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# EXISTENCE AND DIMENSIONALITY OF SIMPLE WEIGHT MODULES FOR QUANTUM ENVELOPING ALGEBRAS

ZHIYONG SHI

We give sufficient and necessary conditions for simple modules of the quantum group or the quantum enveloping algebra $\mathcal{U}_q(\mathfrak{g})$ to have weight space decompositions, where $\mathfrak{g}$ is a semisimple Lie algebra and $q$ is a nonzero complex number. We show that

(i) if $q$ is a root of unity, any simple module of $\mathcal{U}_q(\mathfrak{g})$ is finite dimensional, and hence is a weight module;
(ii) if $q$ is generic, that is, not a root of unity, then there are simple modules of $\mathcal{U}_q(\mathfrak{g})$ which do not have weight space decompositions.

Also the group of units of $\mathcal{U}_q(\mathfrak{g})$ is found.

## 0. INTRODUCTION.

Let $\mathbb{C}$ be the field of complex numbers. The quantum enveloping algebra $\mathcal{U}_q := \mathcal{U}_q(\mathfrak{g})$ is a certain deformation of the universal enveloping algebra $\mathcal{U}(\mathfrak{g})$ of a semisimple algebra $\mathfrak{g}$ over $\mathbb{C}$. For generic $q$, the finite dimensional simple modules of the algebra $\mathcal{U}_q$ are deformations of modules of $\mathcal{U}(\mathfrak{g})$, so that the latter are obtained as $q \to 1$ (Lusztig [5]). But infinite dimensional simple modules of $\mathcal{U}_q$ can not be naturally deformed from that of $\mathcal{U}(\mathfrak{g})$. Indeed, if $\mathfrak{g}$ is simple of type $B_n$, then $\mathcal{U}(\mathfrak{g})$ does not have pointed torsion free modules; that is, modules which are simple, with 1-dimensional weight spaces and with the injective actions of the Chevalley generators $\{e's, f's\}$ (see Britten and Lemire [1]). But $\mathcal{U}_q(B_n)$ does have such representations (Britten, Lemire and Shi [2]). For $q$ equal to a root of unity, the situation is different. Indeed any simple module of $\mathcal{U}_q$ is finite dimensional (Proposition 3.1 below). Nevertheless, all the modules mentioned above are weight modules, that is, they admit weight space decompositions.

With the classification problem of all simple modules for $\mathcal{U}_q$ in mind, in Section 2 we shall give a criteria for a simple module to be a weight module. This is similar to $\mathcal{U}(\mathfrak{g})$ (Lemire [4]). We show in Section 3 that if $q$ is a root of unity, any simple module of $\mathcal{U}_q(\mathfrak{g})$ is finite dimensional, and hence is a weight module, and in Section 4 that if $q$ is generic, that is, not a root of unity, then there are simple modules of $\mathcal{U}_q(\mathfrak{g})$ which do not have weight space decompositions.

We shall follow closely the notation in De Concini and Kac's paper [3].

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## 1. Quantum enveloping algebras

Assume that $(a_{ij})$ be an $n\times n$ integral matrix such that $a_{ii}=2$, $a_{ij}\leqslant 0$ for $i\neq j$, $a_{ij}=0$ implies $a_{ji}=0$, and there exist positive integers $d_1,\ldots,d_n$ such that matrix $(d_ia_{ij})$ is symmetric positive definite. Thus $(a_{ij})$ is a Cartan matrix. Also we assume that $d_1+d_2+\cdots+d_n$ is as small as possible, then the integers $d_1,\ldots,d_n$ are uniquely determined. It is well known that such a matrix is associated with a semisimple Lie algebra $\mathfrak{g}$.

Before giving the definition of the quantum enveloping algebras $\mathcal{U}_q$, we need the following notation. Let $q\in\mathbb{C}^\times := \mathbb{C}\backslash\{0\}$ be such that $q^{2d_i}\neq 1$ for all $i$. For $m\in\mathbb{Z}$ and $d\in\mathbb{N}=\{1,2,\cdots\}$, let

$$[m]_d := (q^{dm}-q^{-dm})/(q^d-q^{-d}),\qquad [m]_d! := [m]_d[m-1]_d\cdots[1]_d,$$

and the Gaussian binomial coefficients

$$\begin{bmatrix} m\\ j\end{bmatrix}_d := \frac{[m]_d[m-1]_d\cdots[m-j+1]_d}{[j]_d!}\quad \text{for } j\in\mathbb{N},\ \begin{bmatrix} m\\ 0\end{bmatrix}_d = 1.$$

We should understand that $\begin{bmatrix} m\\ j\end{bmatrix}_d \in\mathbb{Z}[q,q^{-1}]$ and are well defined by the Gauss binomial formula

$$\prod_{j=0}^{m-1}(1-q^{2j}x) = \sum_{j=0}^{m}(-1)^j\begin{bmatrix} m\\ j\end{bmatrix}_1 q^{j(m-1)}x^j,\ \text{for } m\geqslant 1.$$

Then the *quantum enveloping algebra* $\mathcal{U}_q=\mathcal{U}_q(\mathfrak{g})$ associated with the above matrix $(a_{ij})$ is the associative $\mathbb{C}$-algebra with 1 with generators $E_i, F_i, K_i^{\pm 1}$, $(1\leqslant i\leqslant n)$ and relations

$$K_iK_j=K_jK_i,\ K_iK_i^{-1}=K_i^{-1}K_i=1, \tag{1.1}$$

$$K_iE_jK_i^{-1}=q^{d_ia_{ij}}E_j,\ K_iF_jK_i^{-1}=q^{-d_ia_{ij}}F_j, \tag{1.2}$$

$$E_iF_j-F_jE_i=\delta_{ij}\frac{K_i-K_i^{-1}}{q^{d_i}-q^{-d_i}}, \tag{1.3}$$

$$\sum_{\nu=0}^{1-a_{ij}}(-1)^\nu\begin{bmatrix} 1-a_{ij}\\ \nu\end{bmatrix}_{d_i}E_i^{1-a_{ij}-\nu}E_jE_i^\nu=0,\quad \text{for } i\neq j, \tag{1.4}$$

$$\sum_{\nu=0}^{1-a_{ij}}(-1)^\nu\begin{bmatrix} 1-a_{ij}\\ \nu\end{bmatrix}_{d_i}F_i^{1-a_{ij}-\nu}F_jF_i^\nu=0\quad \text{for } i\neq j. \tag{1.5}$$

Let $\mathcal{U}_0$ be the subalgebra of $\mathcal{U}_q$ generated by $K_i$, $1\leqslant i\leqslant n$. Then from (1.1), we have

$$\mathcal{U}_0=\mathbb{C}[K_1^{\pm1},\cdots,K_n^{\pm1}]. \tag{1.6}$$

We shall need this subalgebra later.

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## 2. Weight modules of $\mathcal{U}_q$

Let $V$ be an arbitrary (left) module of $\mathcal{U}_q$. If $\omega = (c_1, \ldots, c_n) \in (\mathbb{C}^\times)^n$, let $V_\omega = \{v \in V \mid K_i.v = c_i v\}$. If $V_\omega \neq 0$, then $\omega$ is called a *weight* of $V$, $V_\omega$ is called a *weight space* of $V$, and nonzero elements of $V_\omega$ are called *weight vectors* with weight $\omega$. It is clear that the sum $\sum_\omega V_\omega$ $(\omega \in (\mathbb{C}^\times)^n)$ is direct. If this sum is equal to $V$, then we say that $V$ *admits a weight space decomposition*, or simply say that $V$ is a *weight module*. For weight spaces, a simple application of (1.2) gives

$$E_i.V_\omega \subset V_{\omega'} \qquad F_i.V_\omega \subset V_{\omega''}, \tag{2.1}$$

where $\omega' = \left(q^{d_1 a_{1i}} c_1, \cdots, q^{d_n a_{ni}} c_n\right)$ and $\omega'' = \left(q^{-d_1 a_{1i}} c_1, \cdots, q^{-d_n a_{ni}} c_n\right)$ (see [5, (2.5.1)]).

**Proposition 2.1.** *If $V$ is a simple module of $\mathcal{U}_q$, then the following are equivalent.*

- (a) *$V$ admits a weight space decomposition.*
- (b) *For each $v \in V$, $\mathcal{U}_0.v$ is finite dimensional.*
- (b$'$) *For each $v \in V$ and each $i$, $\mathbb{C}[K_i^{\pm 1}].v$ is finite dimensional.*
- (c) *There exists $v \in V \backslash \{0\}$ such that $\mathcal{U}_0.v$ is finite dimensional.*
- (c$'$) *There exists $v \in V \backslash \{0\}$ so that for each $i$, $\mathbb{C}[K_i^{\pm 1}].v$ is finite dimensional.*
- (d) *$V$ has at least one weight vector.*
- (d$'$) *$V$ has at least one weight space.*

Proof: The implications (a) $\Rightarrow$ (b) $\Rightarrow$ (b$'$), (b) $\Rightarrow$ (c) $\Rightarrow$ (c$'$) and (d)$\Leftrightarrow$(d$'$) are clear. For (b$'$) $\Rightarrow$ (b) and (c$'$) $\Rightarrow$ (c), we note that the fact that $\mathbb{C}[K_i^{\pm 1}].v$ is finite dimensional implies there exists $n_i \in \mathbb{N}$ so that

$$\mathbb{C}[K_i^{\pm 1}].v = \sum_{|j| \leqslant n_i} \mathbb{C} K_i^j v.$$

Thus by (1.6), we have that

$$\mathcal{U}_0.v = \sum_{|j_i| \leqslant n_i, i=1,\ldots,n} \mathbb{C} K_1^{j_1} \cdots K_n^{j_n} v,$$

a finite dimensional space.

Assume there exists $v \in V\backslash\{0\}$ such that $\mathcal{U}_0.v$ is finite dimensional, then $K_1, \ldots, K_n$ have a common eigenvector $w \in \mathcal{U}_0.v$ such that $K_i w = c_i w$ for some $c_i \in \mathbb{C}$. In fact, $c_i \neq 0$ since the $K_i$'s are invertible. Thus $w$ is a weight vector of $V$ with weight $\omega$ for $\omega = (c_1, \ldots, c_n)$. This shows (c) $\Rightarrow$ (d).

For (d) $\Rightarrow$ (a), we note the condition in (d) implies that

$$W := \bigoplus\nolimits_{\omega \in (\mathbb{C}^\times)^n} V_\omega \neq 0.$$

By (2.1), we have that $W$ is a nonzero $\mathcal{U}_q$-submodule of $V$, and hence is equal to $V$. ∎

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## 3. The case when $q = \varepsilon$ is a root of unity

Assume $q = \varepsilon$ is a primitive $\ell$-th root of unity throughout this section. We only recall some properties of $\mathcal{U}_\varepsilon$ from [3] for our purposes.

Let $N$ be the number of positive roots of the root system of $\mathfrak{g}$. Let $Z_\varepsilon$ be the centre of $\mathcal{U}_\varepsilon$. From [3, Corollary 3.3], we know that $\mathcal{U}_\varepsilon$ is a finitely generated free module over $Z_0$ of rank $\ell^{2N+n}$, where $Z_0$ ([3, Section 3.3]) is an subalgebra of $Z_\varepsilon$ and is isomorphic to the tensor product of the polynomial algebra of $2N$ variables and the Laurent polynomial algebra of $n$ variables. We may express $Z_0$ as

$$Z_0 = \mathbb{C}[x_i, t_j^{\pm 1}]_{1 \leqslant i \leqslant 2N, 1 \leqslant j \leqslant n},$$

for commuting variables $x_i$'s and $t_j$'s. Now the maximal ideals of $Z_0$ all have codimension 1 in $Z_0$, so any simple module of $Z_0$ is one dimensional.

Proposition 10.1.9 of [6] implies the following

**Fact:** Suppose a $\mathbb{C}$-algebra $R$ is a finitely generated module over $C$, which is a subalgebra of $R$ such that $rc = cr$ for any $r \in R$, $c \in C$. If $M$ is a simple module of $R$, then $M$ as a module of $C$ is a direct sum of finitely many copies of a single simple module of $C$. In other words, $M$ over $C$ is semisimple, of finite length and isotypic.

Applying the Fact to the case of $R = \mathcal{U}_\varepsilon$ and $C = Z_0$, we have the following

**Proposition 3.1.** *Any simple module of $\mathcal{U}_\varepsilon$ is finite dimensional.*

Therefore, in order to determine whether simple modules are weight modules, we need only be concerned about the finite dimensional ones. Then Proposition 2.1 gives

**Corollary 3.2.** *Any simple module of $\mathcal{U}_\varepsilon$ is a weight module.*

Combining this with the result for $\mathcal{U}_\varepsilon(sl_2)$ (see, for instance, [7] or [3]) the classification problem of all simple modules of $\mathcal{U}_\varepsilon(sl_2)$ is solved. Unfortunately, this is the only complete classification available so far.

## 4. The generic case when $q$ is not a root of unity

In this section, we find the group of units of $\mathcal{U}_q(\mathfrak{g})$ for any $q$ and then construct some simple modules for generic $q$, which are not weight modules.

Let us recall briefly some facts about the basis structure of $\mathcal{U}_q$. See Section 1.7 of [3] for more details.

Fix a root basis $\{\alpha_1, \cdots, \cdots, \alpha_n\}$ of the root system $\Delta$ of $\mathfrak{g}$, with respect to the Cartan matrix $(a_{ij})$. A chosen reduced expression of the longest element of the Weyl group of $\mathfrak{g}$ gives an ordering of the set $\Delta^+$ of positive roots, say in this ordering $\Delta^+ = \{\beta_1, \beta_2, \cdots, \beta_N\}$. By using the action of the braid group of the Weyl group of $\Delta$ on

$\mathcal{U}_q$, one defines the root vectors $E_\beta$, $F_\beta$ for all $\beta \in \Delta^+$ having $E_{\alpha_i} = E_i$ and $F_{\alpha_i} = F_i$. Then for $\mathbf{k} = (k_1, \cdots, k_N) \in \mathbb{Z}^N_{\geqslant 0}$, let $E^{\mathbf{k}} = E^{k_1}_{\beta_1} \cdots E^{k_N}_{\beta_N}$ and $F^{\mathbf{k}} = F^{k_N}_{\beta_N} \cdots F^{k_1}_{\beta_1}$.

For $\mathbf{k}$, $\mathbf{r} \in \mathbb{Z}^N_{\geqslant 0}$, $u \in \mathcal{U}_0$, define the monomial $M_{\mathbf{k},\mathbf{r};u} = F^{\mathbf{k}} u E^{\mathbf{r}}$. Define the degree of $M_{\mathbf{k},\mathbf{r};u}$ by

$$d(M_{\mathbf{k},\mathbf{r};u}) = (k_N, \cdots, k_1, r_1, \cdots, r_N \text{ ht } (M_{\mathbf{k},\mathbf{r};u})) \in \mathbb{Z}^{2N+1}_{\geqslant 0},$$

where $\text{ht } (M_{\mathbf{k},\mathbf{r};u}) = \sum_i (k_i + r_i) \text{ ht } (\beta_i)$. The lexicographical order on $\mathbb{Z}^{2N+1}_{\geqslant 0}$ (as semi-group) gives a filtration $\{\mathcal{U}^{(\mathbf{s})}\}$ of $\mathcal{U}_q$ with $\mathcal{U}^{(\mathbf{s})}$ being the linear span of the monomials $M_{\mathbf{k},\mathbf{r};u}$ with $d(M_{\mathbf{k},\mathbf{r};u}) \leqslant \mathbf{s}$. Then we may define a degree $d(x)$ for any element $x \in \mathcal{U}_q$ as the minimal $\mathbf{s}$ such that $x \in \mathcal{U}^{(\mathbf{s})}$. $d(x) = \mathbf{0}$ if and only if $x \in \mathcal{U}_0$ (1.6). Conventionally the only element in $\mathcal{U}_q$ possibly having degree less than $\mathbf{0}$ is 0.

The $q$-analogue PBW theorem is that $\{F^{\mathbf{k}} K_1^{m_1} \cdots K_n^{m_n} E^{\mathbf{r}} \mid \mathbf{k}, \mathbf{r} \in \mathbb{Z}^N_{\geqslant 0}, m_i \in \mathbb{Z}\}$ is a basis of $\mathcal{U}_q$.

Two important formulae used in the proof of the above results (Section 1.7 of [**3**]) are

$$E_{\beta_j} E_{\beta_i} = q^{(\beta_i|\beta_j)} E_{\beta_i} E_{\beta_j} + \sum_{\mathbf{k} \in \mathbb{Z}^N_{\geqslant 0}} c_{\mathbf{k}} E^{\mathbf{k}},$$

for $i < j$, $c_{\mathbf{k}} \in \mathbb{C}$, and $c_{\mathbf{k}} = 0$ unless $d(E^{\mathbf{k}}) < d\left(E_{\beta_i} E_{\beta_j}\right)$ and

$$E_\alpha F_\beta = F_\beta E_\alpha + \text{ terms of degrees less than } d(F_\beta E_\alpha).$$

These formulae and PBW theorem imply that $d(xy) = d(yx)$, $\quad x, y \in \mathcal{U}_q$.

Assume $x = x_1 + x_2$ and $y = y_1 + y_2$ with $x_1 = M_{\mathbf{k}_1,\mathbf{r}_1;u_1}$, $y_1 = M\mathbf{k}_2, \mathbf{r}_2; u_2$, and $d(x_2) < d(x_1)$, $d(y_2) < d(y_1)$. Then we have that $xy = z_1 + z_2$ with $z_1 = M_{\mathbf{k}_1+\mathbf{k}_2,\mathbf{r}_1+\mathbf{r}_2;u}$ and $d(z_2) < d(z_1)$. If $xy = 1$, then $d(z_1) = d(xy) = d(1) = \mathbf{0}$. So $z_2 = 0$ and $\mathbf{k}_1 = \mathbf{k}_2 = \mathbf{r}_1 = \mathbf{r}_2 = \mathbf{0}$. This gives that $x$, $y \in \mathcal{U}_0$. Since $\mathcal{U}_0$ is the Laurent polynomial ring on $K_1, \cdots, K_n$, we have

**Proposition 4.1.** *The units group of $\mathcal{U}_q$ equals $\{K_1^{m_1} \cdots K_n^{m_n} \mid m_i \in \mathbb{Z}, 1 \leqslant i \leqslant n\}$.*

In particular, $E_j - a$ is not invertible for any $a \in \mathbb{C}^\times$. Fix any $j \in \{1, \ldots, n\}$, there exists a maximal left ideal $M$ which contains $E_j - a$. Then $\mathcal{U}_q/M$ is a simple module of $\mathcal{U}_q$. We claim that $\mathcal{U}_q/M$ admits no weight space decomposition.

By Proposition 2.1 (c′), it suffices to show that

$$\{1 + M, K_j + M, K_j^2 + M, \cdots\} \tag{4.1}$$

is linearly independent in $\mathcal{U}_q/M$. Suppose that

$$c_0 1 + c_1 K_j + c_2 K_j^2 + \cdots + c_k K_j^k \in M, \tag{4.2}$$

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for any $k \in \mathbb{N}$ and some $c_i \in \mathbb{C}$. By (1.2),

$$E_j^m K_j^i = q^{-2mi} K_j^i E_j^m.$$

For each $1 \leqslant m \leqslant k$, applying $E_j^m$ to (4.2), we have

$$c_0 1 a^m + c_1 q^{-2m} K_j a^m + c_2 q^{-4m} K_j^2 a^m + \cdots + c_k q^{-2km} K_j^k a^m \in M,$$

and then $$c_0 1 + c_1 q^{-2m} K_j + c_2 q^{-4m} K_j^2 + \cdots + c_k q^{-2km} K_j^k \in M.$$

Since $q$ is not a root of unity, the $(k+1) \times (k+1)$ coefficient matrix $\left(q^{-2mi}\right)$ (a Vandemonde matrix) is non-singular. This implies $c_i K_j^i \in M$, and so $c_i = 0$ since $K_j$ is invertible and $M$ is a maximal left ideal. So the set of (4.1) is linearly independent. This establishes our claim.

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Department of Mathematics
The University of Western Ontario
London, Ontario
Canada N6A 5B7