$$\frac{1}{2}\int_0^{\pi/2} r_2^2\,d\theta - \frac{1}{2}\int_0^{\pi/2} r_1^2\,d\theta = \frac{1}{2}\int_0^{\pi/2}\left(r_2^2 - r_1^2\right)d\theta = \frac{1}{2}\int_0^{\pi/2}\left(2\cos\theta + 3\cos^2\theta\right)d\theta = \frac{3}{8}\pi + 1.$$

Now consider the area of region B. One endpoint is Q with polar coordinates $\left(1, \frac{1}{2}\pi\right)$. The other is given by $r_1 = 0 = r_2$. The left-hand side solves to give $\theta = \pi$ and the right-hand side gives $\theta = \frac{2}{3}\pi$: if you like, the pole P has rival polar coordinates $(0, \pi)$ and $(0, \frac{2}{3}\pi)$ on the two curves. The area of B is then given by two separate integrals that cannot be combined:

$$\frac{1}{2}\int_{\pi/2}^{\pi} r_1^2\,d\theta \;-\; \frac{1}{2}\int_{\pi/2}^{2\pi/3} r_2^2\,d\theta$$

$$= \frac{1}{2}\int_{\pi/2}^{\pi}\left(1 + 2\cos\theta + \cos^2\theta\right)d\theta - \frac{1}{2}\int_{\pi/2}^{2\pi/3}\left(1 + 4\cos\theta + 4\cos^2\theta\right)d\theta$$

$$= \left(\frac{3\pi}{8} - 1\right) - \left(\frac{\pi}{4} + \frac{3\sqrt{3}}{4} - 2\right) = \frac{\pi}{8} - \frac{3\sqrt{3}}{4} + 1.$$

The ambiguity of endpoints exhibited at P can only occur when the curves meet at the pole. This example is a fine one for class discussion and similar ones are also worth investigating: for example, the region corresponding to B for the curves $r_1 = 1 + \sqrt{2}\cos\theta$ and $r_2 = 1 + 2\cos\theta$ has the "π-less" area $\frac{11}{4} - \sqrt{2} - \frac{3\sqrt{3}}{4}$.

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Another appearance of the golden ratio

Professor Anne Watson in her inspirational plenary talk "What school mathematics can be … really" at the 2022 MA Conference mentioned the following problem as a rich one for a discussion of problem-solving strategies and approaches.

Find the area of right-angled triangle ABC, situated in a quadrant of the unit circle as in Figure 1(a).

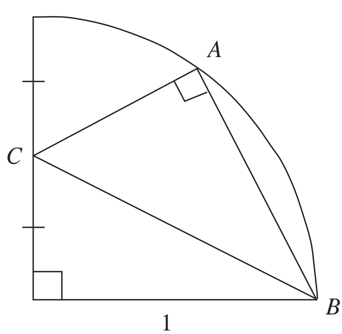

FIGURE 1(a)

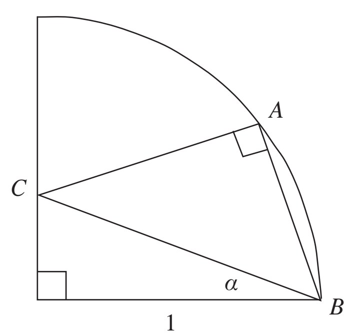

FIGURE 1(b)

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In this Note, we look at the more general configuration in Figure 1(b) and will find an unexpected appearance of the golden ratio when finding the angle α with $0 \leqslant \alpha \leqslant \frac{1}{4}\pi$ that maximises the area of triangle *ABC*.

Extending *AC* gives Figure 2 which embeds Figure 1(b) in a semicircle. Then the area, $f(\alpha)$, of triangle *ABC* is given by "area *ABD* – area *BCD*" which evaluates as

$$f(\alpha) = \tfrac{1}{2}(2 \sin \alpha)(2 \cos \alpha) - \tfrac{1}{2} \times 2 \times \tan \alpha = \sin 2\alpha - \tan \alpha.$$

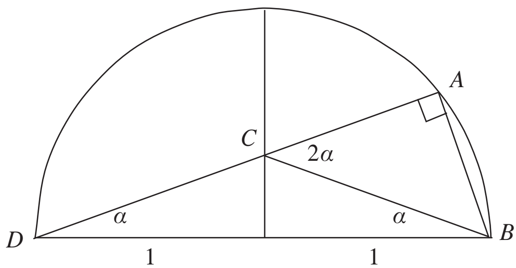

FIGURE 2

In the original problem of Figure 1(a), $\tan \alpha = \frac{1}{2}$, so $f(\alpha) = 2 \cdot \frac{1}{\sqrt{5}} \cdot \frac{2}{\sqrt{5}} - \frac{1}{2} = \frac{3}{10}$. And, since $\sin 2\alpha = \frac{4}{5}$, triangle *ABC* is then similar to a 3, 4, 5 triangle.

The maximum area of triangle *ABC* occurs when

$$0 = f'(\alpha) = 2 \cos 2\alpha - \sec^2 \alpha = 2 \cos 2\alpha - \frac{2}{1 + \cos 2\alpha}.$$

This leads to $\cos^2 \alpha + \cos 2\alpha - 1 = 0$ so that $\cos 2\alpha = \frac{1}{2}(\sqrt{5} - 1) = \frac{1}{\phi}$, where $\phi = \frac{1}{2}(\sqrt{5} + 1)$ is the golden ratio. This means that triangle *ABC* is then similar to the so-called *Kepler triangle* with sides $1, \sqrt{\phi}, \phi$.

Now,

$$\begin{aligned} f(a) &= \sin 2\alpha - \tan \alpha = 2 \sin \alpha \cos \alpha - \tan \alpha \\ &= \tan \alpha (2 \cos^2 \alpha - 1) = \tan \alpha \cos 2\alpha. \end{aligned}$$

But if $\cos 2\alpha = \frac{1}{\phi}$, then $\frac{1 - \tan^2 \alpha}{1 + \tan^2 \alpha} = \frac{1}{\phi}$ and, since $\phi^2 = \phi + 1$, $\tan^2 \alpha = \frac{\phi - 1}{\phi + 1} = \frac{\phi^{-1}}{\phi^2} = \frac{1}{\phi^3}$. Thus the maximum value of $f(\alpha)$ is $\frac{1}{\phi^{3/2}} \cdot \frac{1}{\phi} = \frac{1}{\phi^{5/2}} \approx 0.30028$, attained when $\tan \alpha = \frac{1}{\phi^{3/2}} \approx 0.48587$: these values are actually very close to the corresponding values of 0.3 and 0.5 in Figure 1(a).

Finally, it is worth noting that maximising the perimeter of triangle *ABC* is equivalent to maximising the sum of the lengths $DA + AB$ in Figure 2. This occurs when $\alpha = \frac{1}{4}\pi$ and the triangle has degenerated into a line segment, a nice example of an endpoint maximum. This can be seen analytically since the perimeter is given by

$$DA + AB = 2(\cos\alpha + \sin\alpha) = 2\sqrt{2}\cos\left(\alpha - \tfrac{1}{4}\pi\right).$$

But it can also be seen geometrically by imagining ellipses with foci at D and B generated by a string of length $DA + AB$. As A moves up from B on the circle, the respective ellipses are disjoint and correspond to increasing string-length constants. The maximum such length occurs when $\alpha = \frac{1}{4}\pi$, with the ellipse tangential to the circle.

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