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AN APPROXIMATION THEOREM FOR EXTENDED PRIME SPOTS

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We introduce here a generalization to arbitrary fields of the prime spots of algebraic number theory, essentially by allowing absolute values to take the value ∞. The set of "extended" prime spots of a field admits a natural topology, and an approximation theorem is given here for compact sets of extended prime spots. Among the corollaries of the approximation theorem are the weak approximation theorem for absolute values [**13**, p. 8], Ribenboim's generalization of the approximation theorem for independent valuations [**14**, p. 136], Stone's approximation theorem for type V topologies [**16**, p. 20], and an approximation theorem for Harrison primes.

We now summarize the contents of the paper. After some preliminary definitions (§ 1) the approximation theorem is stated (§ 2). In the next four sections the approximation theorem is applied to obtain an approximation theorem for finite Harrison primes (§ 3), a calculation of the reduced Witt ring of a formally real algebraic extension of the rational numbers (§ 4), and after some preliminary definitions (§ 5), several approximation theorems in the literature (§ 6). The next four sections deal with the space of extended prime spots of a field. After some general observations (§ 7), we give necessary and sufficient conditions for a set of "non-Archimedean" extended prime spots (these are naturally bijective with finite Harrison primes) to be compact (§ 8). We next show that compact sets of non-Archimedean extended prime spots can be lifted through quadratic field extensions (§ 9). The space of extended prime spots of a simple transcendental extension of the rational numbers is examined (§ 10). Finally, we give (mildly) idelic and adelic interpretations of certain compatibility conditions appearing in the statement of the approximation theorem. The proof of the approximation theorem is given in an appendix.

Our hope is that the concepts and results of this paper will be useful in the kind of arithmetic theory of fields visualized in [**6**], and, in particular, in the global part of such a theory. Many "arithmetic" approximation theorems in the literature approximate over finite sets. We have been partly motivated by our feeling that a generalized arithmetic of fields will require approximations over appropriate possibly infinite sets. (For other uses of extended prime spots in the arithmetic of fields see [**3**], where among other things, generalizations

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of the Hasse-Minkowski theorem on quadratic forms are discussed in terms of the "completions" of a field at its extended prime spots.)

Throughout this paper, F will denote a field, regarded as having the discrete topology. $F^{\times}$ denotes the multiplicative group of non-zero elements of F. $\mathbf{Z}$, $\mathbf{Q}$ and $\mathbf{R}$ denote the integers, rationals, and real numbers, respectively. Finally, $A \backslash B$ denotes the set of elements of the set A which are not elements of the set B.

1. The space $S(F)$ of prime values of F. Recall that a "locally finite" field is one which is the union of its finite subfields. Such a field admits only the trivial absolute value (i.e., the absolute value of every non-zero element is 1).

(1.1) *Definition.* A *prime value* on F is a map $\phi : F \to R \cup \infty$ which is the composition of a place of F into either the complex numbers or a locally finite field (say, $\pi : F \to k \cup \infty$) with the usual absolute value on k. (We understand here that $\phi(a) = \infty$ if $\pi(a) = \infty$.)

A more elementary and less ad hoc definition of prime value will be given in § 5 (see especially (5.4)).

Let $S(F)$ denote the set of all prime values of F, given the coarsest topology with

$$W(a) = \{\phi \in S(F) : \phi(a) < 1\}$$

open and closed for all $a \in F$.

Example. If F is a global field, then the prime values of F are naturally bijective with the prime spots [**13**, § 11.D] of F. In this case, $S(F)$ has the discrete topology.

2. The approximation theorem. The theorem below describes when a continuous function from $S(F)$ to F can be approximated (in an additive and a multiplicative sense) on a compact subset of $S(F)$ by a constant function.

(2.1) THEOREM. *Let C be a compact subset of $S(F)$ and let $f : C \to F$ be a continuous function. Let $0 < \epsilon < 1$, $\epsilon \in \mathbf{R}$.*

(A) (*Additive approximation*). *There exists $a \in F$ with $\phi(a - f(\phi)) < \epsilon$, for all $\phi \in C$, if and only if for all ϕ and ψ in C,*

$$f(\phi) + \phi^{-1}(R) \cdot \psi^{-1}(R) = f(\psi) + \phi^{-1}(R) \cdot \psi^{-1}(R).$$

(B) (*Multiplicative approximation*). *Suppose that $f(C) \subseteq F^{\times}$. There exists $a \in F^{\times}$ with $\phi(1 - a^{-1} f(\phi)) < \epsilon$, for all $\phi \in C$, if and only if for all ϕ and ψ in C,*

$$f(\phi) \cdot \phi^{-1}(R) \cdot \psi^{-1}(R) = f(\psi) \cdot \phi^{-1}(R) \cdot \psi^{-1}(R). \tag{1}$$

Recall that for any valuation ring A of F, the set of non-zero principal fractional ideals $\{aA_\phi : a \in F^{\times}\}$ forms in a natural way an ordered group which may be identified with the value group of the valuation canonically associated with A [**14**, p. 34]. The condition (1) above is, thus, essentially the "compatibility" of [**14**, p. 127] (for, $\phi^{-1}(R) \cdot \psi^{-1}(R)$ is the smallest valuation ring containing the valuation rings $\phi^{-1}(R)$ and $\psi^{-1}(R)$). Note that (1) holds for all $\phi, \psi \in C$ if and only if there exists $a \in F$ with $f(\phi) \cdot \phi^{-1}(R) = a \cdot \phi^{-1}(R)$ for all $\phi \in C$ (compare with [**14**, p. 135]). We will interpret the compatibility conditions in (A) and (B) above further in § 11. So as not to interrupt our exposition with a lengthy computational argument, we refer the reader to the appendix for the proof of (2.1).

(2.2) *Remark.* If F is not locally finite, then it admits a non-discrete field topology such that Theorem (2.1) holds with this topology on F. This suggests the possibility of a generalization of (2.1) to topological fields.

3. Harrison primes. We now interpret Theorem (2.1) for finite Harrison primes [**6**]. An equivalent interpretation exists for the minimal valuation rings of F since the finite Harrison primes are simply their maximal ideals [**6**, p. 20].

Give the set $X(F)$ of finite Harrison primes of F the coarsest topology with $\{P \in X(F) : a \in P\}$ open for all $a \in F$. (Caution: this is not the topology of [**6**].) For $P \in X(F)$, let $A_P = \{a \in F : a \cdot P \subseteq P\}$ be the associated valuation ring.

(3.1) Corollary. *Let $f : C \to F$ be continuous, where C is a compact subset of $X(F)$. There exists $a \in F$ with $a - f(P) \in P$, for all $P \in C$, if and only if for all P, $T \in C$,*

$$f(P) - f(T) \in A_P \cdot A_T.$$

An analogous result follows from (B) of Theorem (2.1). To prove the corollary, map $X(F) \to S(F)$ by taking $P \in X(F)$ to the map

$$a \rightsquigarrow \begin{cases} 0, & \text{if } a \in P \\ 1, & \text{if } a \in A_P \backslash P \\ \infty, & \text{otherwise.} \end{cases}$$

This map is a homeomorphism of $X(F)$ *into* $S(F)$. (For each $a \in F^\times$, $\{P \in X(F) : a \in P\}$ is closed since it equals [**6**, p. 6]

$$X(F)\backslash(\{P : a^{-1} \in P\} \cup (\cup_{n>0}\{P : a^n - 1 \in P\})).)$$

The corollary follows immediately.

We have shown above that the finite Harrison primes are naturally bijective with the prime values factoring through locally finite fields (cf. Definition (1.1)). The remaining prime values are bijective with places into the complex numbers, identifying two such if they differ by complex conjugation.

The presence of prime values associated with arbitrary Harrison primes of F [**7**] allows one to include some infinite primes in a generalization of the above corollary. We leave this to the interested reader.

4. Reduced Witt rings. Denote the Witt ring [**17**] of F by Witt(F) and its nil radical by Nil(F). If F is formally real, Nil(F) is the set of elements of Witt(F) of finite additive order [**10**]. Let $\mathscr{O}(F)$ denote the set of orderings of F with the "Harrison topology"; i.e., the coarsest with $\{\leqq \in \mathscr{O}(F) : a \leqq 0\}$ open for all $a \in F$. $\mathscr{O}(F)$ is compact, totally disconnected and Hausdorff [**10**].

(4.1) THEOREM. *Suppose that F is a formally real (possibly infinite dimensional) algebraic extension of the rational numbers. Then the reduced Witt ring of F, i.e.,* Witt(F)/Nil(F), *is isomorphic to* $\mathbf{Z}\cdot 1 + C(\mathscr{O}(F), 2\mathbf{Z})$ *(considered as a subring of* $C(\mathscr{O}(F), \mathbf{Z})$*, the ring of continuous functions from* $\mathscr{O}(F)$ *to* $\mathbf{Z}$*).*

The isomorphism of (4.1) is induced by the function taking a diagonalized quadratic form over F to the map $\mathscr{O}(F) \to Z$ which assigns to $\leqq \in \mathscr{O}(F)$ the number of positive (with respect to $\leqq$) coefficients of the form minus the number of its negative coefficients.

To prove the corollary, recall that for each ordering of F there is a unique order embedding of F into $\mathbf{R}$, and thence a prime value. The resulting map $\mathscr{O}(F) \to S(F)$ is a homeomorphism. Since $\mathscr{O}(F)$ is compact, we may use Theorem (2.1) to apply [**10**, Corollary 13].

The crucial approximation in the proof of (4.1) can also be accomplished by an application to $\mathscr{O}(F)$ of the Stone-Weierstrass theorem; this trick is due to Harrison [**6**, p. 60].

The Pythagorean (i.e. sums of squares are themselves squares) formally real fields are exactly those with torsion free Witt groups [**15**, 2.2.4]. Thus, Theorem (4.1) gives the structure of Witt(F) in terms of $\mathscr{O}(F)$ for any formally real absolutely algebraic Pythagorean field. Sometimes we can say a bit more.

(4.2) COROLLARY. *Suppose that F is the Pythagorean closure of a formally real global field. Then $\mathscr{O}(F)$ is homeomorphic to the Cantor set 2^ω and* Witt(F) *is isomorphic to* $\mathbf{Z}\cdot 1 + C(2^\omega, 2\mathbf{Z})$.

To show that $\mathscr{O}(F)$ is homeomorphic to 2^ω whenever F is the Pythagorean closure of a formally real *finitely generated* field, apply [**8**, Corollary 2–98].

5. Extended absolute values. We enlarge our frame of reference to facilitate relating Theorem (2.1) to some approximation theorems in the literature.

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(5.1) *Definition.* An *extended absolute value* on F is a map $\phi : F \to \mathbf{R} \cup \infty$ such that for all $a, b \in F$ we have $\phi(0) = 0$, $\phi(1) = 1$, $\phi(a) \geqq 0$, $\phi(a + b) \leqq \phi(a) + \phi(b)$, and $\phi(ab) = \phi(a)\phi(b)$ (when defined). (We do not define $0 \cdot \infty$ or $\infty \cdot 0$.)

(5.2) *Remark.* The extended absolute values of F are exactly the compositions of places on F with absolute values on the residue class fields of the places. (For, if ϕ is an extended absolute value, then $\phi^{-1}(R)$ is a valuation ring and ϕ induces an absolute value on its residue class field.) In particular, prime values and absolute values are extended absolute values. Also, the valuation rings of F are naturally bijective with the extended absolute values of F mapping into $\{0, 1, \infty\}$. (Assign to such an extended absolute value the inverse image of $\mathbf{R}$.)

Many of the analogous concepts and results for valuations and absolute values can be unified using the language of extended absolute values. An example of this is the next definition, which unifies the concepts of equivalence for valuations and absolute values. (For more in this direction, see [**3**]; we mention, in particular, that the familiar ramification formula $\sum e_i f_i \leqq n$ generalizes to extended absolute values.)

(5.3) *Definition.* Given an extended absolute value ϕ on F, write $D_\phi = \{a \in F : \phi(a) < 1\}$. Write $\phi \sim \psi$ when $D_\phi = D_\psi$ and $\phi \leqq \psi$ when $D_\phi \subseteq D_\psi$ (ϕ and ψ being extended absolute values on F). Then $\sim$ is an equivalence relation and $\leqq$ a preorder. We call an equivalence class of maximal extended absolute values an *extended prime spot.*

(5.4) *Remark.* Each equivalence class of extended absolute values contains a unique member ϕ with either $\phi(1 + 1) = 2$ or $\phi(F) \subseteq \{0, 1, \infty\}$. We call such ϕ *normalized. The prime values of F are exactly the normalized maximal extended absolute values, and, hence, are naturally bijective with the extended prime spots of F.* (Proof. Let ψ be an extended absolute value, with $\bar{\psi}$ the absolute value induced on the residue class field of $\psi^{-1}(\mathbf{R})$. If $\bar{\psi}$ is Archimedean, let ϕ be the composition of the place induced by $\psi^{-1}(\mathbf{R})$ with the ordinary [**13**, p. 19] absolute value equivalent to $\bar{\psi}$. If $\bar{\psi}$ is not Archimedean, then there is a unique valuation ring with maximal ideal D_ψ; let ϕ be the associated extended absolute value (cf. Remark (5.2)). Then ϕ is clearly the unique normalized extended absolute value equivalent to ψ. The italicized statement above follows from the fact that ϕ is maximal if and only if either $D_\psi \in X(F)$ or $\bar{\phi}$ is Archimedean (use [**13**, 11:4a] to show "Archimedean" extended absolute values are maximal; the rest is elementary).)

6. Corollaries to the approximation theorem. Given an extended absolute value ϕ on F, set $A_\phi = \phi^{-1}(\mathbf{R})$.

(6.1) LEMMA. *Let T be a finite set of pairwise incomparable (with respect to $\leqq$) normalized extended absolute values. Suppose that we are given indexed sets $(e_\phi)_{\phi \in T} \subseteq F^\times$ and $(a_\phi)_{\phi \in T} \subseteq F$ and a positive $\epsilon \in \mathbf{R}$.*

(A) *There exists $e \in F^\times$ with $e \cdot A_\phi = e_\phi \cdot A_\phi$, for all $\phi \in T$, if and only if for all $\phi, \psi \in T$,*

$$e_\phi \cdot A_\phi \cdot A_\psi = e_\psi \cdot A_\psi \cdot A_\phi.$$

(B) *Suppose that the condition of* (A) *holds. Then there exists $a \in F$ with*

$$-\epsilon < 1 - \phi(e_\phi^{-1}(a_\phi - a)) < \epsilon,$$

for all $\phi \in T$, if and only if for all $\phi, \psi \in T$,

$$a_\phi - a_\psi \in e_\phi \cdot A_\phi \cdot A_\psi.$$

Proof. We use Remarks (5.2) and (5.4) implicitly.

For each $\phi \in T$, we can pick a prime value $\phi' \geqq \phi$. Note that $A_{\phi'} \subseteq A_\phi$. If ϕ and ψ are distinct elements of T, then $A_\phi \cdot A_\psi = A_{\phi'} \cdot A_{\psi'}$. (Subproof. We may assume that $A_\phi \subseteq A_\psi$; for, if A_ϕ and A_ψ are incomparable, then the claim follows from elementary valuation theory. If $\psi(F) \subseteq \{0, 1, \infty\}$, then

$$D_\phi \supseteq \phi^{-1}(0) \supseteq \psi^{-1}(0) = D_\psi,$$

contradicting incomparability. Hence, ψ is maximal, so $\psi = \psi'$ and we have finished.)

The condition for the existence of e is clearly necessary. Suppose that it holds. Using the above paragraph, we apply Theorem (2.1B) to find $e \in F^\times$ with $\phi'(1 - e^{-1}e_\phi) < \epsilon/3$ for all $\phi \in T$. We may assume that $\epsilon < 1$. Then, $e \cdot A_\phi (= eA_{\phi'} \cdot A_\phi) = e_\phi A_\phi$ for all $\phi \in T$. One now checks that the condition for the existence of a is necessary; suppose, conversely, that it holds. Again apply (2.1) to find $b \in F$ with $\phi'(b - e^{-1}a_\phi) < \epsilon/3$ for all $\phi \in T$. Set $a = be + e$. Then, for all $\phi \in T$,

$$1 - \epsilon < \phi'(e_\phi^{-1}(a - a_\phi)) < 1 + \epsilon.$$

Let $\phi \in T$. If $\phi = \phi'$, we have finished. Otherwise, $\phi(F) \subseteq \{0, 1, \infty\}$ and $\phi(e_\phi^{-1}(a - a_\phi)) = 1$. This completes the proof.

(6.2) COROLLARY (approximation theorem for independent valuations). *Let $A_1, \ldots, A_n$ be independent valuation rings of F. For any $a_1, \ldots, a_n \in F$ and $e_1, \ldots, e_n \in F^\times$, there exists $a \in F$ with $(a - a_i)A_i = e_i A_i$ for $1 \leqq i \leqq n$.*

Proof. Let T be the set of extended absolute values canonically associated with the valuation rings A_i (cf. Remark (5.2)). Note that $A_i A_j = F$ if $i \neq j$. Now apply (6.1).

A similar (but easier) argument gives P. Ribenboim's approximation theorems for incomparable valuations [**14**, pp. 135–136]. Some cases of [**18**, Proposition (2.5)] can be handled by Theorem (2.1). On the other hand, the results of [**18**] cannot in general be applied to our situation since "most" fields admit compact sets of finite Harrison primes (and, hence, prime values, cf. § 3) whose associated equivalence classes of valuations do not admit an "M_Γ-choix rendant les fonctions $f_{\mathbf{R}x}$ localement constante". (See (10.7).)

(6.3) COROLLARY (weak approximation theorem for absolute values). *Let* $|\ |_1, \ldots, |\ |_n$ *be inequivalent absolute values on* F. *Let* $a_1, \ldots, a_n \in F$ *and* $\delta > 0$. *Then there exists* $a \in F$ *with* $|a - a_i|_i < \delta$ *for* $1 \leqq i \leqq n$.

Proof. Apply (6.1) to the unique normalized extended absolute values equivalent to the $|\ |_i$. (Let $m = 1$ or 2 according as F has characteristic 2 or not. Pick $\epsilon < 1$ and $e_i \in F^\times$ with $|me_i|_i < \delta$, $1 \leqq i \leqq n$.)

We mention one more corollary. Given an extended absolute value ϕ on F, the sets $a + bD_\phi$ ($a \in F$, $b \in F^\times$) together form a base for a topology on F. The topologies obtained are exactly the type V topologies [**4**]. The topologies associated with inequivalent nontrivial extended absolute values ϕ and ψ are distinct if and only if $A_\phi \cdot A_\psi = F$. An immediate consequence is the theorem of A. L. Stone [**16**, Theorem 3.4]: *Let* $T_1, \ldots, T_n$ *be distinct nontrivial type V topologies on* F. *Let* U_i *be a nonempty open set in* T_i, $1 \leqq i \leqq n$. *Then* $\bigcap_{1 \leqslant i \leqslant n} U_i$ *is nonempty.*

7. $S(F)$: **generalities.** For any set A, let $|A|$ denote its cardinality. If F admits a rank two valuation, then $|S(F)| = 2^{|F|}$. (Problem: can $|S(F)|$ assume every cardinal less than $2^{\aleph_0}$?) $S(F)$ is Hausdorff and totally disconnected; therefore, it is completely regular and uniformizable (cf. § 8). $S(F)$ is separable and metrizable if and only if F is countable. (See [**9**] for the basic topology used here.)

8. Compact subsets of $X(F)$. For $P \in X(F)$, let $k_P = A_P/P$; k_P is a locally finite field [**6**, Proposition 1.3].

(8.1) THEOREM. *Let* C *be a closed subset of* $X(F)$. *Then* C *is compact if and only if for each* $a \in F$ *there is a positive integer* $n(a)$ *with* $a^{n(a)} - 1 \in P$, *whenever* $P \in C$ *and* $a \in A_P \backslash P$.

Note that if C is any subset of $X(F)$ satisfying the condition of the last sentence of (8.1), then its closure also satisfies it, and, hence, is compact. Before proving (8.1) we give a corollary generalizing [**13**, 33:7].

(8.2) COROLLARY. *For each positive integer* n, $\{P \in X(F) : |k_P| < n\}$ *is compact.*

Proof of corollary. Identify $X(F)$ with its canonical image in $S(F)$ (cf. § 3); thus, for $a \in F$, $W(a) \cap X(F) = \{P \in X(F) : a \in P\}$. The set in question is closed; indeed, it is exactly

$$X(F) \cap \Big(\bigcup_s \bigcap_{a \in F} (W(a) \cup W(a^{-1}) \cup W(a^{s-1} - 1))\Big),$$

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where s ranges over the prime powers less than n (recall that the multiplicative group of a finite field is cyclic). Now apply (8.1) with $n(a) = n!$, for all $a \in F$.

(8.3) *Remark* If F is finitely generated and is neither global nor locally finite, then there exist $2^{\aleph_0}$ primes $P \in X(F)$ with k_P finite. (Use induction and [**1**] or [**11**].)

We now prove (8.1). Necessity is obvious, since for each $a \in F$, $S(F)$ (and, hence, C) is contained in

$$W(a) \cup W(a^{-1}) \cup (\bigcup_{n>0} W(a^n - 1)).$$

(We are identifying $X(F)$ with a subset of $S(F)$ as in the proof of (8.2).) We prove sufficiency by showing that C is complete in a uniformity on $S(F)$ which makes $S(F)$ (and, hence, C) totally bounded (cf. [**9**, p. 198]).

For each finite set $E \subseteq F$, let

$$\mathscr{W}(E) = \{(\phi, \psi) \in S(F) \times S(F) : D_\phi \cap E = D_\psi \cap E\}.$$

It is routine to verify that the sets $\mathscr{W}(E)$ together form a base for a uniformity on $S(F)$ inducing the given topology on $S(F)$. For each finite set $E \subseteq F$, $S(F)$ is the union of the (finite) collection of sets of the form

$$\{\phi \in S(F) \mid E \cap D_\phi = T\} \quad (T \subseteq E).$$

It follows that $S(F)$ is totally bounded. It remains to show that C is complete.

Let $(P_d : d \in D)$ be a Cauchy net in C (D a directed set). Let P be the set of elements of F eventually in the P_d (i.e., $a \in P$ if and only if for some $d_0 \in D$, $a \in P_d$ for all $d > d_0$). P is easily checked to be a preprime and, indeed, the maximal ideal of a valuation ring of F; call it A_P (for economy, use [**6**, lemma 3.1]). If $a \in A_P \backslash P$, then $a^{n(a)} - 1 \in P$; it follows that $P \in X(F)$. P is now checked to be the limit of the P_d. The theorem is proved.

Note. $X(F)$ is complete in a natural uniformity. One gets another proof of (8.1) by showing that the condition on C implies that it is totally bounded in this uniformity.

9. Quadratic extensions of fields. Let K be a quadratic field extension of F. Let $\pi : X(K) \to X(F)$ be the natural map (namely, intersection with F).

(9.1) THEOREM. *π is open and continuous.*

A generalization of (9.1) to $S(F)$ will be remarked below. Before proving (9.1), we give a corollary.

(9.2) COROLLARY. *If C is a compact (respectively, dense) subset of $X(F)$, then $\pi^{-1}(C)$ is a compact (respectively, dense) subset of $X(K)$.*

Proof of corollary. The "dense" part is obvious. Suppose that C is compact. If K/F is inseparable, then π is an (onto) homeomorphism [**14**, p. 175]. Now suppose that K/F has Galois group $\{1, \sigma\}$. Let $\mathscr{U}$ be an open cover of $\pi^{-1}(C)$. For each $P \in \pi^{-1}(C)$, pick a neighbourhood $U_P \in \mathscr{U}$ of P. Set

$$\mathscr{U}' = \{U_P \cap \sigma U_{\sigma P} : P \in \pi^{-1}(C)\}.$$

(We are also denoting by σ the homeomorphism $X(K) \to X(K)$ induced by σ.) Then, $\mathscr{U}'$ is an open cover of $\pi^{-1}(C)$ refining $\mathscr{U}$ and with $\sigma\mathscr{U}' = \mathscr{U}'$. Since π is open (Theorem (9.1)) and C is compact, there exists a finite subset T of $\pi^{-1}(C)$ with

$$C \subseteq \bigcup_{P \in T} \pi(U_P \cap \sigma U_{\sigma P}).$$

Consequently,

$$\pi^{-1}(C) \subseteq \bigcup_{P \in T} ((U_P \cap \sigma U_{\sigma P}) \cup (\sigma U_P \cap U_{\sigma P})).$$

The corollary is proved.

When $F = \mathbf{Q}(x)$, the set of $P \in X(F)$ with k_P finite and A_P Noetherian gives an example of a dense subset of $X(F)$ (see (10.3) below).

We now prove (9.1). The continuity of π is obvious. As in the proof of (9.2), we may suppose that the extension K/F is separable and that the nontrivial F-automorphism of K, call it σ, operates on $X(K)$. For E equal to either K or F, and A a finite subset of E, let $W_E(A)$ denote $\{P \in X(E) : A \subseteq P\}$; the sets $W_E(A)$ form a basis for $X(E)$.

Let A be a finite subset of K and let $T \in W_K(A)$. It suffices to find a finite subset E of F with

$$T \in W_K(E) \subseteq W_K(A) \cup \sigma W_K(A)$$

since, then,

$$\pi(T) \in W_F(E) \subseteq \pi(W_K(A)).$$

Let $\delta \in K \backslash F$, so $K = F[\delta]$. Let N and tr denote the norm and trace maps from K to F. We may suppose that A is not contained in F (otherwise, simply take $E = A$). We can pick $\alpha_0 = a_0 + b_0\delta \in A$ with the fractional ideal $b_0 \cdot A_T$ as large as possible. If $\sigma(\alpha_0) \in T$, then for each $\alpha = a + b\delta \in A$ we have

$$\sigma(\alpha) = \alpha + bb_0^{-1}(\sigma(\alpha_0) - \alpha_0) \in T,$$

so it suffices to set

$$E = \{\mathrm{N}(\alpha) : \alpha \in A\} \cup \{\mathrm{tr}(\alpha) : \alpha \in A\}.$$

Now, suppose that $\sigma(\alpha_0) \notin T$. Then

$$\{0\} \neq \sigma(\alpha_0) \cdot A_T = \mathrm{tr}(\alpha_0) \cdot A_T.$$

Hence, for each $\alpha = a + b\delta \in A$, we have

$$b\,\mathrm{N}(\alpha_0)/b_0\,\mathrm{tr}(\alpha_0) \tag{2}$$

and

$$a + b(-a_0/b_0) \quad (= \alpha - (b/b_0)\alpha_0) \tag{3}$$

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both in T. Let E be the set of elements of the form (2) or (3). Let $P \in W_K(E)$; we show that $P \in W_K(A) \cup W_K(\sigma A)$. We may assume that $\alpha_0 \cdot A_P \subseteq \sigma(\alpha_0) \cdot A_P$ (otherwise, replace P by σP), so $\mathrm{tr}(\alpha_0) \cdot A_P \subseteq \sigma(\alpha_0) \cdot A_P$. Each $a + b\delta \in A$ equals

$$(a + b(-a_0/b_0)) + (b\,\mathrm{N}(\alpha_0)/b_0\,\mathrm{tr}(\alpha_0)) \cdot (\mathrm{tr}(\alpha_0)/\sigma(\alpha_0)),$$

and is, therefore, in P. The theorem is proved.

(9.3) *Remark.* Theorem (9.1) and its corollary generalize to $S(F)$ if we modify the topology on $S(F)$ so as to require only that the sets $W(a)$ be open (cf. § 1). This is the topology one gets by considering $S(F)$ as a subspace of $[0, \infty]^F$ (given the product topology; here, $[0, \infty]$ denotes the closed interval in $\mathbf{R} \cup \infty$, with the order topology). In the new topology, $X(F)$ is still homeomorphically embedded into $S(F)$ but now $S(F) \backslash X(F)$ is compact. Consequently, this topology appears more suitable for generalizing Theorem (4.1) (if an appropriate approximation theorem can be proved).

10. Example: $\mathbf{Q}(x)$. We sketch an example which has frequently guided us in this paper. Let $F = \mathbf{Q}(x)$ be a simple transcendental extension of the rational numbers.

Let C denote the set of complex numbers $a + bi$ with $b \geqq 0$ $(a, b \in \mathbf{R})$ together with ∞ (so C lies in the extended complex plane). For each $c \in C$, there is a place from $\mathbf{Q}(x)$ to the complex numbers with x mapped to c. This correspondence induces a bijection from C to the set of "Archimedean" prime values of $\mathbf{Q}(x)$ (i.e, $S(\mathbf{Q}(x)) \backslash X(\mathbf{Q}(x))$). The bijection carries the set $W(f)$ (where $f \in \mathbf{Q}(x)$) to the inverse image of the open unit disc under f, where f is thought of as a rational function on C. The topology induced on $\mathbf{R}$ by the bijection is the coarsest refinement of the order topology on $\mathbf{R}$ with all algebraic numbers isolated; in this topology, $\mathbf{R}$ obviously admits uncountable compact subsets.

The set $X(\mathbf{Q}(x))$ was computed in [**1**]. We restate the computation in slightly modified language.

Give $\Gamma = \mathbf{Q} \oplus \mathbf{R} \oplus \mathbf{Q}$ (the direct sum of the ordered additive groups) the lexicographic order. Give Γ the coarsest T_1 topology with

$$\{\gamma \in \Gamma : \gamma \geqq (0, q, 0)\}$$

and $\{\gamma \in \Gamma : \gamma \leqq (0, q, 0)\}$ open for all $q \in \mathbf{Q}$. (The reader may find it helpful to notice that each element of Γ cannot be separated from exactly one element of Γ of one of the forms $(q \in \mathbf{Q}, r \in \mathbf{R})$

$$(-1, 0, 0),\ (0, q, -1),\ (0, r, 0),\ (0, q, 1),\ (1, 0, 0).$$

If $r = q$, the above elements are listed in ascending order. The set of such elements is compact and Hausdorff).

p always denotes a prime number. $\mathbf{Z}_p^{\text{alg}}$ will denote the algebraic closure of the field $\mathbf{Z}/p\mathbf{Z}$. Let $0 \leqq n \leqq \infty$. A *presignature* (of *length* n and *characteristic* p) is a sequence in $\mathbf{Z}_p^{\text{alg}} \times \Gamma$ with n terms, i.e., of the form

$$\langle(\theta_i, q_i)\rangle_{0 \leqq i < n} \text{ (abbreviated: } \langle\theta_i, q_i\rangle_{i<n}).$$

Let $\mathscr{P}$ denote the set of presignatures (of all lengths and characteristics).

Γ is completely regular and, hence, admits a uniformity; pick one and call it $\mathscr{U}(\Gamma)$ (cf. [**9**, p. 188]; the interested reader can easily construct such a uniformity). Let $U \in \mathscr{U}(\Gamma)$ and let $m > 0$ ($m \in \mathbf{Z}$). Let $\mathscr{W}(U, m)$ denote the set of pairs

$$(\langle\theta_i, q_i\rangle_{i<n}, \langle\theta_i', q_i'\rangle_{i<n'}) \in \mathscr{P} \times \mathscr{P},$$

where n and n' are larger than m, $(q_i, q_i') \in U$ for all $i \leqq m$, both presignatures have the same characteristic (say, p), and there exists an automorphism of $\mathbf{Z}_p^{\text{alg}}$ carrying θ_i to θ_i' for all $i < m$. The sets $\mathscr{W}(U, m)$ together form a base for a uniformity on $\mathscr{P}$. Call two presignatures *equivalent* if and only if they cannot be separated in the uniform topology on $\mathscr{P}$.

(10.1) *Definition.* A *signature* is an equivalence class of presignatures $S = \langle\theta_i, q_i\rangle_{i<n}$ having

(i) for all $m < n$, $q_m \notin 0 \oplus \mathbf{Q} \oplus 0$ if and only if $\theta_m = 0$, and $\theta_m = 0$ if and only if $m + 1 = n$;

(ii) $\langle q_m/e_m f_m\rangle_{m<n}$ is strictly increasing.

Here, e_m denotes the index of $0 \oplus \mathbf{Z} \oplus 0$ in $(0 \oplus \mathbf{Z} \oplus 0) + \sum_{i<m}\mathbf{Z} \cdot q_i$ and f_m denotes the dimension of $\mathbf{Z}/p\mathbf{Z}[\{\theta_i : i < m\}]$ over $\mathbf{Z}/p\mathbf{Z}$ (where p is the characteristic of S).

Let $\mathscr{S}$ denote the set of signatures, given the topology inherited from $\mathscr{P}$. Now, a different definition of "signature" was given in [**1**]. However, $\mathscr{S}$ is naturally bijective with the set of "signatures" of [**1**, Corollary 6.5, taking $F = Q$]. Treating this bijection as an identification, we have

(10.2) PROPOSITION. *The bijection of* [**1**, *Corollary* 6.5] *gives a homeomorphism of* $\mathscr{S}$ *onto* $X(\mathbf{Q}(x))$.

Proof (sketch). Let $\Phi : \mathscr{S} \to X(\mathbf{Q}(x))$ be our bijection. We use [**1**, Lemma 6. 1] implicitly.

Suppose that $P \in X(\mathbf{Q}(x))$ has $\Phi^{-1}(P) = S$ with $n = \infty$ (notation for S as in Definition (10.1)). Let $1 < m < \infty$. Write $(e_m f_m/e_{m-1}f_{m-1})q_m$ in the form $(0, r/s, 0)$ where $r, s \in \mathbf{Z}$, $s > 0$. If g_m is the generator of $\langle\theta_i, q_i\rangle_{i<m}$, then $\Phi^{-1}(W(p, p^{-r}g_m^{\,s}))$ is contained in the set of signatures with the same first m terms as S [**1**, Lemma 3.5]. That is, Φ^{-1} is continuous at P. The proof that Φ^{-1} is continuous in other cases is similar.

Now suppose that S (notation again as in (10.1)) is a signature with $n = \infty$ and such that $e_m f_m \to \infty$ as $m \to \infty$. Let A be a finite subset of $\Phi(S)$. Pick

$m < \infty$ with $e_m f_m$ larger than the degrees of the numerators and denominators of the elements of A (thought of as quotients of polynomials in x). Then the set of signatures with the same first m terms as S is mapped into $W(A)$ [**1**, Supplement 4.2]. That is, Φ is continuous at S. The proof that Φ is continuous in general is similar (but also uses [**11**, p. 372]).

To obtain more than purely topological facts about $X(\mathbf{Q}(x))$ using Proposition (10.2), one must use special properties of the map Φ. Two such facts are given below.

(10.3) COROLLARY. *The following sets are dense in* $X(\mathbf{Q}(x))$*:*

(A) *The set of* $P \in X(\mathbf{Q}(x))$ *with* k_P *finite and* A_P *Noetherian.* (*Recall that a nontrivial valuation ring is Noetherian if and only if its associated valuation is discrete rank one.*)

(B) *The set of* $P \in X(\mathbf{Q}(x))$ *with* k_P *algebraically closed and*

$$\{a \cdot A_P : a \in \mathbf{Q}(x)^\times\}$$

divisible. (*We are identifying* $\{a \cdot A_P : a \in \mathbf{Q}(x)^\times\}$ *with the "value group" of* A_P.)

This corollary follows from the proposition and [**1**, Remark 6.9]. It is easy to pick out other dense sets.

Call a polynomial $f \in Q[x]$ *p-irreducible* when it is irreducible over $\mathbf{Q}_p$ (= the p-adic numbers). The next corollary was illustrated in the proof of (10.2).

(10.4) COROLLARY. $X(\mathbf{Q}(x))$ *admits a basis of sets of the form* $W(p^s g^t, p^u g^v)$, *where g is a p-irreducible polynomial and* $s, t, u, v \in \mathbf{Z}$.

Proof. Apply (10.2) and [**1**, Lemma 3.5].

(10.5) *Remark.* Proposition (10.2) and its two corollaries generalize in a straightforward way when $\mathbf{Q}$ is replaced by any global field.

(10.6) *Remark.* Suppose that F is a locally finite field, an arbitrary algebraic extension of a global field, or a simple transcendental extension of an arbitrary algebraic extension of a global field. Let $t - 1$ be the transcendence degree of F if F has zero characteristic, and let t be the transcendence degree, otherwise. Then $X(F)$ admits a basis of sets of the form $W_F(E)$ where E has at most t elements. Are there other fields with this property?

(10.7) *Remark.* We give an example mentioned in § 6. Let $\langle r_i \rangle_{i<\infty}$ be a strictly increasing sequence of irrational real numbers converging to an irrational real number r_∞. For $0 \leqq i \leqq \infty$, let $v^i : \mathbf{Q}(x) \to \mathbf{R} \cup \infty$ be the unique valuation (additively written) with $v^i(2) = 1$, $v^i(x) = r_i$. Then the set of finite Harrison primes associated with the v^i is compact. However, the family M of equivalence classes of the valuations v^i does not admit an M_Λ-choice (for any ordered group Λ) with $f_{x\mathbf{R}}$ and $f_{2\mathbf{R}}$ locally constant at v_∞ (cf. [**18**]).

11. Compatibility. We interpret further the "compatibility" conditions in Theorem (2.1). Our motivation is a connection with adeles and ideles.

Let $S(F)^*$ denote the set $S(F)$ given the coarsest topology having $W(a)$ closed for all $a \in F$. Give

$$\Gamma_F = \{aA_\phi : a \in F^\times, \phi \in S(F)\}$$

the coarsest topology with $\{aA_\phi : \phi \in S(F)\}$ open for all $a \in F^\times$. Thus, one may think of Γ_F as the disjoint union of the value groups of the valuation rings A_ϕ, given the "open path topology".

(11.1) *Definition.* A continuous map $f : S(F) \to F^\times$ is an *I-map* when the induced map $S(F)^* \to \Gamma_F$ (namely, $\phi \rightsquigarrow f(\phi) \cdot A_\phi$) is also continuous.

(11.2) PROPOSITION. *Let f be as in Theorem* (2.1B). *Then*

$$f(\psi)A_\phi A_\psi = f(\phi)A_\phi A_\psi,$$

for all $\phi, \psi \in C$, if and only if f is the restriction to C of an I-map.

Proof. Suppose that g is an I-map extending f. Let $\phi, \psi \in C$. Let k denote the residue class field of $A_\phi \cdot A_\psi$. Each $\phi \in S(k)$ induces a $\phi' \in S(F)$ (by composition with the place $F \to k \cup \infty$). Using elementary valuation theory, we see that the map $\phi \rightsquigarrow \phi'$ is a homeomorphism from $S(k)^*$ onto

$$L = \{\theta \in S(F)^* : A_\theta \subseteq A_\phi \cdot A_\psi\}.$$

Let $X(k)^*$ denote the set $X(k)$ with the topology of [**6**, p. 10]. The natural map $X(k) \to S(k)$ (cf. § 3) maps $X(k)^*$ homeomorphically onto a dense subset of $S(k)^*$ (use [**6**, Lemma 3.2]). We conclude that $S(k)^*$ and, hence, L are irreducible [**6**, Proposition 2.8]. Now, suppose that $g(\phi) = aA_\phi$ and $g(\psi) = bA_\psi$. By irreducibility, there exists

$$\theta \in g^{-1}(\{aA_\rho : \rho \in S(F)\} \cap \{bA_\rho : \rho \in S(F)\})$$

so

$$f(\phi)A_\phi A_\psi = aA_\theta A_\phi A_\psi = bA_\theta A_\phi A_\psi = f(\psi)A_\phi A_\psi.$$

Now, suppose that $f(\phi)A_\phi A_\psi = f(\psi)A_\phi A_\psi$ for all $\phi, \psi \in C$. There exists a cover $W_1, \ldots, W_n$ of C of open and closed sets, and elements $a_1, \ldots, a_n \in F^\times$ with $f(W_i \cap C) = a_i$ for $1 \leqq i \leqq n$. (Recall that C is compact and $S(F)$ admits a subbasis of open and closed sets.) By Theorem (2.1), there exists $b \in F^\times$ with $W_i \cap C \subseteq W(1 - b^{-1}a_i)$ for $1 \leqq i \leqq n$. Thus, the map, call it g, taking $W_i \cap W(1 - b^{-1}a_i)$ to a_i for $1 \leqq i \leqq n$ and the rest of $S(F)$ to b, is continuous. Indeed, it is an I-map since the induced map $S(F)^* \to \Gamma_F$ is the map $\phi \rightsquigarrow bA_\phi$. (The inverse image of any path $\{aA_\phi : \phi \in S(F)\}$ under this map is the open set

$$\cup_{n \in \mathbf{Z}} S(F)^* \backslash (W(n \cdot 1) \cup W(nab^{-1}) \cup W(na^{-1}b)).)$$

But g clearly extends f. The proposition is proved.

(11.3) *Remarks.* (A) The set of I-maps of F forms a group under pointwise multiplication. $F^\times$ may be identified with the subgroup of constant I-maps (see the proof of (11.2)).

(B) Suppose that F is a global field. Then $S(F)^*$ has the cofinite topology. The I-maps of F may be identified with the ideles of F which map everywhere into F. The group of I-maps of F may be substituted for the idele group of F with virtually no other change in the proof of the Hasse-Minkowski theorem on quadratic forms in [**13**]. (Essentially, because every idele differs from some I-map by an idele which is at every spot a local square and a unit congruent to 1; but such ideles enter only trivially into the index computations of [**13**, § 65].)

(C) The compatibility conditions of Theorem (2.1A) can be given a treatment analogous to that above for (2.1B). Namely, $f : C \to F$ satisfies the condition of (2.1A) if and only if it is the restriction of a continuous map $g : S(F) \to F$ inducing a continuous map

$$S(F)^* \to \{a + A_\phi : a \in F, \phi \in S(F)\}$$

(the right hand set given the "open path" topology). When F is global, these maps g are the adeles which map everywhere into F. (For more on the group F/A_ϕ, see [**12**].)

(D) It is natural to ask for a definition of an idele (and adele) group for F for which the approximation theorem has an interpretation as in the global case. One would hope that a generalization of Theorem (2.1) to locally compact subsets of $S(F)$ would also have such an interpretation (see [**13**, 33:11] and [**5**, p. 20] for such "strong approximation theorems"). (Suppose that F has finite characteristic. We sketch a candidate for "the group of ideles of F modulo those congruent everywhere to 1". Give

$$C_F = \{a(1 + P) : a \in F^\times, P \in X(F)\}$$

the coarsest topology with the "path" $\{a(1 + P) : P \in X(F)\}$ open for all $a \in F^\times$. Then the set of continuous maps $f : X(F) \to C_F$ with

$$f(P) \in F^\times/(1 + P)$$

for all $P \in X(F)$ and inducing a continuous map $X(F)^* \to \Gamma_F$ (notation as in the proof of (11.2)) forms a group. Elements of this group map compact subsets of $X(F)$ into paths (apply Theorem (2.1)). For more on the groups $F^\times/(1 + P)$, see [**2**].)

Appendix.

Proof of Theorem (2.1). For any positive rational number q and finite subset A of F, the sets

$$W_q(A) = \{\phi \in S(F) : \phi(A) < q\},$$
$$U(A) = \{\phi \in S(F) : \phi(A) = 1\}$$

are both open and closed (note that $W_q(a)\backslash X(F) = W(a/q, 1/2)$ for all $a \in F$; as usual, we are identifying $X(F)$ with a subset of $S(F)$ – cf. § 3).

In the following lemmas, δ will denote a rational, $0 < \delta < 1$. D and D' will denote non-empty disjoint compact subsets of $S(F)$ (so we implicitly assume that F is not locally finite).

LEMMA A. *Let $a \in F^\times$. Then there exists $b \in F^\times$ and an integer $s > 0$ with $D \cap W(a) \subseteq W_{\delta/s}(b)$, $D \cap W(1/a) \subseteq W_{\delta/s}(1 - b)$, and $D \cap U(a) \subseteq W_s(b)$.*

Proof. We may suppose that D is not contained in $U(a)$ (otherwise, set $b = a$, $s = 2$). Hence, $a^n + a - 1$ is non-zero for sufficiently large n. There exists $m > 0$ with $D \cap U(a) \cap X(F) \subseteq W(a^m - 1)$ (Theorem (8.1)). Since $D \cap U(a)$ is compact, it is contained in

$$U(a-1) \cup W_{(s-1)/s}(a-1) \cup W_{s/(s+1)}(1/(a-1)),$$

for some integer $s > 1$. Taking n to be a sufficiently large integer divisible by $6m$ and setting $b = a^n/(a^n - a + 1)$, we have $W(a) \cap D \subseteq W_{\delta/s}(b)$ and $W(a^{-1}) \cap D \subseteq W_{\delta/s}(b - 1)$. (E.g., the sets $W(a) \cap W_{\delta/2s}(a^r/(a-1))$ contain $D \cap W(a)$ for sufficiently large r, since together they form a nested cover of $D \cap W(a)$; but each of these sets is contained in $W_{\delta/s}(a^r/(a^r + a - 1))$.) Let $\phi \in D \cap U(a)$. If $\phi(a-1) \neq 1$, then $\phi(b) < s$ (by the choice of s). If $\phi \in U(a-1)$, then $\phi(a^n - 1) = 0$ (since $6m$ divides n; when ϕ factors through a complex place π, observe that $|\pi(a) - 1| = |\pi(a)| = 1$, so $\pi(a)$ is a 6th root of unity). Hence, $\phi(b) = 1 < s$.

LEMMA B. *Suppose that $A \subseteq F$, $D \subseteq \bigcup_{a \in A} W(a)$, and $D' \subseteq \bigcap_{a \in A} W(1/a)$. Then for some $a \in F^\times$, $D \subseteq W(a)$ and $D' \subseteq W(1/a)$.*

Proof. Since D is compact, we may suppose that A is finite. By induction on $|A|$, we may suppose that A has exactly two elements, a_1 and a_2. Apply Lemma A to find b_1, $b_2 \in F^\times$ and $s > 1$ with $D \subseteq W_{\delta/2s}(b_1) \cup W_{\delta/2s}(b_2)$, $D' \subseteq W_{\delta/2s}(1 - b_i)$, and $D \subseteq W_s(b_i)$ for $i = 1, 2$. It suffices to set

$$a = b_1 b_2/(1 - b_1 b_2).$$

LEMMA C. *There exists $a \in F^\times$ with $D \subseteq W_\delta(a)$ and $D' \subseteq W_\delta(a^{-1})$.*

Proof. Let $\phi \in D$, $\psi \in D'$. By Remark (4.3), we can find $b \in D_\phi \backslash D_\psi$ and $b' \in D_\psi \backslash D_\phi$ so that $\phi \in W(b/b')$ and $\psi \in W(b'/b)$. Apply Lemma B to find $c \in F^\times$ with $\{\phi\} \subseteq W(c^{-1})$, $D' \subseteq W(c)$. Now apply Lemma B again to find $d \in F^\times$ with $D \subseteq W(d)$, $D' \subseteq W(d^{-1})$. It suffices to let a be a sufficiently large power of d (use that $D \cup D'$ is compact).

LEMMA D. *There exists $a \in F^\times$ with $D \subseteq W_\delta(a)$ and $D' \subseteq W_\delta(1 - a)$.*

Proof. Apply Lemma C and then Lemma A.

LEMMA E. *Let $b, b' \in F^\times$. Suppose that $A \subseteq F^\times$, $D \subseteq \bigcup_{a \in A} W_\delta(b/a, b'/a)$, and $D' \subseteq \bigcap_{a \in A} W_\delta(a/b, a/b')$. Then there exists $a \in F^\times$ with $D \subseteq W_\delta(b/a, b'/a)$ and $D' \subseteq W_\delta(a/b, a/b')$.*

Proof. As in the proof of Lemma B, we may suppose that A has exactly two elements, a_1 and a_2. By Lemma D we can find $c_1, c_2 \in F^\times$ with

$$D' \cup (D \backslash W(a_2/a_1)) \subseteq W_{1/4}(c_1, c_2 - 1)$$

and

$$D \cap W(a_2/a_1) \subseteq W_{1/4}(c_2, c_1 - 1).$$

By Lemma C, we have $d \in F^\times$ with $D \subseteq W_{1/2}(1/d)$, $D' \subseteq W_{1/2}(d)$. It suffices to set $a = d(c_1a_1 + c_2a_2)$. (E.g., if $\phi \in D \cap W(a_2/a_1)$, then $\phi(b/a_1) < \delta$. Hence,

$$\phi(a/b) = \phi(d)\phi(a_1/b)\phi(c_2a_2a_1^{-1} + c_1) > 1/\delta.)$$

LEMMA F. *Let $b, b' \in F^\times$. Suppose that $bA_\phi A_\psi = b'A_\phi A_\psi$ for all $\phi \in D$, $\psi \in D'$. Then there exists $a \in F^\times$ with $D \subseteq W_\delta(b/a, b'/a)$ and*

$$D' \subseteq W_\delta(a/b, a/b').$$

Proof. Let $\phi \in D$, $\psi \in D'$. By our hypotheses,

$$(b^{-1}A_\phi \cap b'^{-1}A_\phi) \cdot A_\psi \supseteq b^{-1}A_\phi A_\psi$$

and, hence (also using symmetry),

$$(b^{-1}A_\phi \cap b'^{-1}A_\phi)A_\psi = (b^{-1}A_\psi \cup b'^{-1}A_\psi) \cdot A_\phi.$$

If A_ϕ and A_ψ are incomparable, there exists $c \in F^\times$ with $cA_\phi = b^{-1}A_\phi \cap b'^{-1}A_\phi$ and $cA_\psi = b^{-1}A_\psi \cup b'^{-1}A_\psi$ [**14**, p. 135, Theorem 1]. But if A_ϕ and A_ψ are comparable, such $c \in F^\times$ still exists (e.g., if $A_\phi \subseteq A_\psi$, take any c with $cA_\phi = b^{-1}A_\phi \cap b'^{-1}A_\phi$). Thus, $\phi(bc) < \infty$, $\phi(b'c) < \infty$, $\psi(bc) > 0$, and $\psi(b'c) > 0$. Hence, by Lemma C we can pick $d \in F^\times$ with $\phi(dbc) < \delta$, $\phi(db'c) < \delta$, $\psi(dbc) > 1/\delta$, and $\psi(db'c) > 1/\delta$. Now apply Lemma E (with b^{-1}, b'^{-1}, D' and $\{\phi\}$ in place of b, b', D and D', respectively) to obtain $a_\phi \in F_q$ with $\phi \in W_\delta(b/a_\phi, b'/a_\phi)$ and $D' \subseteq W_\delta(a_\phi/b, a_\phi/b')$. A second application of Lemma E gives the required element a.

Note. The reader who objects to using the strong result [**14**, Theorem 1, p. 135] above (perhaps because we claim it as a corollary to Theorem (2.1)), will find it not much more work to use instead the much weaker [**14**, Lemma 2, p. 128].

Proof of Theorem (2.1B). The necessity of the condition for the existence of b is obvious; we prove sufficiency. Let us pick δ so that

$$0 < 2\delta^2/(1 - 3\delta^2) < \epsilon.$$

Since F is discrete and C compact, $f(C)$ is finite. By induction, we may suppose that $f(C)$ contains exactly two elements, b and b'. Set $D = f^{-1}(b)$, $D' = f^{-1}(b')$. By Lemma F we can pick $s, t \in F^\times$ with

$$D \subseteq W_\delta(s/b, s/b', b/t, b'/t),$$
$$D' \subseteq W_\delta(t/b, t/b', b/s, b'/s).$$

Note that $s + t \neq 0$. Let $a = (bt + b's)/(s + t)$. Write

$$1 - ab^{-1} = sb^{-1}((sb^{-1} + tb^{-1})^{-1} + (sb'^{-1} + tb'^{-1})^{-1})$$

and

$$1 - a^{-1}b = (1 - ab^{-1})/(-1 + (1 - ab^{-1})).$$

Then, for $\phi \in D$, $\phi(1 - ab^{-1}) < 2\delta/(1 - \delta^2)$ and, hence,

$$\phi(1 - a^{-1}b) < 2\delta^2/(1 - 3\delta^2) < \epsilon.$$

Similarly, $\psi(1 - a^{-1}b') < \epsilon$ for $\psi \in D'$.

Theorem (2.1B) is proved.

Note G. Observe from the above proof that $af(\phi)^{-1}$ may be substituted for $a^{-1}f(\phi)$ in the statement of Theorem (2.1B).

LEMMA H. *Suppose that $b, b' \in F$ and $b - b' \in A_\phi A_\psi$ for all $\phi \in D$, $\psi \in D'$. There exists $a \in F$ with $D \subseteq W_\delta(a - b)$ and $D' \subseteq W_2(a - b')$.*

Proof. Let $c = b - b' + 1$. Then $cA_\phi A_\psi = A_\phi A_\psi$ for all $\phi \in D$, $\psi \in D'$. (Sub-proof. $c \in A_\phi A_\psi$, since either $\phi \in D$ or $\phi(b - b') \leqq 1$. But since $\psi(c) \neq 0$, c must be invertible in $A_\phi A_\psi$.) By Lemma F, we can find $s, t \in F^\times$ with

$$D \cup (D' \backslash W((c - 1)^{-1})) \subseteq W_{\delta/2}(c/s, 1/s, t/c, t)$$

and

$$D' \cap W((c - 1)^{-1}) \subseteq W_{\delta/2}(s/c, s, c/t, 1/t).$$

Since D is nonempty, $s + t \neq 0$. It suffices to set $a = (sb + tb')/(s + t)$. (E.g., write

$$a - b = (s/(sc^{-1} + tc^{-1})) + (s/(s + t)) + (b - b').$$

Then, clearly, $\phi(a - b) < 2$ for $\phi \in D' \backslash W((b - b')^{-1})$. The other cases are similar, but easier.)

Proof of Theorem (2.1A). The necessity of the condition for the existence of a is obvious; we prove sufficiency. Let $\delta < \epsilon/8$. As in the proof of (2.1B), we may assume that $f(C)$ has exactly two elements b and b', and we set $D = f^{-1}(b)$, $D' = f^{-1}(b')$. By Lemma H, there exists $c \in F$ with

$$D \subseteq W_2(b + c) \quad \text{and } D' \subseteq W_2(b' + c).$$

Apply Lemma H and Lemma D each twice to get $d, d' \in F$ and $s, t \in F^\times$ with

$$D \subseteq W_\delta(s, 1 - t, d - b - c) \cap W_2(d' - b - c),$$
$$D' \subseteq W_\delta(t, 1 - s, d' - b' - c) \cap W_2(d - b' - c).$$

Set $a = sd' + td - c$. Then

$$a - b = t(d - b - c) + s(d' - b - c) + s(c + b) + (t - 1)(c + b),$$

so, clearly, $\phi(a - b) < \epsilon$ if $\phi \in D$. Symmetrically, $\phi(a - b') < \epsilon$ if $\phi \in D'$.

Theorem (2.1) is proved.

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