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SUM OF TWO INNER FUNCTIONS AND EXPOSED POINTS IN H^1

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If h is an outer function in H^1 then it is shown that $h=(q_1+q_2)g$ where both q_1 and q_2 are inner functions with $\operatorname{Im}\bar{q}_1q_2\leqq 0$ almost everywhere, and g is a strong outer function (equivalently, $g/\|g\|_1$ is an exposed point of the unit ball of H^1). If q_1+q_2 is nonconstant then such an h is not strongly outer. Moreover a sum of two inner functions is studied.

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1. Introduction

Let U be the open unit disc in the complex plane and let ∂U be the boundary of U. A function in U is said to be of class N if the integrals

$$\int_{-\pi}^{\pi} \log^+|f(re^{i\theta})|\,d\theta$$

are bounded for $r<1$. If f is in N, then $f(e^{i\theta})$ which we define to be $\lim_{r\to 1} f(re^{i\theta})$, exists almost everywhere on ∂U. If

$$\lim_{r\to 1}\int_{-\pi}^{\pi} \log^+|f(re^{i\theta})|\,d\theta=\int_{-\pi}^{\pi} \log^+|f(e^{i\theta})|\,d\theta$$

then f is said to be the class N_+. The set of all boundary functions in N or N_+ is denoted by N or N_+, respectively. For $0<p\leqq\infty$, the Hardy space H^p, is defined by $N_+\cap L^p$.

We call q in N_+ an inner function if $|q(e^{i\theta})|=1$ a.e. on ∂U. A function h in N_+ is called outer if it is not divisible in N_+ by a nonconstant inner function. A function g in H^1 is strongly outer if the only functions f in H^1 such that f/g is positive are scalar multiples of g. (If g has norm 1, it has this property if and only if it is an exposed point of the unit ball of H^1.) Every strongly outer function is outer.

H. Helson [7] recently gave a necessary and sufficient condition for strong outer

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functions. That is related to the divisibility by the square of the sum of two inner functions. We can prove similarly the following result. Let g be outer in H^1. Then g is strongly outer in H^1 if and only if $g/(q_1+q_2)$ is not in H^1 for any inner functions q_1, q_2 such that q_1+q_2 is not constant and $-i(q_1-q_2)/(q_1+q_2)$ is nonnegative almost everywhere. A. Beurling [1] established a celebrated factorization theorem. That is, a nonzero function f in N_+ can be factorized as in the following: $f=qh$ where q is inner and h is outer in N_+. Thus it is desirable to show that if f is not a strong outer function in H^1 then $f=(q_1+q_2)g$ where both q_1 and q_2 are inner functions, $-i(q_1-q_2)/(q_1+q_2)$ is nonnegative almost everywhere and g is a strong outer function in H^1. In Section 2 we show this factorization is true. In the proof, a theorem of E. Hayashi ([4, 5]) and theorems of H. Helson ([6, 7]) are crucial. It is more desirable to show that if f is not strongly outer then $f=(q_1+q_2)^2 g$ where both q_1 and q_2 are inner and g is strongly outer by a theorem of Helson [7]. In Section 3 we consider this problem. A strong outer function is important in a solution set of an extremal problem of H^1. In Section 4 we try to describe the solution sets intelligibly. In Section 5 a characterization of absolute values of strong outer functions is given.

2. A sum of two inner functions

Suppose f is a function in H^1 which has the form: $f=(q_1+q_2)g$ where q_1, q_2 are inner functions, q_1+q_2 is not constant, $\operatorname{Im}\bar{q}_1 q_2 \leqq 0$ and g is a strong outer function. By the following lemma, f is not strongly outer. In this section we show the converse.

Lemma. *Let q_1, q_2 be inner functions with $q_1 \neq -q_2$. Then $-i(q_1-q_2)/(q_1+q_2)$ is nonnegative if and only if* $\operatorname{Im}\bar{q}_1 q_2 \leqq 0$.

Proof. Use the following equality:

$$\frac{-i(q_1-q_2)}{q_1+q_2}=\frac{-2\operatorname{Im}\bar{q}_1 q_2}{|1+\bar{q}_1 q_2|^2}.$$

Theorem 1. *If f is not a strong outer function in H^1 then $f=(q_1+q_2)g$ where both q_1 and q_2 are inner functions,* $\operatorname{Im}\bar{q}_1 q_2 \leqq 0$ *almost everywhere, $(q_1-q_2)^{-1}$ is summable and g is a strong outer function. If f is outer then q_1+q_2 is also outer. If q_1 is a finite Blaschke product of degreee n then so is q_2.*

Proof. Suppose $f=q_0 h^2$ where q_0 is inner and h is outer in H^2. By a theorem of E. Hayashi ([4, 5])

$$H^2 \cap (h/\bar{h})\bar{H}^2 = g_0(H^2 \ominus zqH^2)$$

and

$$\bar{h}/h = \bar{q}\bar{g}_0/g_0,$$

where q is an inner function and g_0^2 is strongly outer. Hence $h=sg_0$ where $s\in H^2\ominus zqH^2$ and $\bar{q}s^2\geqq 0$. Since $|s^2\pm iq|^2=|s|^4+1$,

$$s^2+iq=p_1h_1 \quad\text{and}\quad s^2-iq=p_2h_1$$

where p_1, p_2 are inner functions and h_1 is a function in H^1 with $h_1^{-1}\in H^\infty$. Hence

$$s^2=(p_1+p_2)\frac{h_1}{2} \quad\text{and}\quad q=-i(p_1-p_2)\frac{h_1}{2}$$

and

$$\bar{q}s^2=(p_1+p_2)/-i(p_1-p_2).$$

Put $g=h_1g_0^2/2$, $q_1=q_0p_1$ and $q_2=q_0p_2$, then $f=(q_1+q_2)g$, g is strongly outer, $-i(q_1-q_2)/(q_1+q_2)$ is nonnegative and $(q_1-q_2)^{-1}$ is summable. By the lemma above $\operatorname{Im}\bar{q}_1q_2\leqq 0$.

If q_1 is a finite Blashke product of degree n then so is q_2. For $-i(p_1-p_2)/(p_1+p_2)$ is a nonnegative in N_+ because p_1+p_2 is outer, and

$$\frac{-i(p_1-p_2)}{p_1+p_2}-i=\frac{-i2p_1}{p_1+p_2}$$

and

$$\frac{-i(p_1-p_2)}{p_1+p_2}+i=\frac{-i2p_2}{p_1+p_2}.$$

This shows degree $p_1=$degree p_2 [6, 7] and hence $q_2=q_0p_2$ is a finite Blaschke product of degree n.

In the theorem above

$$-i(q_1-q_2)g=q_0qg_0^2.$$

In general even if $f=(q_1+q_2)g$ and g is strongly outer, $-i(q_1-q_2)$ does not have such properties.

For any pair (q_1,q_2) of inner functions q_1,q_2, $-i(q_1-q_2)/(q_1+q_2)$ is real. However there exists a pair (q_1,q_2) such that $-i(q_1-q_2)/(q_1+q_2)$ is nonnegative. Suppose a, α_j and β_j are complex numbers, $0<|a|\leqq 1$, $|\alpha_1|<1$ and $|\alpha_j|\leqq|\beta_j|$ $(j=1,2)$. If

$$\alpha_1\beta_1=a/\bar{a}, \alpha_1+\beta_1=(1+|a|^2+i)/\bar{a}$$

and

$$\alpha_2\beta_2=a/\bar{a}, \alpha_2+\beta_2=(1+|a|^2-i)/\bar{a}$$

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then $-i(q_1-q_2)/(q_1+q_2)$ is nonnegative where

$$q_1=\frac{z-\alpha_1}{1-\bar{\alpha}_1 z} \quad \text{and} \quad q_2=\frac{z-\alpha_2}{1-\bar{\alpha}_2 z}.$$

For $|(z-a)(1-\bar{a}z)\pm iz|^2=|1-\bar{a}z|^2+1$. Hence $(z-a)(1-\bar{a}z)+iz=q_1h$ and $(z-a)(1-\bar{a}z)-iz=q_2h$ where h is outer. Therefore $(z-a)(1-\bar{a}z)=(q_1+q_2)h$ and $z=-i(q_1-q_2)h$. Since $z/(z-a)(1-\bar{a}z)$ is nonnegative, $-i(q_1-q_2)/(q_1+q_2)$ is nonnegative.

3. A sum of two inner functions and the square

In this section we wish to factorize the part q_1+q_2 in Theorem 1 which is not strongly outer. This problem is suggested by a theorem of Helson [7] (see Section 1).

Theorem 2. *Let q_1 and q_2 be inner functions such that q_1+q_2 is outer, $\operatorname{Im}\bar{q}_1q_2\leqq 0$ almost everywhere and $(q_1-q_2)^{-1}$ is summable. If the inner part q of q_1-q_2 has one of the following three properties then there exist two inner functions p_1,p_2 such that*

$$q_1+q_2=(p_1+p_2)^2k$$

where k is strongly outer.

(1) *$q=q_0^2$ for some inner function q_0.*

(2) *$q(U)\subsetneqq U$.*

(3) *q is a finite Blaschke product.*

Proof. Since (1) is a special case of the proof of (2), we will show (2). If $q(U)\subsetneqq U$ then there exists $\alpha\in U$ with $\alpha\notin q(U)$. Put $q'=(q-\alpha)/(1-\bar{\alpha}q)$ then $q'=q_0^2$ for some inner function q_0 because q' is a singular inner function. Moreover

$$\bar{q}q_0^2=\frac{\bar{h}_0}{h_0} \quad \text{and} \quad h_0=1-\bar{\alpha}q.$$

Hence $\bar{q}(q_0h_0)^2=|q_0h_0|^2$ and $q_0h_0/(1+q)$ is real because $\bar{q}(1+q)^2$ is nonnegative. Let $s^2=q(q_1+q_2)/-i(q_1-q_2)$; then $s/(1+q)$ is real and $s\in H^2$ because $(q_1-q_2)^{-1}$ is summable. Since $|s\pm iq_0h_0|^2=|s|^2+|h_0|^2$,

$$s+iq_0h_0=p_1h_1 \quad \text{and} \quad s-iq_0h_0=p_2h_1$$

where p_1,p_2 are inner functions and h_1 is a function in H^2 with $|h_0|\leqq|h_1|$. Hence $h_1^{-1}\in H^\infty$ and $s=(p_1+p_2)h_1$. Thus

$$q_1+q_2=(p_1+p_2)^2\frac{-i(q_1-q_2)}{q}h_1^2.$$

Put $k=-i(q_1-q_2)h_1^2/q$; then k is strongly outer and hence (2) follows.

Suppose q is a finite Blaschke product of degree n. If $f\in H^1$ and $\bar{q}f$ is nonnegative then $f=\gamma\prod_{j=1}^n(z-a_j)(1-\bar{a}_jz)l^2$ where γ is positive constant, $|a_j|\leqq 1$ $(1\leqq j\leqq n)$ and l^2 is strongly outer in H^1 with $l^{-1}\in H^\infty$ ([**2**, **8**]). Since $s^2=q(q_1+q_2)/-i(q_1-q_2)$ is an outer function with $\bar{q}s^2\geqq 0$,

$$s=\gamma_1\prod_{j=1}^n(-\bar{a}_j)^{1/2}(z-a_j)l$$

where $\gamma_1>0$ and $|a_j|=1$ $(1\leqq j\leqq n)$. Put $f=\prod_{j=1}^n(-\bar{c}_j)^{1/2}(z-c_j)l$ where $|c_j|=1$ $(1\leqq j\leqq n)$ and $\{c_j\}_{j=1}^n$ is disjoint from $\{a_j\}_{j=1}^n$. Then

$$|s\pm if|^2=|s|^2+|f|^2\geqq\varepsilon|l|^2$$

for some $\varepsilon>0$ because $\{c_j\}_{j=1}\cap\{a_j\}_{j=1}^n=\phi$. Therefore $s+if=2p_1h_1$ and $s-if=2p_2h_1$ where p_1,p_2 are inner functions and h_1 is a function in H^2 with $|l|\leqq|h_1|$. Now as in the proof of (2), (3) follows.

By Theorem 1 and the remark after it, if f is outer and not strongly outer then

$$f=(q_1+q_2)\frac{q}{-i(q_1-q_2)}g_0.$$

The proof of Theorem 2 implies that if q satisfies one of (1)–(3) then f can be factorized as in the following: $f=(p_1+p_2)^2h_1^2g_0$ and $h_1^2g_0$ is strongly outer. In Theorem 2 if $\bar{q}q_0^2=\bar{h}_0/h_0$ and h_0^2 is strongly outer then the proof of (2) works and hence its conclusion is still valid.

4. The solution sets of extremal problems

For each f in H^2 we define a subset of H^2 in the following:

$$\mathscr{A}_f=\{g\in H^2: gf^{-1}\text{ is real}\}.$$

Again by virtue of Theorem 1, we consider $\mathscr{A}_{q_1+q_2}$ for two inner functions q_1,q_2. If $\mathscr{A}_{q_1+q_2}$ contains a function whose absolute value is that of a strong outer function and the inner part satisfies the conditions in Theorem 2, then the conclusion of Theorem 2 is still valid. This is one of the motivations for the study of $\mathscr{A}_{q_1+q_2}$. For each f in H^1 we define a subset of H^1 as follows:

$$\mathscr{S}_f=\{g\in H^1: gf^{-1} \text{ is nonnegative}\}.$$

The intersection of $\mathscr{S}_f$ and the unit ball S^1 of H^1 is a solution set of an extremal problem of H^1. $\mathscr{S}_f\cap S^1$ was described by the author [8] when it is weak-* compact and E. Hayashi ([4, 5]) described it completely. However the solution may not be understandable enough when it is not weak-* compact. If $f\in H^2$ then $(\mathscr{A}_f)^2\subset\mathscr{S}_{f^2}$ and this is another motivation for the study of $\mathscr{A}_{q_1+q_2}$.

Theorem 3. *Let q_1,q_2 be inner functions and $p=q_1q_2$.*

(1) $\mathscr{A}_{q_1+q_2}+i\mathscr{A}_{q_1+q_2}=H^2\ominus pzH^2$.

(2) *Let b be a Blaschke product such that $b=(p-\alpha)/(1-\bar{\alpha}p)$ and $\alpha\in U$ (such an α exists always). Then*

$$\mathscr{A}_{q_1+q_2}=(1-\bar{\alpha}p)\mathscr{A}_{1+b}.$$

(3) *If b is a finite Blaschke product of degree n then*

$$\mathscr{A}_{1+b}=g^{-1}\mathscr{A}_{1+zn}$$

$$\textit{where } g=\prod_{j=1}^{n}\bar{a}_j(1-\bar{a}_jz) \quad \textit{and} \quad \{a_j\}_{j=1}^n \textit{ are the zeros of } b.$$

(4) *If n is finite then* $\mathscr{A}_{1+zn}=\{\alpha\prod_{j=1}^{l}(z-a_j)(1-\bar{a}_jz)\prod_{j=2l+1}^{n}(-\bar{a}_j)^{1/2}(z-a_j);\ |a_j|<1$ *if* $1\leqq j\leqq l$ *and* $|a_j|=1$ *if* $2l+1\leqq j\leqq n$, *and α is real*$\}$.

(5) $\mathscr{A}_{1+b}\supseteq\mathscr{A}=\{f\in H^2: f=\sum_{j=0}^{\infty}[\alpha_j(b_j+b'_j)-i\beta_j(b_j-b'_j)]\}$ *and* $\mathscr{A}+i\mathscr{A}=\mathscr{A}_{1+b}+i\mathscr{A}_{1+b}$. *Here* $b=\prod_{j=1}^{\infty}b_j$ *where*

$$b_j=\frac{|z_j|}{z_j}\frac{z-z_j}{1-\bar{z}_jz} \textit{ for } j\geqq 1,$$

$b_0=1$ *and* $b'_j=b\bar{b}_j$. *α_j and β_j are real numbers and $\sum$ denotes the L^2-limit of the finite sums.*

Proof. (1) $\mathscr{A}_{q_1+q_2}=\mathscr{A}_{1+p}$ because

$$\frac{q_1+q_2}{1+p}=\frac{(q_1+q_2)\bar{p}}{(1+p)\bar{p}}=\frac{\bar{q}_1+\bar{q}_2}{\bar{p}+1}.$$

If $f\in\mathscr{A}_{1+p}$ then $f(1+p)^{-1}=\bar{f}(1+\bar{p})^{-1}$ and hence

$$(1+p)^{-1}H^2\cap(1+\bar{p})^{-1}\bar{H}^2\supseteq(1+p)^{-1}\{\mathscr{A}_{1+p}+i\mathscr{A}_{1+p}\}.$$

If $(1+p)^{-1}f=(1+\bar{p})^{-1}\bar{g}\in(1+p)^{-1}H^2\cap(1+\bar{p})^{-1}\bar{H}^2$ then $pf=\bar{g}$ because $1+\bar{p}=\bar{p}(1+p)$.

Hence $f=p_1h$ and $g=p_2h$ where h is outer, and p_1 and p_2 are inner. This implies that $\bar{p}p_1h=\overline{p_2h}$ and $\bar{p}p_2h=\overline{p_1h}$. Therefore

$$\bar{p}(p_1+p_2)h=\overline{(p_1+p_2)h}$$

and

$$\bar{p}\{-i(p_1-p_2)h\}=\overline{-i(p_1-p_2)h}.$$

This implies that $(p_1+p_2)h/2$ and $-i(p_i-p_2)h/2$ belong to $\mathscr{A}_{1+p}$. Since

$$f=(p_1+p_2)h/2+i\{-i(p_1-p_2)h/2\},\ (1+p)^{-1}f(1+p)^{-1}\mathscr{A}_{1+p}+i(1+p)^{-1}\mathscr{A}_{1+p}.$$

Thus

$$(1+p)^{-1}H^2\cap(1+\bar{p})^{-1}\bar{H}^2=(1+p)^{-1}\{\mathscr{A}_{1+p}+i\mathscr{A}_{1+p}\}.$$

On the other hand

$$H^2\ominus pzH^2=H^2\cap p\bar{H}^2=(1+p)\{(1+p)^{-1}H^2\cap(1+\bar{p})^{-1}\bar{H}^2\}.$$

This shows (1) because $\mathscr{A}_{q_1+q_2}=\mathscr{A}_{1+p}$.

(2) If $l=1-\bar{\alpha}p$ then

$$b=p\frac{|l|^2}{l^2}$$

and hence $(1+p)^2l^2/(1+b)^2$ is nonnegative a.e.. Therefore $(1+p)l/(1+b)$ is real and $\mathscr{A}_{1+p}=\mathscr{A}_{(1+b)l}=l\mathscr{A}_{1+b}$ because $l^{-1}\in H^\infty$.

(3) $\bar{z}^n=\bar{b}|g|^2/g^2$ and hence we can show $\mathscr{A}_{1+z^n}=g\mathscr{A}_{1+b}$ as in the proof of (2).

(4) Since $(\mathscr{A}_{1+z^n})^2\subset\mathscr{S}_{z^n}$ and

$$\mathscr{S}_{z^n}=\left\{\gamma\prod_{j=1}^{n}(z-a_j)(1-\bar{a}_jz):|a_j|\leqq 1\ (1\leqq j\leqq n)\text{ and }\gamma>0\right\},$$

(4) follows.

(5) If $f=\sum_{j=0}^{\infty}[\alpha_j(b_j+b'_j)-i\beta_j(b_j-b'_j)]$ then it belongs to $\mathscr{A}_{1+b}$ by the first line of (1) and

$$\frac{-i(b_j-b'_j)}{1+b}=\frac{-i(b_j-b'_j)\bar{b}}{(1+b)\bar{b}}=\frac{i(\bar{b}_j-\bar{b}'_j)}{\bar{b}+1}.$$

If $f\in H^2\ominus bzH^2$ and f is orthogonal to $\{b_j+b'_j,-i(b_j-b'_j)\}_{j=0}^{\infty}$ then it is orthogonal to $\{b_j,b'_j\}_{j=0}^{\infty}$. We will show that f is zero almost everywhere. Since $(f,b_0)=0$ and $b_0=1$,

$f=zf_0$ and $f_0\in H^2\ominus bzH^2$. $(f,b_1)=0$ implies $f_0(z_1)=0$, and hence $f_0=b_1f_1$ and $f_1\in H^2$, where $(\ ,\ )$ denotes the inner product in H^2. Since

$$0=(f_0,b_2)=(f_1,b_2\bar{b}_1),$$

$f_1(z_2)=0$ and $f_0(z_2)=0$. By the same argument, $f_0(z_j)=0$ for $j\geqq 3$ and hence $f\in bzH^2$. This implies that f is zero because $f\in H^2\ominus bzH^2$. This completes the proof of (5).

5. A characterization of strong outer functions

Several characterizations of strong outer functions are known (cf. [**8**, Theorem 3], [**4**, Theorem 8], [**3**, Chapter IV, Exercise 18] and [**7**]). See Theorem 1. In this section we give a characterization. For a real valued measurable function u on ∂U, if $f\in N$ and $\operatorname{Re} f=u$ on ∂U, $\tilde{u}$ denotes the real part of f on $\bar{U}$, that is, the harmonic extension of u, and $^*u$ denotes the imaginary part of $f-\tilde{f}(0)$ on $\bar{U}$.

If f is in H^1 and u is a nonnegative function in L^1 such that $|f(e^{i\theta})|\leqq u(e^{i\theta})$ almost everywhere on ∂U, then $|\tilde{f}(z)|\leqq \tilde{u}(z)$ on U. The inner outer factorization of f and the Poisson integrals of u and $|f|$ give the above inequality in U. If u is nonnegative then $u+i^*u$ belongs to $\bigcap_{p<1}H^p$ and is an outer function. Let $f(e^{i\theta})=(e^{i\theta}-1)^2$ and $u(e^{i\theta})=-2(e^{i\theta}-1)^2(e^{i\theta}+1)^{-2}(1+\cos\theta)$; then $u+i^*u=-2(e^{i\theta}-1)^2(e^{i\theta}+1)^{-1}$ is an outer function in $\bigcap_{p<1}H^p$ and $|f(e^{i\theta})|\leqq u(e^{i\theta})$ a.e. on ∂U. However it is not true that $|\tilde{f}(z)|\leqq\tilde{u}(z)$ on U. Hence the condition of the integrability of u is needed to show the inequality on U.

Theorem 6. *Let g be an outer function in H^1. The following are equivalent.*

(1) *g is a strong outer function.*

(2) *For every nonnegative function u on ∂U such that $u+i^*u$ is an outer function in N_+ and $|g(e^{i\theta})|\leqq u(e^{i\theta})$ a.e. on ∂U, we have $|\tilde{g}(z)|\leqq\tilde{u}(z)$ on U.*

(3) *For every nonnegative function u on ∂U such that $u+i^*u$ is an outer function in N_+ and $|g(e^{i\theta})|\leqq u(e^{i\theta})$ a.e. on ∂U, we have $|\tilde{g}(0)|\leqq\tilde{u}(0)$.*

Proof. The proof is suggested by the proof of [**3**, Chapter IV, Lemma 5.4].

(1) $\Rightarrow$ (2). Put $\phi=u+i^*u$. Let α be a measurable function on ∂U such that

$$|\alpha|\leqq\pi/2 \text{ a.e.} \quad \text{and} \quad \alpha=\arg\phi\,(\operatorname{mod} 2\pi).$$

Then $u=\operatorname{Re}\phi=|\phi|\cos\alpha$ and $|g/\phi|\leqq\cos\alpha$ a.e. on ∂U. Let $\psi=\exp(-^*\alpha+i\alpha)$ then ψ belongs to H^p for some $p<1$ by Zygmund's theorem (cf. [**3**, Chapter III, Corollary 2.6]) and

$$|\psi g/\phi|\leqq|\psi|\cos\alpha=\operatorname{Re}\psi \text{ a.e. on } \partial U$$

and $\mathrm{Re}\,\psi$ is in L^1. Since ϕ is an outer function, $\psi g/\phi$ is in N_+ and hence $\psi g/\phi$ belongs to H^1. Put $h=\psi g/\phi$. Then $\arg h=\arg g$ a.e. $(\mathrm{mod}\, 2\pi)$ and hence $h=\gamma g$ for some $\gamma>0$ because g is strongly outer. Thus $\psi=\gamma\phi$ and u is in L^1. By the remark above Theorem 6 $|\tilde{g}(z)|\leqq\tilde{u}(z)$ on U. This implies (2).

(2) $\Rightarrow$ (3) is trivial.

(3) $\Rightarrow$ (1). If g is not a strong outer function, then there exists an outer function h_0 in H^1 such that $\arg g=\arg h_{a,\gamma}$ a.e. $(\mathrm{mod}\, 2\pi)$ for any $|a|=1$ and $\gamma>0$ where $h_{a,\gamma}=\gamma(z-a)(1-\bar{a}z)h_0$. We can choose a and γ such that

$$\mathrm{Re}\, h_{a,\gamma}^{-1}(0)\tilde{g}(0)<-|\tilde{g}|(0).$$

Hence, if $u=|g|+\mathrm{Re}\, h_{a,\gamma}^{-1}g=h_{a,\gamma}^{-1}g(|g|+1)$ then

$$\tilde{u}(0)=|\tilde{g}|(0)+\mathrm{Re}\,\tilde{h}_{a,\gamma}^{-1}(0)\tilde{g}(0)<0\leqq|\tilde{g}(0)|.$$

On the other hand,

$$h_{a,\gamma}^{-1}g(|h_{a,\gamma}|+1)+ih_{a,\gamma}^{-1}g^*(|h_{a,\gamma}|+1)$$

is analytic and outer in N_+. This contradicts (3).

REFERENCES

1. A. Buerling, On two problems concerning linear transformations in Hilbert space, *Acta Math.* **81** (1949), 239–255.

2. K. de Leeuw and W. Rudin, Extreme points and extreme problems in H^1, *Pacific J. Math.* **8** (1958), 467–485.

3. J. Garnett, *Bounded Analytic Functions* (Academic Press, New York, N.Y., 1981).

4. E. Hayashi, The solution sets of extremal problems in H^1, *Proc. Amer. Math. Soc.* **93** (1985), 690–696.

5. E. Hayashi, The kernel of a Toeplitz operator, Integral E_q. *Operator Th.* **9** (1986), 589–591.

6. H. Helson, Large analytic functions, *Oper. Theory: Adv. Appl.* **43** (1989), 209–216.

7. H. Helson, Large analytic functions II, in *Analysis and Partial Differential Equations* (C. Sadosky, ed., Marcel Dekker, Basel, 1990).

8. T. Nakazi, Exposed points and extremal problems in H^1, *J. Funct. Anal.* **53** (1983), 224–230.

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