1)} \tag{1.1}$$

have been considered in the literature. Best known are two related q-Bessel functions denoted $J_\nu^{(1)}(x;q)$ and $J_\nu^{(2)}(x;q)$ by Ismail [6], but first introduced by Jackson in a series of papers during the years 1903–1905 (see references in [6]) and also studied by Hahn [4]. A third q-Bessel function was introduced by Hahn [5] (in a special case) and by Exton [2] (in full). They obtained these functions as the solutions of a special basic Sturm-Liouville equation. Later Koornwinder and Swarttouw [7] also studied this third q-Bessel function. In the notation of Koornwinder and Swarttouw it is defined as:

$$J_\nu(x;q) = \frac{x^\nu (q^{\nu+1};q)_\infty}{(q;q)_\infty} \sum_{k=0}^{\infty} \frac{(-1)^k q^{k(k+1)/2} x^{2k}}{(q^{\nu+1};q)_k (q;q)_k}, \tag{1.2}$$

where

$$\begin{cases} (a;q)_0 = 1, \\ (a;q)_k = \prod_{j=0}^{k-1}(1-aq^j), \quad k = 1,2,\ldots \\ (a;q)_\infty = \prod_{j=0}^{\infty}(1-aq^j), \end{cases} \tag{1.3}$$

when the product converges.

The main objective in this paper is the derivation of the addition formula

$$\begin{aligned} &\sum_{n=0}^{\infty} \frac{(a^{-1}bq^{(1-n)/2}e^{i\varphi}, a^{-1}bq^{(1-n)/2}e^{-i\varphi}; q)_n (-a^2x^2)^n q^{n(n+1)/2}}{(q^{\nu+1}, q; q)_n} \\ &\quad = \frac{(q;q)_\infty^2}{(q^{\nu+1};q)_\infty^2} (abx^2)^{-\nu} \sum_{k=0}^{\infty} \frac{1-q^{\nu+k}}{1-q^\nu} \cdot q^{k(1-\nu)/2} \\ &\qquad \times J_{\nu+k}(axq^{k/4};q) J_{\nu+k}(bxq^{k/4};q) C_k(\cos\varphi; q^\nu|q), \end{aligned} \tag{1.4}$$

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where $a, b > 0$, $x \geq 0$, $\mathrm{Re}(\nu) > 0$, $0 \leq \varphi \leq \pi$ and $C_k(z;\beta|q)$ is Rogers' q-Ultraspherical polynomial defined by

$$C_n(z;\beta|q) = \sum_{k=0}^{n} \frac{(\beta;q)_k(\beta;q)_{n-k}}{(q;q)_k(q;q)_{n-k}} \cdot \cos(n-2k)\varphi, \tag{1.5}$$

where $z = \cos\varphi$, see [1].

Rahman [8] has derived a similar addition formula for the Jackson q-Bessel functions $J_\nu^{(1)}(x;q)$ and $J_\nu^{(2)}(x;q)$. Formula (1.4) is a q-analogue of formula [9, (2), p. 363]:

$$\omega^{-\nu}J_\nu(\omega) = 2^\nu\Gamma(\nu)\sum_{k=0}^{\infty}(\nu+k)Z^{-\nu}J_{\nu+k}(Z)z^{-\nu}J_{\nu+k}(z)C_k^\nu(\cos\varphi),$$

where $\omega = \sqrt{(Z^2+z^2-2zZ\cos\varphi)}$ and $C_k^\nu(\cos\varphi)$ are Gegenbauer's Ultraspherical polynomials.

In order to derive formula (1.4) we shall first derive a product formula in Section 3. In the fourth section we will give an integral representation for this product. The results from Sections 3 and 4 can be used to derive the addition formula (1.4), which we shall prove in Section 5.

2. **Some formulas and notations.** We will always assume that $0 < q < 1$. The general q-hypergeometric series is defined by

$$ {}_r\Phi_s\left[\begin{matrix} a_1,\ldots,a_r \\ b_1,\ldots,b_s \end{matrix}\middle| q, z\right] = \sum_{k=0}^{\infty} \frac{(a_1,\ldots,a_r;q)_k}{(b_1,\ldots,b_s;q)_k} \cdot \frac{\left((-1)^k q^{k(k-1)/2}\right)^{1+s-r} z^k}{(q;q)_k}, \tag{2.1}$$

where the q-shifted factorial is defined by (1.3) and

$$(a_1,\ldots,a_r;q)_k = (a_1;q)_k(a_2;q)_k\cdots(a_r;q)_k.$$

The power series, in the non-terminating case of (2.1), has radius of convergence ∞, 1 or 0 according to whether $r-s < 1$, $= 1$ or > 1, respectively. The series is called *balanced* or *Saalschützian* if $r = s+1$, $z = q$ and $b_1b_2\cdots b_s = qa_1a_2\cdots a_{s+1}$. The series is called *well-poised* if $r = s+1$ and its parameters satisfy the relations $qa_1 = a_2b_1 = a_3b_2 = \cdots = a_{s+1}b_s$.

In the next sections we shall need some summation and transformation formulas. We shall list them here, but the reader is referred to [3] for further details.

The q-Gamma function is defined by [3, p. 16]:

$$\Gamma_q(x) = \frac{(q;q)_\infty}{(q^x;q)_\infty} \cdot (1-q)^{1-x} \tag{2.2}$$

When $q \uparrow 1$ the q-Gamma function tends to the ordinary Gamma function.

Sears' transformation formula for a terminating balanced ${}_4\Phi_3$ [3, p. 41]:

$$ {}_4\Phi_3\left[\begin{matrix} q^{-n},a,b,c \\ d,e,f \end{matrix}\middle| q,q\right] = \frac{(e/a,f/a;q)_n}{(e,f;q)_n} \cdot a^n {}_4\Phi_3\left[\begin{matrix} q^{-n},a,d/b,d/c \\ d,aq^{1-n}/e,aq^{1-n}/f \end{matrix}\middle| q,q\right], \tag{2.3}$$

with $def = abcq^{1-n}$.

A transformation formula for a terminating well-poised ${}_2\Phi_1$ [1, p. 63]:

$$(2.4)\quad {}_2\Phi_1\left[\begin{matrix} q^{-n},\beta \\ q^{1-n}/\beta \end{matrix}\Big| q, qz^2/\beta\right] = \frac{(\beta^2;q)_n}{(\beta;q)_n}\cdot z^n\beta^{-n/2}{}_4\Phi_3\left[\begin{matrix} q^{-n},\beta^2q^n,\beta^{1/2}z,\beta^{1/2}/z \\ \beta q^{1/2},-\beta q^{1/2},-\beta \end{matrix}\Big| q,q\right].$$

The q-Saalschütz summation formula for a terminating balanced ${}_3\Phi_2$ [3, p. 13]:

$$(2.5).\qquad {}_3\Phi_2\left[\begin{matrix} q^{-n},a,b \\ c,d \end{matrix}\Big| q,q\right] = \frac{(c/a,c/b;q)_n}{(c,c/ab;q)_n},\ \text{with } cd = abq^{1-n}$$

The Askey-Wilson integral [3, p. 140]:

$$(2.6)\qquad \int_0^\pi \frac{h(\cos\varphi;1,-1,q^{1/2},-q^{1/2}|q)}{h(\cos\varphi;a,b,c,d|q)}\,d\varphi = \frac{2\pi(abcd;q)_\infty}{(q,ab,ac,ad,bc,bd,cd;q)_\infty},$$

where $\max(|q|,|a|,|b|,|c|,|d|) < 1$.

We have written $h(z;a,b,c,d|q) = h(z;a|q)\cdots h(z;d|q)$, where $z = \cos\varphi$ and

$$(2.7)\qquad h(z;a|q) = (ae^{i\varphi},ae^{-i\varphi};q)_\infty.$$

We will also use some easily verified identities:

$$(2.8)\qquad (a,-a,aq^{1/2},-aq^{1/2};q)_n = (a^2;q)_{2n},\quad n = 0,1,2,\ldots \text{ or } \infty$$

$$(2.9)\qquad (aq^{-n};q)_n = (-a)^nq^{-n(n+1)/2}(q/a;q)_n$$

$$(2.10)\qquad (a;q)_{n-k} = (-1)^ka^{-k}q^{k(k+1)/2}q^{-nk}\frac{(a;q)_n}{(a^{-1}q^{1-n};q)_k}$$

$$(2.11)\qquad (a;q)_{n+k} = (a;q)_k(aq^k;q)_n = (a;q)_n(aq^n;q)_k$$

$$(2.12)\qquad (aq^n;q)_\infty = \frac{(a;q)_\infty}{(a;q)_n}$$

3. **A product formula.** In this section we will derive a product formula for the Hahn-Exton q-Bessel functions. Let us always assume that a,b,x are real and that $a>0$, $b>0$ and $x\geq 0$. Starting with (1.2) we have for $\mathrm{Re}(\mu,\nu) > -1$:

$$J_\nu(ax;q)J_\mu(bx;q)$$
$$= \frac{(ax)^\nu(q^{\nu+1};q)_\infty}{(q;q)_\infty}\sum_{k=0}^{\infty}\frac{(-a^2x^2)^k q^{k(k+1)/2}}{(q^{\nu+1};q)_k(q;q)_k}\cdot\frac{(bx)^\mu(q^{\mu+1};q)_\infty}{(q;q)_\infty}\sum_{m=0}^{\infty}\frac{(-b^2x^2)^m q^{m(m+1)/2}}{(q^{\mu+1};q)_m(q;q)_m}$$
$$= K_{\nu,\mu}(x)\sum_{k=0}^{\infty}\sum_{n=k}^{\infty}\frac{(-1)^n a^{2k}b^{2(n-k)}x^{2n}q^{[k(k+1)+(n-k)(n-k+1)]/2}}{(q^{\nu+1};q)_k(q;q)_k(q^{\mu+1};q)_{n-k}(q;q)_{n-k}}$$
$$\overset{(2.10)}{=} K_{\nu,\mu}(x)\sum_{n=0}^{\infty}\frac{(-1)^n(bx)^{2n}q^{n(n+1)/2}}{(q^{\mu+1};q)_n(q;q)_n}\sum_{k=0}^{n}\frac{(q^{-n};q)_k(q^{-n-\mu};q)_k}{(q^{\nu+1};q)_k(q;q)_k}\left(\frac{a^2}{b^2}q^{\mu+n+1}\right)^k.$$

With (2.1) the general product formula for the Hahn-Exton q-Bessel functions becomes:

$$J_\nu(ax;q)J_\mu(bx;q)$$
(3.1)
$$= K_{\nu,\mu}(x)\sum_{n=0}^{\infty}\frac{(-1)^n(bx)^{2n}q^{n(n+1)/2}}{(q^{\mu+1};q)_n(q;q)_n}\,{}_2\Phi_1\left[\begin{matrix}q^{-n},q^{-n-\mu}\\ q^{\nu+1}\end{matrix}\middle| q,\frac{a^2}{b^2}q^{\mu+n+1}\right],$$

where

$$K_{\nu,\mu}(x) = \frac{a^\nu b^\mu x^{\nu+\mu}(q^{\nu+1};q)_\infty(q^{\mu+1};q)_\infty}{(q;q)_\infty(q;q)_\infty}.$$

If we replace in (1.2) x by $x(1-q)/2$, take the limit $q\uparrow 1$ and use (2.2), we obtain the ordinary Bessel function:

(3.2)
$$\lim_{q\uparrow 1} J_\nu\big(x(1-q)/2;q\big) = \sum_{k=0}^{\infty}\frac{(-1)^k(x/2)^{\nu+2k}}{k!\,\Gamma(\nu+k+1)} = J_\nu(x).$$

If we replace in (3.1) a and b by $a(1-q)/2$ and $b(1-q)/2$ respectively, take the limit $q\uparrow 1$ and use (3.2), we find the formula [9, (2), p. 148]:

$$J_\nu(ax)J_\mu(bx) = \frac{(ax/2)^\nu(bx/2)^\mu}{\Gamma(\nu+1)}\sum_{n=0}^{\infty}\frac{(-1)^n(bx/2)^{2n}\,{}_2F_1\left[\begin{matrix}-n,-n-\mu\\ \nu+1\end{matrix}\middle| a^2b^{-2}\right]}{\Gamma(n+\mu+1)\,n!},$$

where the hypergeometric series ${}_2F_1$ is defined by

$${}_2F_1\left[\begin{matrix}a,b\\ c\end{matrix}\middle| z\right] = \sum_{k=0}^{\infty}\frac{(a)_k(b)_k z^k}{(c)_k\,k!},$$

with $(a)_k = a(a+1)\cdots(a+k-1)$ and $(a)_0 = 1$.

Let us choose $\mu=\nu$. Now the ${}_2\Phi_1$ in (3.1) becomes well-poised and can thus be transformed into a balanced ${}_4\Phi_3$ by formula (2.4). After that we apply (2.3) twice: in the first time a and d in (2.3) are chosen as $ab^{-1}q^{-(n+\nu)/2}$ and $-q^{-n-\nu}$ respectively; in the second time a and d are chosen as $-q^{\nu}$ and $ab^{-1}q^{(1-n+\nu)/2}$ respectively. This yields

$$\begin{aligned}
&J_\nu(ax;q)J_\nu(bx;q)\\
&=K_{\nu,\nu}(x)\sum_{n=0}^{\infty}\frac{(-1)^n(bx)^{2n}q^{n(n+1)/2}}{(q^{\nu+1};q)_n(q;q)_n}\,{}_2\Phi_1\left[\begin{matrix}q^{-n},q^{-n-\nu}\\ q^{\nu+1}\end{matrix}\,\middle|\,q,\frac{a^2}{b^2}q^{\nu+n+1}\right]\\
&=K_{\nu,\nu}(x)\sum_{n=0}^{\infty}\frac{(-1)^n(abx^2)^nq^{n(2n+\nu+1)/2}(q^{-2n-2\nu};q)_n}{(q^{\nu+1};q)_n(q;q)_n(q^{-n-\nu};q)_n}\\
&\quad\times{}_4\Phi_3\left[\begin{matrix}q^{-n},q^{-n-2\nu},ab^{-1}q^{-(n+\nu)/2},a^{-1}bq^{-(n+\nu)/2}\\ q^{-n-\nu+1/2},-q^{-n-\nu+1/2},-q^{-n-\nu}\end{matrix}\,\middle|\,q,q\right]\\
&=K_{\nu,\nu}(x)\sum_{n=0}^{\infty}\frac{(-abx^2)^nq^{n(2n+\nu+1)/2}(q^{-2n-2\nu},a^{-1}bq^{(1-n-\nu)/2},-a^{-1}bq^{(1-n-\nu)/2};q)_n}{(q^{\nu+1},q^{-n-\nu},q^{-n-\nu+1/2},-q^{-n-\nu+1/2},q;q)_n}\\
&\quad\times(ab^{-1}q^{-(n+\nu)/2})^n\,{}_4\Phi_3\left[\begin{matrix}q^{-n},-q^{\nu},ab^{-1}q^{-(n+\nu)/2},-ab^{-1}q^{-(n+\nu)/2}\\ ab^{-1}q^{(1-n+\nu)/2},-ab^{-1}q^{(1-n+\nu)/2},-q^{-n-\nu}\end{matrix}\,\middle|\,q,q\right]\\
&=K_{\nu,\nu}(x)\sum_{n=0}^{\infty}\frac{(-1)^n(ax)^{2n}q^{n(n+1)/2}(q^{-2n-2\nu},a^{-1}bq^{(1-n-\nu)/2},-a^{-1}bq^{(1-n-\nu)/2};q)_n}{(q^{\nu+1},q^{-n-\nu},q^{-n-\nu+1/2},-q^{-n-\nu+1/2},q;q)_n}\\
&\quad\times\frac{(-q^{\nu})^n(q^{-n-2\nu},ab^{-1}q^{(1-n-\nu)/2};q)_n}{(-q^{-n-\nu},-ab^{-1}q^{(1-n+\nu)/2};q)_n}\\
&\quad\times{}_4\Phi_3\left[\begin{matrix}q^{-n},-q^{\nu},q^{\nu+1/2},-q^{\nu+1/2}\\ ab^{-1}q^{(1-n+\nu)/2},a^{-1}bq^{(1-n+\nu)/2},q^{2\nu+1}\end{matrix}\,\middle|\,q,q\right].
\end{aligned}$$

When we apply (2.8) and (2.9) we have found for $\mathrm{Re}(\nu)>-1$ a product formula for the Hahn-Exton q-Bessel function in terms of balanced ${}_4\Phi_3$'s:

$$\begin{aligned}
&J_\nu(ax;q)J_\nu(bx;q)\\
&\qquad=\frac{(abx^2)^{\nu}(q^{\nu+1};q)_\infty(q^{\nu+1};q)_\infty}{(q;q)_\infty(q;q)_\infty}\\
&\qquad\quad\times\sum_{n=0}^{\infty}\frac{(-1)^n(ax)^{2n}q^{n(n+1)/2}(a^{-1}bq^{(1-n-\nu)/2},a^{-1}bq^{(1-n+\nu)/2};q)_n}{(q,q^{\nu+1};q)_n}\\
&\qquad\quad\times{}_4\Phi_3\left[\begin{matrix}q^{-n},-q^{\nu},q^{\nu+1/2},-q^{\nu+1/2}\\ ab^{-1}q^{(1-n+\nu)/2},a^{-1}bq^{(1-n+\nu)/2},q^{2\nu+1}\end{matrix}\,\middle|\,q,q\right].
\end{aligned}\tag{3.3}$$

4. **An integral representation.** With the aid of the Askey-Wilson integral (2.6) we can find an integral representation for a product of two q-Bessel functions. Since by (2.5)

$${}_3\Phi_2\left[\begin{matrix}q^{-n},q^{\nu/2}e^{i\varphi},q^{\nu/2}e^{-i\varphi}\\ a^{-1}bq^{(1+\nu-n)/2},ab^{-1}q^{(1+\nu-n)/2}\end{matrix}\,\middle|\,q,q\right]=\frac{(a^{-1}bq^{(1-n)/2}e^{i\varphi},a^{-1}bq^{(1-n)/2}e^{-i\varphi};q)_n}{(a^{-1}bq^{(1-n+\nu)/2},a^{-1}bq^{(1-n-\nu)/2};q)_n},$$

and since by (2.7) and (2.12)

$$\frac{(q^{\nu/2}e^{i\varphi},q^{\nu/2}e^{-i\varphi};q)_k}{h(z;q^{\nu/2}|q)}=\frac{1}{h(z;q^{\nu/2+k}|q)},\tag{4.1}$$

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we have for $\mathrm{Re}(\nu) > 0$

$$\begin{aligned}
&\int_0^\pi \frac{h(\cos\varphi;1,-1,q^{1/2},-q^{1/2}|q)(a^{-1}bq^{(1-n)/2}e^{i\varphi},a^{-1}bq^{(1-n)/2}e^{-i\varphi};q)_n}{h(\cos\varphi;q^{\nu/2},-q^{\nu/2},q^{(\nu+1)/2},-q^{(\nu+1)/2}|q)}\,d\varphi \\
&= \sum_{k=0}^{n} \frac{(q^{-n};q)_k(a^{-1}bq^{(1-n+\nu)/2},a^{-1}bq^{(1-n-\nu)/2};q)_n q^k}{(a^{-1}bq^{(1+\nu-n)/2},ab^{-1}q^{(1+\nu-n)/2},q;q)_k} \\
&\quad\times \int_0^\pi \frac{h(\cos\varphi;1,-1,q^{1/2},-q^{1/2}|q)\,d\varphi}{h(\cos\varphi;q^{\nu/2+k},-q^{\nu/2},q^{(\nu+1)/2},-q^{(\nu+1)/2}|q)} \\
&= \sum_{k=0}^{n} \frac{(q^{-n};q)_k(a^{-1}bq^{(1-n+\nu)/2},a^{-1}bq^{(1-n-\nu)/2};q)_n q^k}{(a^{-1}bq^{(1+\nu-n)/2},ab^{-1}q^{(1+\nu-n)/2},q;q)_k} \\
&\quad\times \frac{2\pi(q^{k+2\nu+1};q)_\infty}{(q,-q^{\nu+k},q^{\nu+k+1/2},-q^{\nu+k+1/2},q^{\nu+1/2},-q^{\nu+1/2},-q^{\nu+1};q)_\infty} \\
&\overset{(2.12)}{=} \frac{2\pi(q^{2\nu+1};q)_\infty(a^{-1}bq^{(1-n+\nu)/2},a^{-1}bq^{(1-n-\nu)/2};q)_n}{(q,-q^{\nu},q^{\nu+1/2},-q^{\nu+1/2},q^{\nu+1/2},-q^{\nu+1/2},-q^{\nu+1};q)_\infty} \\
&\quad\times {}_4\Phi_3\left[\begin{matrix} q^{-n},-q^{\nu},q^{\nu+1/2},-q^{\nu+1/2} \\ ab^{-1}q^{(1-n+\nu)/2},a^{-1}bq^{(1-n+\nu)/2},q^{2\nu+1}\end{matrix}\middle| q,q\right].
\end{aligned}$$

After applying (2.8) this becomes

$$\begin{aligned}
&\int_0^\pi \frac{h(\cos\varphi;1,-1,q^{1/2},-q^{1/2}|q)(a^{-1}bq^{(1-n)/2}e^{i\varphi},a^{-1}bq^{(1-n)/2}e^{-i\varphi};q)_n}{h(\cos\varphi;q^{\nu/2},-q^{\nu/2},q^{(\nu+1)/2},-q^{(\nu+1)/2}|q)}\,d\varphi \\
&= \frac{2\pi(q^{\nu+1},q^{\nu};q)_\infty}{(q^{2\nu},q;q)_\infty}(a^{-1}bq^{(1-n+\nu)/2},a^{-1}bq^{(1-n-\nu)/2};q)_n \\
&\quad\times {}_4\Phi_3\left[\begin{matrix} q^{-n},-q^{\nu},q^{\nu+1/2},-q^{\nu+1/2} \\ ab^{-1}q^{(1-n+\nu)/2},a^{-1}bq^{(1-n+\nu)/2},q^{2\nu+1}\end{matrix}\middle| q,q\right].
\end{aligned} \tag{4.2}$$

When we compare (4.2) with (3.3), we see that we have found an integral representation for a product of Hahn-Exton q-Bessel functions. Since

$$\frac{h(\cos\varphi;1,-1,q^{1/2},-q^{1/2}|q)}{h(\cos\varphi;q^{\nu/2},-q^{\nu/2},q^{(\nu+1)/2},-q^{(\nu+1)/2}|q)} = \frac{(e^{2i\varphi},e^{-2i\varphi};q)_\infty}{(q^{\nu}e^{2i\varphi},q^{\nu}e^{-2i\varphi};q)_\infty} \tag{4.3}$$

the integral representation is:

$$\begin{aligned}
&J_\nu(ax;q)J_\nu(bx;q) \\
&= \frac{(abx^2)^{\nu}(q^{\nu+1},q^{2\nu};q)_\infty}{2\pi(q^{\nu},q;q)_\infty}\int_0^\pi \frac{(e^{2i\varphi},e^{-2i\varphi};q)_\infty}{(q^{\nu}e^{2i\varphi},q^{\nu}e^{-2i\varphi};q)_\infty} \\
&\quad\times \sum_{n=0}^{\infty} \frac{(a^{-1}bq^{(1-n)/2}e^{i\varphi},a^{-1}bq^{(1-n)/2}e^{-i\varphi};q)_n(-1)^n(ax)^{2n}q^{n(n+1)/2}}{(q^{\nu+1},q;q)_n}\,d\varphi,
\end{aligned} \tag{4.4}$$

with $\mathrm{Re}(\nu) > 0$.

5. **The addition formula.** Askey and Ismail [1] proved the following orthogonality relation

$$\int_0^\pi \frac{(e^{2i\varphi}, e^{-2i\varphi}; q)_\infty}{(q^\nu e^{2i\varphi}, q^\nu e^{-2i\varphi}; q)_\infty} C_m(\cos\varphi; q^\nu|q) C_n(\cos\varphi; q^\nu|q)\, d\varphi = \frac{2\pi\Gamma_q(2\nu)}{\Gamma_q(\nu)\Gamma_q(\nu+1)} \cdot \frac{1-q^\nu}{1-q^{\nu+n}} \cdot \frac{(q^{2\nu};q)_n}{(q;q)_n} \cdot \delta_{mn}, \tag{5.1}$$

where $\mathrm{Re}(\nu) > 0$ and where $C_n(\cos\varphi; \beta|q)$ are the q-Ultraspherical polynomials of Rogers (1.5). By (2.1) and (2.10) these polynomials can be written as a well-poised ${}_2\Phi_1$:

$$C_n(\cos\varphi;\beta|q) = \frac{(\beta;q)_n}{(q;q)_n} e^{in\varphi}{}_2\Phi_1\left[\begin{matrix} q^{-n}, \beta \\ \beta^{-1}q^{1-n}\end{matrix}\middle| q, \beta^{-1}qe^{-2i\varphi}\right].$$

By (2.4) this can be transformed into a balanced ${}_4\Phi_3$:

$$C_n(\cos\varphi;\beta|q) = \frac{(\beta^2;q)_n}{(q;q)_n}\beta^{-n/2}{}_4\Phi_3\left[\begin{matrix} q^{-n}, \beta^2 q^n, \beta^{1/2}e^{i\varphi}, \beta^{1/2}e^{-i\varphi} \\ \beta q^{1/2}, -\beta q^{1/2}, -\beta \end{matrix}\middle| q, q\right] \tag{5.2}$$

With the orthogonality relation (5.1) and the integral representation (4.4) it is natural to look for a series expansion of the form

$$\sum_{n=0}^\infty \frac{(a^{-1}bq^{(1-n)/2}e^{i\varphi}, a^{-1}bq^{(1-n)/2}e^{-i\varphi}; q)_n (-1)^n (ax)^{2n} q^{n(n+1)/2}}{(q^{\nu+1}, q; q)_n} = \sum_{k=0}^\infty A_k(x) C_k(\cos\varphi; q^\nu|q) \tag{5.3}$$

The series on the left hand side of (5.3) is a differentiable function in $\cos\varphi$ (see Appendix). Hence the Fourier series on the right hand side converges pointwise to the function on the left hand side of (5.3).

If $\mathrm{Re}(\nu) > 0$ we find with (5.1) for $A_k(x)$

$$\begin{aligned} A_k(x) = {} & \frac{\Gamma_q(\nu)\Gamma_q(\nu+1)}{2\pi\Gamma_q(2\nu)} \cdot \frac{1-q^{\nu+k}}{1-q^\nu} \cdot \frac{(q;q)_k}{(q^{2\nu};q)_k} \\ & \cdot \int_0^\pi \frac{(e^{2i\varphi}, e^{-2i\varphi}; q)_\infty}{(q^\nu e^{2i\varphi}, q^\nu e^{-2i\varphi}; q)_\infty} C_k(\cos\varphi; q^\nu|q) \\ & \times \sum_{n=0}^\infty \frac{(a^{-1}bq^{(1-n)/2}e^{i\varphi}, a^{-1}bq^{(1-n)/2}e^{-i\varphi}; q)_n (-1)^n (ax)^{2n} q^{n(n+1)/2}}{(q^{\nu+1}, q; q)_n}\, d\varphi \end{aligned}$$

For fixed non-negative integers n and k, let us first compute the integral

$$\begin{aligned} I_{n,k} = {} & \int_0^\pi \frac{(e^{2i\varphi}, e^{-2i\varphi}; q)_\infty}{(q^\nu e^{2i\varphi}, q^\nu e^{-2i\varphi}; q)_\infty} \cdot (a^{-1}bq^{(1-n)/2}e^{i\varphi}, a^{-1}bq^{(1-n)/2}e^{-i\varphi}; q)_n \\ & \times {}_4\Phi_3\left[\begin{matrix} q^{-k}, q^{2\nu+k}, q^{\nu/2}e^{i\varphi}, q^{\nu/2}e^{-i\varphi} \\ q^{\nu+1/2}, -q^{\nu+1/2}, -q^\nu \end{matrix}\middle| q, q\right] d\varphi. \end{aligned}$$

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When we use the q-Saalschütz formula (2.5), we have

$$\frac{(a^{-1}bq^{(1-n)/2}e^{i\varphi},a^{-1}bq^{(1-n)/2}e^{-i\varphi};q)_n}{(a^{-1}bq^{(2+\nu-n)/2},a^{-1}bq^{-(n+\nu)/2};q)_n} = \sum_{j=0}^{n}\frac{(q^{-n};q)_j(q^{(\nu+1)/2}e^{i\varphi},q^{(\nu+1)/2}e^{-i\varphi};q)_jq^j}{(a^{-1}bq^{(2+\nu-n)/2},ab^{-1}q^{(2+\nu-n)/2},q;q)_j}.$$

When we insert this in the integral using (4.1) and (4.3) it becomes

$$\begin{aligned}
I_{n,k} &= \sum_{j=0}^{n}\frac{(q^{-n};q)_jq^j(a^{-1}bq^{(2-n+\nu)/2},a^{-1}bq^{-(n+\nu)/2};q)_n}{(a^{-1}bq^{(2-n+\nu)/2},ab^{-1}q^{(2-n+\nu)/2},q;q)_j} \\
&\quad\times\sum_{m=0}^{k}\frac{(q^{-k},q^{2\nu+k};q)_mq^m}{(q^{\nu+1/2},-q^{\nu+1/2},-q^{\nu},q;q)_m} \\
&\quad\cdot\int_0^{\pi}\frac{h(\cos\varphi;1,-1,q^{1/2},-q^{1/2}|q)\,d\varphi}{h(\cos\varphi;q^{\nu/2+m},-q^{\nu/2},q^{(\nu+1)/2+j},-q^{(\nu+1)/2}|q)} \\
&= \sum_{j=0}^{n}\frac{(q^{-n};q)_jq^j(a^{-1}bq^{(2-n+\nu)/2},a^{-1}bq^{-(n+\nu)/2};q)_n}{(a^{-1}bq^{(2-n+\nu)/2},ab^{-1}q^{(2-n+\nu)/2},q;q)_j}\sum_{m=0}^{k}\frac{(q^{-k},q^{2\nu+k};q)_mq^m}{(q^{\nu+1/2},-q^{\nu+1/2},-q^{\nu},q;q)_m} \\
&\quad\times\frac{2\pi(q^{2\nu+m+j+1};q)_\infty}{(q,-q^{\nu+m},q^{\nu+m+j+1/2},-q^{\nu+m+1/2},-q^{\nu+j+1/2},q^{\nu+1/2},-q^{\nu+j+1};q)_\infty}.
\end{aligned}$$

Now by formula (2.12) and formula (2.8) we can rewrite the last factor as

$$\begin{aligned}
&\frac{2\pi(q^{2\nu+1};q)_\infty(-q^{\nu},q^{\nu+j+1/2},-q^{\nu+1/2};q)_m(q^{\nu+1/2},-q^{\nu+1/2},-q^{\nu+1};q)_j}{(q,-q^{\nu},q^{\nu+1/2},-q^{\nu+1/2},-q^{\nu+1/2},q^{\nu+1/2},-q^{\nu+1};q)_\infty(q^{2\nu+1};q)_j(q^{2\nu+j+1};q)_m} \\
&= \frac{2\pi(q^{\nu+1};q)_\infty(q^{\nu};q)_\infty}{(q^{2\nu};q)_\infty(q;q)_\infty}\cdot\frac{(-q^{\nu},-q^{\nu+1/2},q^{\nu+j+1/2},q)_m(q^{\nu+1/2},-q^{\nu+1/2},-q^{\nu+1};q)_j}{(q^{2\nu+1};q)_j(q^{2\nu+j+1};q)_m}.
\end{aligned}$$

When we also use definition (2.2) we find

$$\begin{aligned}
I_{n,k} &= \frac{2\pi\Gamma_q(2\nu)}{\Gamma_q(\nu)\Gamma_q(\nu+1)}(a^{-1}bq^{(2-n+\nu)/2},a^{-1}bq^{-(n+\nu)/2};q)_n \\
&\quad\times\sum_{j=0}^{n}\frac{(q^{-n},q^{\nu+1/2},-q^{\nu+1/2},-q^{\nu+1};q)_jq^j}{(a^{-1}bq^{(2-n+\nu)/2},ab^{-1}q^{(2-n+\nu)/2},q^{2\nu+1},q;q)_j} \\
&\quad\times\sum_{m=0}^{k}\frac{(q^{-k},q^{2\nu+k},q^{\nu+j+1/2};q)_mq^m}{(q^{\nu+1/2},q^{2\nu+1+j},q;q)_m}.
\end{aligned}$$

The sum over m is a balanced ${}_3\Phi_2$ and can thus be summed by (2.5). With (2.9) this yields:

$${}_3\Phi_2\left[\begin{matrix}q^{-k},q^{2\nu+k},q^{\nu+j+1/2}\\ q^{\nu+1/2},q^{2\nu+1+j}\end{matrix}\middle|q,q\right] = \frac{(q^{1+j-k},q^{\nu+1/2};q)_k}{(q^{2\nu+1+j},q^{-\nu-k+1/2};q)_k} = \frac{(q^{-j};q)_kq^{k(\nu+j+1/2)}}{(q^{2\nu+1+j};q)_k}.$$

This gives

$$I_{n,k} = \frac{2\pi\Gamma_q(2\nu)}{\Gamma_q(\nu)\Gamma_q(\nu+1)}(a^{-1}bq^{(2-n+\nu)/2}, a^{-1}bq^{-(n+\nu)/2}; q)_n$$
$$\times \sum_{j=0}^{n} \frac{(q^{-n}, q^{\nu+1/2}, -q^{\nu+1/2}, -q^{\nu+1}; q)_j q^j}{(a^{-1}bq^{(2-n+\nu)/2}, ab^{-1}q^{(2-n+\nu)/2}, q^{2\nu+1}, q; q)_j} \cdot \frac{(q^{-j}; q)_k}{(q^{2\nu+j+1}; q)_k} q^{k(\nu+j+1/2)}.$$

Because $(q^{-j}; q)_k = 0$ if $j < k$, the integral vanishes unless $n \geq k$. So we can start summing by $j = k$. After a shift in the summation index and after applying identities (2.9) and (2.11), the integral becomes

$$I_{n,k}$$
$$= \frac{2\pi\Gamma_q(2\nu)}{\Gamma_q(\nu)\Gamma_q(\nu+1)}(a^{-1}bq^{(2-n+\nu)/2}, a^{-1}bq^{-(n+\nu)/2}; q)_n$$
$$\times \sum_{j=0}^{n-k} \frac{(q^{-n}, q^{\nu+1/2}, -q^{\nu+1/2}, -q^{\nu+1}; q)_{j+k} q^{j+k}}{(a^{-1}bq^{(2-n+\nu)/2}, ab^{-1}q^{(2-n+\nu)/2}, q^{2\nu+1}, q; q)_{j+k}} \cdot \frac{(q^{-j-k}; q)_k}{(q^{2\nu+j+k+1}; q)_k} q^{k(\nu+j+k+1/2)}$$
$$= \frac{2\pi\Gamma_q(2\nu)(a^{-1}bq^{(2-n+\nu)/2}, a^{-1}bq^{(-n+\nu)/2}; q)_n (q^{-n}, q^{\nu+1/2}, -q^{\nu+1/2}, -q^{\nu+1}; q)_k}{\Gamma_q(\nu)\Gamma_q(\nu+1)(q^{2\nu+1}, a^{-1}bq^{(2-n+\nu)/2}, ab^{-1}q^{(2-n+\nu)/2}; q)_k}$$
$$\times \frac{(-1)^k q^{k(k+2\nu+2)/2}}{(q^{2\nu+k+1}; q)_k} \sum_{j=0}^{n-k} \frac{(q^{-n-k}, q^{\nu+k+1/2}, -q^{\nu+k+1/2}, -q^{\nu+k+1}; q)_j q^j}{(a^{-1}bq^{k+(2-n+\nu)/2}, ab^{-1}q^{k+(2-n+\nu)/2}, q^{2\nu+1+2k}, q; q)_j}$$
$$= \frac{2\pi\Gamma_q(2\nu)(a^{-1}bq^{(2-n+\nu)/2}, a^{-1}bq^{-(n+\nu)/2}; q)_n (q^{-n}, q^{\nu+1/2}, -q^{\nu+1/2}, -q^{\nu+1}; q)_k}{\Gamma_q(\nu)\Gamma_q(\nu+1)(q^{2\nu+1}, q^{2\nu+k+1}, a^{-1}bq^{(2-n+\nu)/2}, ab^{-1}q^{(2-n+\nu)/2}; q)_k}$$
$$\times (-1)^k q^{k(k+2\nu+2)/2} {}_4\Phi_3 \left[\begin{matrix} q^{-n+k}, q^{\nu+k+1/2}, -q^{\nu+k+1/2}, -q^{\nu+k+1} \\ q^{2\nu+2k+1}, a^{-1}bq^{k+(2-n+\nu)/2}, ab^{-1}q^{k+(2-n+\nu)/2} \end{matrix} \middle| q, q \right].$$

This leads to

$$A_k(x) = \frac{1-q^{k+\nu}}{1-q^\nu} \cdot \frac{(q^{\nu+1/2}, -q^{\nu+1/2}, -q^{\nu+1}; q)_k}{(q^{2\nu+1}, q^{2\nu+k+1}; q)_k} (-1)^k q^{k(k+\nu+2)/2}$$
$$\times \sum_{n=0}^{\infty} \frac{(-1)^n (ax)^{2n} q^{n(n+1)/2} (a^{-1}bq^{(2-n+\nu)/2}, a^{-1}bq^{-(n+\nu)/2}; q)_n (q^{-n}; q)_k}{(q^{\nu+1}, q; q)_n (a^{-1}bq^{(2-n+\nu)/2}, ab^{-1}q^{(2-n+\nu)/2}; q)_k}$$
$$\times {}_4\Phi_3 \left[\begin{matrix} q^{-n+k}, q^{\nu+k+1/2}, -q^{\nu+k+1/2}, -q^{\nu+k+1} \\ q^{2\nu+2k+1}, a^{-1}bq^{k+(2-n+\nu)/2}, ab^{-1}q^{k+(2-n+\nu)/2} \end{matrix} \middle| q, q \right].$$

Because $(q^{-n}; q)_k = 0$ if $k > n$, we can start summing at $n = k$. Then we shift the summation index and apply identities (2.9) and (2.11). This yields

$$A_k(x) = \frac{1-q^{k+\nu}}{1-q^\nu} \cdot \frac{(q^{\nu+1/2}, -q^{\nu+1/2}, -q^{\nu+1}; q)_k}{(q^{\nu+1}, q^{2\nu+1}, q^{2\nu+k+1}; q)_k} (ax)^{2k} (-1)^k q^{k(k+\nu+2)/2}$$
$$\times \sum_{n=0}^{\infty} \frac{(-1)^n (ax)^{2n} q^{n(n+1)/2} (a^{-1}bq^{(2-n-k+\nu)/2}, a^{-1}bq^{-(n+k+\nu)/2}; q)_{n+k}}{(q^{\nu+1+k}, q; q)_n (a^{-1}bq^{(2-n-k+\nu)/2}, ab^{-1}q^{(2-n+k+\nu)/2}; q)_k}$$
$$\times {}_4\Phi_3 \left[\begin{matrix} q^{-n}, q^{\nu+k+1/2}, -q^{\nu+k+1/2}, -q^{\nu+k+1} \\ q^{2\nu+2k+1}, ab^{-1}q^{(2-n+\nu+k)/2}, a^{-1}bq^{(2-n+\nu+k)/2} \end{matrix} \middle| q, q \right].$$

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Now we apply Sears' transformation formula (2.3), where a and d are chosen as $q^{\nu+k+1/2}$ and $q^{2\nu+2k+1}$ respectively. Then we have

$$\begin{aligned}
&A_k(x) \\
&= \frac{1-q^{k+\nu}}{1-q^{\nu}}\cdot\frac{(q^{\nu+1/2},-q^{\nu+1/2},-q^{\nu+1};q)_k}{(q^{\nu+1},q^{2\nu+1},q^{2\nu+k+1};q)_k}(ax)^{2k}(-1)^k q^{k(k+\nu+2)/2} \\
&\quad\times\sum_{n=0}^{\infty}\frac{(-a^2x^2)^n q^{n(1+k+\nu+n/2)}}{(q^{\nu+1+k},q;q)_n}\,{}_4\Phi_3\left[\begin{matrix}q^{-n},-q^{\nu+k},q^{\nu+k+1/2},-q^{\nu+k+1/2}\\ q^{2\nu+2k+1},ab^{-1}q^{(1-n+\nu+k)/2},a^{-1}bq^{(1-n+\nu+k)/2}\end{matrix}\middle|q,q\right] \\
&\quad\times\frac{(a^{-1}bq^{(2-n-k+\nu)/2},a^{-1}bq^{-(n+k+\nu)/2};q)_{n+k}(a^{-1}bq^{(1-n-k-\nu)/2},ab^{-1}q^{(1-n-k-\nu)/2};q)_n}{(a^{-1}bq^{(2-n-k+\nu)/2},ab^{-1}q^{(2-n+k+\nu)/2};q)_k(a^{-1}bq^{(2-n+k+\nu)/2},ab^{-1}q^{(2-n+k+\nu)/2};q)_n}.
\end{aligned}$$

The last factor of the formula above can be simplified by the following identities (here (2.9) and (2.11) are used frequently):

i) $$\frac{(a^{-1}bq^{(2-n-k+\nu)/2};q)_{n+k}}{(a^{-1}bq^{(2-n-k+\nu)/2};q)_k(a^{-1}bq^{(2-n+k+\nu)/2};q)_n}=1,$$

ii) $$\begin{aligned}\frac{(a^{-1}bq^{-(n+k+\nu)/2};q)_{n+k}}{(ab^{-1}q^{(2-n-k+\nu)/2};q)_k}&=\frac{(a^{-1}bq^{-(n+k+\nu)/2};q)_n(a^{-1}bq^{(n-k-\nu)/2};q)_k}{(ab^{-1}q^{(2-n-k+\nu)/2};q)_k}\\&=(-a^{-1}b)^k q^{k(n-\nu-1)/2}(a^{-1}bq^{-(n+k+\nu)/2};q)_n,\end{aligned}$$

iii) $$\frac{(a^{-1}bq^{-(n+k+\nu)/2};q)_n}{(ab^{-1}q^{(2-n+k+\nu)/2};q)_n}=(-a^{-1}b)^n q^{-n(1+k+\nu)/2},$$

iv) $$(ab^{-1}q^{(1-n-k-\nu)/2};q)_n=(-1)^n(ab^{-1})^n q^{-n(k+\nu)/2}(a^{-1}bq^{(1-n+k+\nu)/2};q)_n$$

.

Moreover by (2.8) we have

v) $$\frac{(q^{\nu+1/2},-q^{\nu+1/2},-q^{\nu+1};q)_k}{(q^{\nu+1},q^{2\nu+1},q^{2\nu+k+1};q)_k}=\frac{(q^{2\nu+1};q)_{2k}}{(q^{2\nu+1};q)_{2k}}\cdot\frac{1}{(q^{\nu+1};q)_k^2}=\frac{1}{(q^{\nu+1};q)_k^2}.$$

Applying identities i) to v) we finally have

$$\begin{aligned}
A_k(x)&=\frac{1-q^{k+\nu}}{1-q^{\nu}}\cdot\frac{1}{(q^{\nu+1};q)_k^2}\cdot(abx^2)^k q^{k(k+1)/2}\\
&\quad\times\sum_{n=0}^{\infty}\frac{(-1)^n(ax)^{2n}q^{n(n+k+1)/2}(a^{-1}bq^{(1-n+k+\nu)/2},a^{-1}bq^{(1-n-k-\nu)/2};q)_n}{(q^{\nu+1+k},q;q)_n}\\
&\quad\times{}_4\Phi_3\left[\begin{matrix}q^{-n},-q^{\nu+k},q^{\nu+k+1/2},-q^{\nu+k+1/2}\\ q^{2\nu+2k+1},ab^{-1}q^{(1-n+\nu+k)/2},a^{-1}bq^{(1-n+\nu+k)/2}\end{matrix}\middle|q,q\right].
\end{aligned}$$

If we compare this last sum with the sum in the product formula (3.3), we see that they are the same if we replace in (3.3) ν and x by $\nu+k$ and $xq^{k/4}$ respectively. Substituting this into (5.3) we have an addition formula for the Hahn-Exton q-Bessel functions. For

$a, b > 0, x \geq 0, 0 \leq \varphi \leq \pi$ and $\mathrm{Re}(\nu) > 0$

$$\sum_{n=0}^{\infty} \frac{(a^{-1}bq^{(1-n)/2}e^{i\varphi}, a^{-1}bq^{(1-n)/2}e^{-i\varphi}; q)_n(-a^2x^2)^n q^{n(n+1)/2}}{(q^{\nu+1}, q; q)_n}$$
$$(5.4) \qquad = \frac{(q;q)^2_\infty}{(q^{\nu+1};q)^2_\infty}(abx^2)^{-\nu} \sum_{k=0}^{\infty} \frac{1-q^{\nu+k}}{1-q^\nu} \cdot q^{k(1-\nu)/2}$$
$$\times J_{\nu+k}(axq^{k/4}; q)\, J_{\nu+k}(bxq^{k/4}; q)\, C_k(\cos\varphi; q^\nu | q).$$

This addition formula is a q-analogue of Gegenbauer's addition formula [9, (2), p. 363]. In order to find this formula we will first take the limit

$$(5.5) \qquad \begin{aligned} \lim_{q\uparrow 1} C_k(\cos\varphi; q^\nu|q) &= \lim_{q\uparrow 1} \sum_{k=0}^{n} \frac{(q^\nu;q)_k(q^\nu;q)_{n-k}}{(q;q)_k(q;q)_{n-k}} \cdot \cos(n-2k)\varphi \\ &= \sum_{k=0}^{n} \frac{(\nu)_k(\nu)_{n-k}}{k!\,(n-k)!} \cdot \cos(n-2k)\varphi = C_k^\nu(\cos\varphi). \end{aligned}$$

$C_k^\nu(\cos\varphi)$ are Gegenbauer's Ultraspherical polynomials.

When we replace in (5.4) a and b by $a(1-q)/2$ and $b(1-q)/2$ respectively and let $q \uparrow 1$ we find with (3.2) and (5.5)

$$\sum_{n=0}^{\infty} \frac{(1-a^{-1}be^{i\varphi})^n(1-a^{-1}be^{-i\varphi})^n(-a^2x^2/4)^n}{(\nu+1)_n\, n!}$$
$$= \frac{4^\nu\Gamma(\nu+1)\Gamma(\nu+1)}{(abx^2)^\nu} \sum_{k=0}^{\infty} \frac{\nu+k}{\nu} J_{\nu+k}(ax)\, J_{\nu+k}(bx)\, C_k^\nu(\cos\varphi).$$

The series on the left hand side can also be represented as a Bessel function. This yields (in the notation of Watson)

$$\omega^{-\nu}J_\nu(\omega) = 2^\nu\Gamma(\nu)\sum_{k=0}^{\infty}(\nu+k)Z^{-\nu}J_{\nu+k}(Z)\, z^{-\nu}J_{\nu+k}(z)\, C_k^\nu(\cos\varphi),$$

where $\omega = \sqrt{(Z^2+z^2-2zZ\cos\varphi)}$.

APPENDIX

THEOREM. *The series*

$$(A.1) \qquad \sum_{n=0}^{\infty} \frac{(a^{-1}bq^{(1-n)/2}e^{i\varphi}, a^{-1}bq^{(1-n)/2}e^{-i\varphi}; q)_n(-1)^n(ax)^{2n}q^{n(n+1)/2}}{(q^{\nu+1}, q; q)_n}$$

is a differentiable function in $\cos\varphi$.

To prove this we shall first show the uniform convergence in φ of (A.1).

$$\begin{aligned}\max_{\varphi}\left|(a^{-1}bq^{(1-n)/2}e^{i\varphi},a^{-1}bq^{(1-n)/2}e^{-i\varphi};q)_n\right| \\ &= \max_{\varphi}\left|\prod_{j=0}^{n-1}(1-a^{-1}bq^{(1-n)/2+j}e^{i\varphi})(1-a^{-1}bq^{(1-n)/2+j}e^{-i\varphi})\right| \\ &= \max_{\varphi}\left|\prod_{j=0}^{n-1}(1-2a^{-1}bq^{(1-n)/2+j}\cos\varphi+a^{-2}b^2q^{(1-n)+2j})\right| \\ &\leq \prod_{j=0}^{n-1}\max_{\varphi}\left|(1-2a^{-1}bq^{(1-n)/2+j}\cos\varphi+a^{-2}b^2q^{(1-n)+2j})\right| \\ &= \prod_{j=0}^{n-1}(1+a^{-1}bq^{(1-n)/2+j})^2.\end{aligned}$$

Since $0<q<1$ we have

$$\prod_{j=[n/2]}^{n-1}(1+a^{-1}bq^{(1-n)/2+j})^2\leq(1+a^{-1}b)^{n+1},$$

where $[x]$ is the largest integer less than or equal to x.

If $j\leq[n/2]$ we have

$$\begin{aligned}\prod_{j=0}^{[n/2]-1}(1+a^{-1}bq^{(1-n)/2+j})^2 &= \prod_{j=0}^{[n/2]-1}q^{1-n+2j}(q^{(n-1)/2-j}+a^{-1}b)^2 \\ &\leq (1+a^{-1}b)^n\prod_{j=0}^{[n/2]-1}q^{1-n+2j} \\ &\leq (1+a^{-1}b)^n q^{-n^2/4}.\end{aligned}$$

Using these upper bounds we have

$$\begin{aligned}&\left|(a^{-1}bq^{(1-n)/2}e^{i\varphi},a^{-1}bq^{(1-n)/2}e^{-i\varphi};q)_n(-1)^n(ax)^{2n}q^{n(n+1)/2}\right| \\ &\qquad\leq (a+b)^{2n+1}a^{-1}x^{2n}q^{n(n+2)/4}.\end{aligned}$$

Now by Cauchy's theorem we see that (A.1) converges uniformly in φ. Hence it is a continuous function.

Now Bernstein's inequality ([10, Vol. II, p. 276])

$$\max_{\varphi}|T'(\varphi)|\leq n\max_{\varphi}|T(\varphi)|,$$

where $T(\varphi)$ is a trigonometric polynomial of the form $T(\varphi)=\sum_{k=-n}^{n}c_ke^{ik\varphi}$, gives us an upper bound for each term when we differentiate (A.1) termwise. Using the same estimates as in the proof above, it is easy to show that the sum of these derivatives also converge uniformly in φ. Since that last sum is equal to the derivative of (A.1), we have shown that (A.1) is differentiable.

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REFERENCES

1. R. Askey and M. E. H. Ismail, *A generalization of Ultraspherical polynomials*. In: Studies in pure mathematics, Birkhäuser-Verlag, Basel, (1983), 55–78.

2. H. Exton, *q-Hypergeometric Functions and Applications*, Ellis Horwood, 1983.

3. G. Gasper and M Rahman, *Basic Hypergeometric Series*, Cambridge University Press, 1990.

4. W. Hahn, *Beiträge zur Theorie der Heineschen Reihen*, Math. Nachr., **2**(1949), 340–379.

5. ———, *Die mechanische Deutung einer geometrischen Differenzengleichung*, Z. Angew. Math. Mech., **33**(1953), 270–272.

6. M. E. H. Ismail, *The zeros of basic Bessel functions, the functions $J_{\nu+ax}(x)$ and associated orthogonal polynomials*, J. Math. Anal. Appl., **86**(1982), 1–19.

7. T. H. Koornwinder and R. F. Swarttouw, *On q-analogues of the Fourier and Hankel transforms*, Trans. Amer. Math. Soc. (to appear).

8. M. Rahman, *An addition theorem and some product formulas for q-Bessel functions*, Canad. J. Math., **40**(1988), 1203–1221.

9. G. N. Watson, *A treatise on the theory of Bessel functions*, Cambridge University Press, 1966.

10. A. Zygmund, *Trigonometric Series*, Cambridge University Press, 1968.

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