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Certain Operators with Rough Singular Kernels

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Abstract. We study the singular integral operator

$$T_{\Omega,\alpha} f(x) = \text{p.v.} \int_{R^n} b(|y|)\Omega(y')|y|^{-n-\alpha} f(x-y)\,dy,$$

defined on all test functions f,where b is a bounded function, $\alpha \geq 0$, $\Omega(y')$ is an integrable function on the unit sphere S^{n-1} satisfying certain cancellation conditions. We prove that, for $1 < p < \infty$, $T_{\Omega,\alpha}$ extends to a bounded operator from the Sobolev space L^p_α to the Lebesgue space L^p with Ω being a distribution in the Hardy space $H^q(S^{n-1})$ where $q = \frac{n-1}{n-1+\alpha}$. The result extends some known results on the singular integral operators. As applications, we obtain the boundedness for $T_{\Omega,\alpha}$ on the Hardy spaces, as well as the boundedness for the truncated maximal operator $T^*_{\Omega,m}$.

1 Introduction

Let S^{n-1} be the unit sphere in R^n, $n \geq 2$, with normalized Lebesgue measure $d\sigma = d\sigma(x')$. Let $\Omega(x)$ be a homogeneous function of degree 0, with $\Omega \in L^1(S^{n-1})$ and

$$\int_{S^{n-1}} \Omega(x')\,d\sigma(x') = 0, \tag{1.1}$$

where $x' = x/|x|$ for any $x \neq 0$.

Suppose $b(|x|)$ is an L^∞ function; the singular integral operator $\mathrm{SI}_b(f)$ is defined by

$$\mathrm{SI}_b(f)(x) = \text{p.v.} \int_{R^n} b(|y|)\Omega(y')|y|^{-n} f(x-y)\,dy \tag{1.2}$$

for all test functions f, where $y' = y/|y| \in S^{n-1}$.

We denote $\mathrm{SI}_b(f)$ by $\mathrm{SI}(f)$ if $b = 1$. This operator SI was first studied by Calderon and Zygmund in [CZ1, CZ2]. They proved that if $\Omega \in L\log^+ L(S^{n-1})$ satisfies the mean zero condition (1.1) then the operator SI with kernel $\Omega(x')|x|^{-n}$ is a bounded operator on $L^p(R^n)$, $1 < p < \infty$. Below let us recall briefly the idea used in Calderon-Zygmund's proof.

Suppose that $\Omega \in L^1(S^{n-1})$is an odd function; then one can easily show that

$$\mathrm{SI}(f)(x) = \frac{1}{2} \int_{S^{n-1}} \Omega(y') \Big\{ \int_{-\infty}^{\infty} f(x-ty')t^{-1}\,dt \Big\}\,d\sigma(y'). \tag{1.3}$$

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By the method of rotation and the well-known L^p-boundedness of the Hilbert transform one then obtains the L^p-boundedness of SI under the weak condition $\Omega \in L^1(S^{n-1})$.

For even kernels, the condition $\Omega \in L^1(S^{n-1})$ is insufficient. It turns out the right condition is $\Omega \in L\log^+ L(S^{n-1})$ (as far as the size of Ω is concerned). The idea of Calderon-Zygmund is to compose the operator SI with the Riesz transform R_j, $1 \le j \le n$, and show that $R_j(\mathrm{SI})$ is a singular integral operator with an appropriate odd kernel. Thus

$$\|R_j(\mathrm{SI})(f)\|_p \le C_p \|f\|_p$$

for all test functions $f \in \mathcal{S}$. Furthermore, one can obtain

$$\begin{aligned}\|\,\mathrm{SI}(f)\|_p &= \Big\| \Big(\sum_{j=1}^n R_j^2 \Big)\ \mathrm{SI}(f) \Big\|_p \le \sum_{j=1}^n \big\| R_j \big(R_j\, \mathrm{SI}(f) \big) \big\|_p \\ &\le nC \sum_{j=1}^n \|R_j\, \mathrm{SI}(f)\|_p \le n^2 C C_p \|f\|_p\end{aligned}$$

for all test functions $f \in \mathcal{S}$, since $-\sum_{j=1}^n R_j^2$ is the identity map. Using the above method, Ricci and Weiss [RW] obtained the same L^p-boundedness of $\mathrm{SI}(f)$ under a weaker condition $\Omega \in H^1(S^{n-1})$, where $H^1(S^{n-1})$is the Hardy space which contains $L\log^+ L(S^{n-1})$ as a proper subspace.

In [Fe], R. Fefferman generalized this Calderon-Zygmund singular integral by replacing the kernel $\Omega(x')|x|^{-n}$ by $b(|x|)\Omega(x')|x|^{-n}$, where b is an arbitrary L^∞ function. This allows the kernel to be rough not only on the sphere, but also in the radial direction. For the singular integral operator $\mathrm{SI}_b f(x)$ with the kernel $K(x) = b(|x|)\Omega(x')|x|^{-n}$, the formula (1.3) now is

$$\mathrm{SI}_b(f)(x) = \int_{S^{n-1}} \Omega(y') \Big\{ \int_0^\infty f(x - ty')t^{-1}\, dt \Big\}\, d\sigma(y'). \tag{1.3$'$}$$

Clearly, the method by Calderon and Zygmund can no longer be used to estimate the above integral in (1.3$'$) even if Ω is odd, since the integral in the parenthesis can not be reduced to the Hilbert transform for an arbitrary $b(t)$. Thus one needs to find a new approach.

Using a method which is different from Calderon and Zygmund, R. Fefferman showed in [Fe] that if Ω satisfies a Lipschitz condition then SI_b is bounded on $L^p(R^n)$ for $1 < p < \infty$. Later, in [DR], using Littlewood- Paley theory and Fourier transform methods, Duoandikoetxea and Rubio De Francia improved Fefferman's result by assuming a roughness condition $\Omega \in L^q(S^{n-1})$ (see also [Ch],[Na] and a recent survey paper [GS]). By modifying the method in [DR], recently, Fan and Pan obtained the following theorem.

***Theorem A* [FP1]** *Let SI_b be the singular integral operator defined by (1.2). If $\Omega \in H^1(S^{n-1})$ satisfies (1.1), then SI_b is bounded on $L^p(R^n)$, $1 < p < \infty$.*

In this paper, we will study the singular integral operator $T_{\Omega,\alpha} f(x)$ (formally) defined by

$$
(1.4)\qquad
\begin{aligned}
T_{\Omega,\alpha} f(x) &= \text{p.v.} \int_{R^n} b(|y|)\Omega(y')|y|^{-n-\alpha} f(x-y)\,dy \\
&= \lim_{\varepsilon\to 0} T_{\Omega,\alpha,\varepsilon} f(x) = \lim_{\varepsilon\to 0}\int_{|y|>\varepsilon} b(|y|)\Omega(y')|y|^{-n-\alpha} f(x-y)\,dy
\end{aligned}
$$

for all functions $f \in \mathcal{S}(R^n)$, where $b \in L^\infty$, $\alpha \geq 0$ and $\Omega \in H^q(S^{n-1})$ satisfies

$$
(1.5)\qquad \int_{S^{n-1}} \Omega(y')Y_m(y')\,d\sigma(y') = 0
$$

for all spherical polynomials $Y_m(y')$ with degrees $m \leq [\alpha]$. We recall that the Hardy space $H^q(S^{n-1})$ is a distribution space if $0 < q < 1$. So (1.4) and (1.5) are well-defined only if $\Omega \in L^1(S^{n-1})$. In general, the way in which the above integrals have to be interpreted.

Let $\chi_{(a,b)}(t)$ be the characteristic function of the interval (a, b), and let

$$
C_\varepsilon(|x|) = b(|x|)|x|^{-n-\alpha}\chi_{(\varepsilon,\infty)}(|x|), \quad L_\varepsilon(t) = b(t)\chi_{(\varepsilon,\infty)}(t)t^{-1-\alpha}.
$$

For $f \in \mathcal{S}(R^n)$, write $f(x-y) = f_{x,t}(y')$ with $t = |y|$. Denote $\langle\Omega, \phi\rangle$ the pairing between Ω and a C^∞ function ϕ on S^{n-1}. The operators $T_{\Omega,\alpha,\varepsilon}$ and $T_{\Omega,\alpha}$ are defined on the test function space $\mathcal{S}(R^n)$ as

$$
(1.4')\qquad
\begin{aligned}
T_{\Omega,\alpha,\varepsilon} f(x) &= \int_0^\infty L_\varepsilon(t)\langle\Omega, f_{x,t}\rangle\,dt, \\
T_{\Omega,\alpha} f(x) &= \lim_{\varepsilon\to 0} T_{\Omega,\alpha,\varepsilon} f(x)
\end{aligned}
$$

where $\Omega \in H^q$ satisfies

$$
(1.5')\qquad \langle\Omega, Y_m\rangle = 0
$$

for all sphere polynomials Y_m with degrees $\leq [\alpha]$.

As we will mention in the second section, a distribution $\Omega \in H^q(S^{n-1})$ has an atomic decomposition $\Omega = \sum \lambda_j a_j$ in the distribution sense, where each a_j is called a (q, r) *atom* that is an integrable function having the same cancellation conditions as that of Ω, and $\sum |\lambda_j|^q \cong \|\Omega\|^q_{H^q(S^{n-1})}$. Thus, the pairing in (1.4′) is

$$
\langle\Omega, f_{x,t}\rangle = \sum \lambda_j\langle a_j, f_{x,t}\rangle.
$$

It is known (see [Co2]) that the dual space of H^q is the Lipschitz space $\Lambda^\beta(S^{n-1})$ with $\beta = (1/q-1)(n-1)$. Let k be an integer greater than β. If $\beta > 0$, the space $\Lambda^\beta(S^{n-1})$ is the set of all functions $g \in L^\infty(S^{n-1})$ with norm

$$
\|g\|_{\Lambda^\beta(S^{n-1})} = \|g\|_{L^\infty(S^{n-1})} + \Big\{ \sup \Big\| \frac{d^k}{dr^k} g(r\cdot)\Big\|_{L^\infty(S^{n-1})}(1-r)^{k-\beta} : 0 \leq r < 1\Big\} < \infty.
$$

Λ^0 is the BMO space. So

$$|\langle \Omega, f_{x,t} \rangle| \leq \sum |\lambda_j| \, \|a_j\|_{H^q(S^{n-1})} \|f_{x,t}\|_{\Lambda^\beta(S^{n-1})}.$$

It is known from [Co1, Co2] or [CTW] that $\|a_j\|_{H^q(S^{n-1})} \leq C$ uniformly for atoms a_j. It is also not difficult to check that $\|f_{x,t}\|_{\Lambda^\beta(S^{n-1})} \leq C$ uniformly on $t > 0$ (see Appendix). By the Lebesgue dominated convergence theorem we have, for $\varepsilon > 0$,

$$T_{\Omega,\alpha,\varepsilon} f(x) = \sum \lambda_j \int_0^\infty L_\varepsilon(t) \langle a_j, f_{x,t} \rangle \, dt.$$

We define the distribution kernel

$$K_{\Omega,\varepsilon} = \sum \lambda_j C_\varepsilon a_j.$$

Now the operators $T_{\Omega,\alpha,\varepsilon}$ can be written as

$$T_{\Omega,\alpha,\varepsilon} f(x) \cong K_{\Omega,\varepsilon} * f(x) = \sum \lambda_j (C_\varepsilon a_j) * f(x).$$

The definition in $(1.4')$ is well-defined, since $\lim_{\varepsilon \to 0} T_{\Omega,\alpha,\varepsilon}(f)(x)$ exists for all $x \in R^n$. To see this fact, we use the Taylor's expansion

$$f(x-y) = \sum_{|K| \leq [\alpha]} C_K y^K (D^K f)(x) + \sum_{|K|=[\alpha]+1} C_K y^K \int_0^1 (1-s)^{[\alpha]} (D^K f)(x-sy) \, ds,$$

where C_K are certain constant coefficients depending on the multi-indices K. Let

$$g_{x,t}(y') = \sum_{|K|=[\alpha]+1} C_K y'^K \int_0^1 (1-s)^{[\alpha]} (D^K f)(x-sty') \, ds.$$

By the cancellation condition of Ω, we have

$$\langle \Omega, f_{x,t} \rangle = \langle \Omega, g_{x,t} \rangle t^{[\alpha]+1}$$

which gives, in $(1.4')$, that

$$T_{\Omega,\alpha,\varepsilon} f(x) = \int_0^\infty L_\varepsilon(t) \big(\chi_{(0,1)}(t) t^{[\alpha]+1} \langle \Omega, g_{x,t} \rangle + \chi_{(1,\infty)}(t) \langle \Omega, f_{x,t} \rangle \big) \, dt.$$

Since f is a test function, there is a constant C (perhaps depending on x) such that

$$\chi_{(0,1)}(t) \|g_{x,t}\|_{\Lambda^\beta(S^{n-1})} \leq C, \quad \chi_{(1,\infty)}(t) \|f_{x,t}\|_{\Lambda^\beta(S^{n-1})} \leq C$$

uniformly for all $t > 0$. So

$$\begin{aligned} &\big| L_\varepsilon(t) \{ \chi_{(0,1)}(t) t^{[\alpha]+1} \langle \Omega, g_{x,t} \rangle + \chi_{(1,\infty)}(t) \langle \Omega, f_{x,t} \rangle \} \big| \\ &\quad \leq C \|\Omega\|_{H^q(S^{n-1})} \big| \chi_{(0,1)}(t) b(t) t^{[\alpha]-\alpha} + \chi_{(1,\infty)}(t) L_1(t) b(t) t^{-1-\alpha} \big|. \end{aligned}$$

By the Lebesgue dominated convergence theorem we obtain

$$\lim_{\varepsilon\to 0} T_{\Omega,\alpha,\varepsilon} f(x) = \int_0^\infty b(t)\{\chi_{(0,1)}(t)\langle\Omega, g_{x,t}\rangle t^{[\alpha]-\alpha} + \chi_{(1,\infty)}(t)t^{-1-\alpha}\langle\Omega, f_{x,t}\rangle\}\, dt.$$

Clearly (1.4′) (1.5′) are consistent with (1.4) (1.5) if $\Omega \in L^1(S^{n-1})$.

We have the following theorem.

Theorem 1 *Let $1 < p < \infty$ and $\tilde{p} = \max\{p, p/(p-1)\}$ and let $T_{\Omega,\alpha}$ be defined in (1.4′). Suppose that $\Omega \in H^q(S^{n-1})$ with $q = (n-1)/(n-1+\alpha)$ and satisfies*

$$\langle\Omega, Y_m\rangle = 0 \tag{1.5′′′}$$

for all $Y_m(y')$ with degree $m \le N$, where N is an integer larger than $\alpha\tilde{p}/2 - 1$. Then we have

$$\|T_{\Omega,\alpha} f\|_{L^p(R^n)} \le C\|\Omega\|_{H^q(S^{n-1})}\|f\|_{L^p_\alpha(R^n)} \tag{1.6}$$

for all $f \in \mathcal{S}(R^n)$, where L^p_α is the homogeneous Sobolev space whose definition can be found in the appendix. Thus the operator $T_{\Omega,\alpha}$ can be extended to the full $L^p_\alpha(R^n)$ in the usual manner.

It is clear that Theorem 1 is an extension of Theorem A, since $L^p_\alpha = L^p$ if $\alpha = 0$. On the other hand, although $T_{\Omega,\alpha}$ is a translation-invariant operator, it is hard to see whether it belongs to the symbol class S^α (see [St] for the definition of S^α). One can not use the results in [T] to cover our result.

Let $\mathcal{L}^p_\alpha$ be the inhomogeneous Sobolev space which is the set of all functions f satisfying $\|f\|_{\mathcal{L}^p_\alpha} = \|f\|_{L^p} + \|f\|_{L^p_\alpha} < \infty$. Define the operator $S_{\Omega,\alpha}$ on $\mathcal{S}(R^n)$ by

$$S_{\Omega,\alpha} f(x) = \text{p.v.} \int_{R^n} \Gamma(|y|)\Omega(y')f(x-y)\, dy,$$

where $|\Gamma(t)| \le Ct^{-n-\alpha}$ if $t \in (0,1)$ and $|\Gamma(t)| \le Ct^{-n}$ if $t \ge 1$. Then by Theorem A and Theorem 1, we easily obtain the following result on $S_{\Omega,\alpha}$.

Theorem 1′ *Let $1 < p < \infty$. Suppose that $\Omega \in H^1(R^n)$ satisfies (1.5′). Then*

$$\|S_{\Omega,\alpha} f\|_{L^p} \le C\|f\|_{\mathcal{L}^p_\alpha}.$$

We recall also a result by Muckenhoupt and Wheeden in the following.

Theorem B **[MW]** *If $-n < \alpha < 0$, $q = n/(n+\alpha)$, and $\Omega \in L^q(R^n)$ (no cancellation condition needed), then*

$$\|T_{\Omega,\alpha} f\|_{L^r(R^n)} \le C\|f\|_{L^p(R^n)} \tag{1.7}$$

with $1/r = 1/p + \alpha/n$. In order to clarify the relations among Theorem 1 and Theorems A–B, we remark that on S^{n-1}, $L^q \subset L\,\mathrm{Log}^+ L \subset H^1 \subset L^1 \subset H^r$, $0 < r < 1 < q$ and all the inclusions are proper, while $L^q = H^q$ if $1 < q < \infty$. Let $\dot{F}^{\alpha,q}_p$ be the Triebel-Lizorkin space defined in [FJW]. It is known that on R^n, $\dot{F}^{0,2}_p = L^p$, $\dot{F}^{\alpha,2}_p = L^p_\alpha$ if $\alpha \ge 0$; $L^p \subset \dot{F}^{\alpha,2}_p$ if $\alpha < 0$ and $1/r = 1/p + \alpha/n$. Thus, our theorem also can be viewed as a partial extension of Theorem B.

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Define the truncated maximal operator by $T^*_{\Omega,\alpha} f(x) = \sup_{\varepsilon>0} |T_{\Omega,\alpha,\varepsilon} f(x)|$.
When $\alpha = m$ is an integer, we have a stronger result.

Theorem 2 *Let $m = 0, 1, 2, \dots$, and $\Omega \in H^q(S^{n-1})$ satisfy (1.5′′) in Theorem 1 with $q = \frac{n-1}{n-1+m}$. For $1 < p < \infty$, we have*

$$\|T^*_{\Omega,m}(f)\|_{L^p(R^n)} \le C\|\Omega\|_{H^q(S^{n-1})}\|f\|_{L^p_m(R^n)}. \tag{1.8}$$

Below we briefly outline the strategy of our proofs.
From (1.6) we have

$$T^*_{\Omega,\alpha}(f)(x) \le \sum |\lambda_j| T^*_{a_j,\alpha}(f)(x). \tag{1.9}$$

so by the Minkowski inequality

$$\|T^*_{\Omega,\alpha}(f)\|_{L^p} \le \sum |\lambda_j| \, \|T^*_{a_j,\alpha}(f)\|_{L^p}.$$

To show the boundedness of $T^*_{\Omega,\alpha}(f)$, it suffices to show

$$\|T^*_{a_j,\alpha}(f)\|_{L^p} \le C\|f\|_{L^p_\alpha}, \tag{1.10}$$

where C is independent of f and atoms a_j; because by (1.10) one has

$$\|T^*_{\Omega,\alpha}(f)\|_{L^p} \le C\|f\|_{L^p_\alpha}\Big(\sum |\lambda_j|^q\Big)^{\frac{1}{q}} = C\|f\|_{L^p_\alpha(R^n)}\|\Omega\|_{H^q(S^{n-1})}.$$

To establish (1.10), in the third section, we will prove for $\alpha \ge 0$,

$$\|T_{a_j,\alpha}(f)\|_{L^p} \le C\|f\|_{L^p_\alpha}, \tag{1.11}$$

$$\|T_{a_j,\alpha,\varepsilon}(f)\|_{L^p} \le C\|f\|_{L^p_\alpha}, \tag{1.11′}$$

where C is independent of f, a_j and $\varepsilon > 0$.

In Section 4, we will use (1.11) to establish (1.10) for $\alpha = 0, 1, 2, \dots$, which proves Theorem 2. (Recently, we proved this theorem for all $\alpha \ge 0$ [CF].)

To prove Theorem 1, since $\lim_{\varepsilon\to 0} T_{\Omega,\alpha,\varepsilon} f(x)$ exists for all x, by the Fatou Lemma

$$\begin{aligned}\|T_{\Omega,\alpha}(f)\|_{L^p} &= \Big\{\int_{R^n} \liminf_{\varepsilon\to 0} |T_{\Omega,\alpha,\varepsilon}(f)(x)|^p\,dx\Big\}^{\frac{1}{p}}\\ &\le \liminf_{\varepsilon\to 0}\Big\{\int_{R^n} |T_{\Omega,\alpha,\varepsilon}(f)(x)|^p\,dx\Big\}^{\frac{1}{p}}\end{aligned}$$

Here without loss of generality we assume that $\{\varepsilon\}$ is a sequence of positive numbers going to zero. Thus, by the atomic decomposition of Ω,

$$\|T_{\Omega,\alpha}(f)\|_{L^p} \le \liminf_{\varepsilon\to 0} \sum |\lambda_j| \|T_{a_j,\alpha,\varepsilon}(f)\|_{L^p},$$

which with (1.11′) proves Theorem 1.

We will introduce some known lemmas in the second section. The proof of (1.11) and (1.11′) can be found in Section 3. We prove (1.10) for integers α in Section 4. As applications, we will study the boundedness property of $T_{\Omega,\alpha}$ on the Hardy spaces in the fifth section. In the sixth section(Appendix), we will review the definitions of Triebel-Lizorkin spaces and the Sobolev spaces L^p_α.

Throughout this paper, the letter C will denote a positive constant that may vary at each occurrence but is independent of the essential variables.

2 Definitions and Lemmas

Recall that the Poisson kernel on S^{n-1} is defined by

$$P_{ry'}(x') = \frac{1-r^2}{|ry'-x'|^n},$$

where $0 \le r < 1$ and $x', y' \in S^{n-1}$.

For any $f \in \mathcal{S}'(S^{n-1})$, we define the radial maximal function $P^+ f(x')$ by

$$P^+ f(x') = \sup_{0\le r<1} |\langle f, P_{rx'}\rangle|,$$

where $\mathcal{S}'(S^{n-1})$ is the space of Schwartz distributions on S^{n-1}.

The Hardy space $H^q(S^{n-1}), 0 < q \le 1$, is the linear space of distributions $f \in \mathcal{S}'(S^{n-1})$ with the finite norm $\|f\|_{H^q(S^{n-1})} = \|P^+ f\|_{L^q(S^{n-1})} < \infty$. The space $H^q(S^{n-1})$ was studied in [Co1, Co2] (see also [CTW]). S^1 and S^3 are compact Lie groups. For H^q on a compact Lie group, the reader can see [BF].

An important property of $H^q(S^{n-1})$ is the atomic decomposition, which will be reviewed below.

An exceptional atom $E(x)$ is an $L^\infty(S^{n-1})$ function bounded by 1. A (q, r) regular atom is an $L^r(S^{n-1}), r > 1$ function $a(x')$ that satisfies

(2.1) $\operatorname{supp}(a) \subset \{x' \in S^{n-1}, |x'-x'_0| < \rho\}$ for some $x'_0 \in S^{n-1}$ and $0 < \rho \le 2$;

(2.2) $$\int_{S^{n-1}} a(x')Y_m(x')\,d\sigma(x') = 0,$$

for any sphere polynomial Y_m with degree $m \le N$, where N is any fixed integer larger than $[(n-1)(\frac{1}{q}-1)]$;

(2.3) $$\|a\|_{L^r(S^{n-1})} \le \rho^{-(n-1)/(1/q-1/r)},$$

From [Co1, Co2] or [CTW], we find that any $\Omega \in H^q(S^{n-1})$ has an atomic decomposition

$$\Omega = \sum_{j=1}^{\infty} \lambda_j a_j + \sum_{i=1}^{\infty} \mu_i E_i,$$

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where each a_j is a regular (q,∞) atom; each E_i is an exceptional atom, and

$$\sum |\lambda_j|^q + \sum |\mu_i|^q \le C\|\Omega\|^q_{H^q(S^{n-1})}.$$

we note that for any $x \in S^{n-1}$,

$$\Big|\sum \mu_i E_i(x)\Big| \le \sum |\mu_i| \le \Big\{\sum |\mu_i|^q\Big\}^{\frac{1}{q}}.$$

Without loss of generality, we can assume

$$\Big|\sum \mu_i E_i(x)\Big| \le \|\Omega\|_{H^q(S^{n-1})}.$$

Thus we write

$$\sum_{i=1}^{\infty} \mu_i E_i(x) = \|\Omega\|_{H^q(S^{n-1})} A(x),$$

with

$$A(x) = \frac{\sum \mu_i E_i(x)}{\|\Omega\|_{H^q(S^{n-1})}}.$$

In this new definition,

$$\Omega = \sum \lambda_j a_j + \|\Omega\|_{H^q(S^{n-1})} A, \quad \text{and } \|A\|_{L^\infty} \le 1$$

If Ω has the cancellation conditions for all Y_m whose degrees $\le N$, (since a_j's are regular atoms whose cancellation property (2.2) can be chosen for any large N), we see that $A(x)$ has the same cancellation conditions as that of Ω. In other words, $A(x)$ can be viewed as a regular (q,∞) atom whose support is S^{n-1}. As a conclusion, if $\Omega \in H^q(S^{n-1})$ has the cancellation condition (1.5$''$), then all the atoms satisfy (2.2) uniformly for the N in (1.5$''$). Furthermore, we can see

$$\Omega_k = \sum_{j=1}^{k} \lambda_j a_j \in H^q(S^{n-1}) \cap L^1(S^{n-1})$$

and

$$\lim_{k\to\infty} \|\Omega_k - \Omega\|_{H^q(S^{n-1})} = 0.$$

Throughout this paper, we always assume that N is a fixed integer larger than $\alpha\tilde{p}/2 - 1$.

In the rest of the paper, for any non-zero $\xi = (\xi_1, \dots, \xi_n) \in R^n$, we write $\xi/|\xi| = \xi' = (\xi'_1, \dots, \xi'_n) = (\zeta_1, \dots, \zeta_n) = \zeta$. Thus $\zeta \in S^{n-1}$. Also we use ζ_* to denote $(\zeta_2, \dots, \zeta_n)$ and use ξ_* to denote $(\xi_2, \dots, \xi_n)$.

Lemma 2.1 *Suppose $n \ge 3$ and $a(\cdot)$ is a (q,∞) atom on S^{n-1} supported in $S^{n-1} \cap B(\zeta,\rho)$, where $B(\zeta,\rho)$ is the ball with radius ρ and center $\zeta = \xi' \in S^{n-1}$. Let*

$$F_a(s) = (1-s^2)^{\frac{n-3}{2}} \chi_{(-1,1)}(s) \int_{S^{n-2}} a\big(s, (1-s^2)^{\frac{1}{2}} \tilde{y}\big)\, d\sigma(\tilde{y}).$$

Then there exist $s_0 \in R$ and a constant C independent of $a(\cdot)$ such that

$$\operatorname{supp}(F_a) \subseteq \big(s_0 - 2r(\xi'), s_0 + 2r(\xi')\big); \tag{2.4}$$

$$\|F_a\|_\infty \le C\rho^{(1-1/q)(n-1)} r(\xi')^{-1}; \tag{2.5}$$

$$\int_R F_a(s)s^k\,ds = 0 \quad \text{for any integer } k \in [0, N], \tag{2.6}$$

where $r(\xi') = |A_\rho \xi'| = |\xi|^{-1}|A_\rho \xi|$ and $A_\rho \xi = (\rho^2\xi_1, \rho\xi_2, \dots, \rho\xi_n)$.

Lemma 2.2 *Suppose $n = 2$ and $a(\cdot)$ is a (q, ∞) atom supported in $S^1 \cap B(\zeta, \rho)$. Let*

$$F_a(s) = (1 - s^2)^{-\frac{1}{2}} \chi_{(-1,1)}(s) \Big(a(s, (1 - s^2)^{\frac{1}{2}} + a(s, -(1 - s^2)^{\frac{1}{2}} \Big).$$

Then $F_a(s)$ satisfies (2.4), (2.6) and

$$\|F_a\|_r \le C|A_\rho(\xi')|^{-1+\frac{1}{r}} \rho^{1-\frac{1}{q}}.$$

for some $r \in (1, 2)$.

Lemma 2.1 and Lemma 2.2 can be found in [FP2, Lemma 2.1 and 2.2 and their remarks]. Or see [FP3] for the case $q = 1$.

3 Proof of Theorem 1

Let $a(x')$ be a (q, ∞) atom with $q = \frac{n-1}{n-1+\alpha}$, recall that $T_{a,\alpha}(f) = \lim_{\varepsilon \to 0} T_{a,\alpha,\varepsilon}(f)$ and $T_{a,\alpha,\varepsilon}(f)$ is defined by

$$T_{a,\alpha,\varepsilon}(f)(x) = \int_{|y|>\varepsilon} b(|y|)|y|^{-n-\alpha} a(y) f(x - y)\,dy.$$

The main purpose of this section is to prove that for any (q, ∞) atom $a(x')$ with the cancellation conditions in Theorem 1, one has

$$\|T_{a,\alpha}(f)\|_{L^p(R^n)} \le C\|b\|_\infty \|f\|_{L^p_\alpha(R^n)}, \tag{3.1}$$

$$\|T_{a,\alpha,\varepsilon}(f)\|_{L^p(R^n)} \le C\|b\|_\infty \|f\|_{L^p_\alpha(R^n)}, \tag{3.2}$$

where C is a constant independent of $a(x')$, $\varepsilon > 0$, $b \in L^\infty$ and $f \in \mathcal{S}(R^n)$. As we mentioned in the introduction, Theorem 1 will be proved as soon as we establish the inequality (3.2).

Without loss of generality we may assume that $\operatorname{supp}(a)$ is the ball $B(\mathbf{1}, \rho) \cap S^{n-1}$, where $\mathbf{1} = (1, 0, \dots, 0)$. Let I_k be the interval $(2^k, 2^{k+1})$, $k = 1, 2, \dots$. Then

$$T_{a,\alpha} f(x) = \sum_{k=-\infty}^{\infty} T_k f(x), \tag{3.3}$$

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where

$$T_k f(x) = \int_{R^n} b(|y|)|y|^{-n-\alpha} a(y') \chi_{I_k}(|y|) f(x-y)\, dy.$$

It is easy to see $(T_k f)^\wedge(\xi) = \hat{\sigma}_k(\xi)\hat{f}(\xi)$, where

$$\sigma_k(x) = b(|x|)|x|^{-n-\alpha} a(x') \chi_{I_k}(|x|). \tag{3.4}$$

We have the following L^2 estimates for σ_k.

Lemma 3.5

$$(*) \qquad |\hat{\sigma}_k(\xi)| \le C\|b\|_\infty 2^{-k\alpha} |2^k A_\rho \xi|^{N+1} \rho^{(1-1/q)(n-1)}$$

$$(**) \qquad |\hat{\sigma}_k(\xi)| \le C\|b\|_\infty 2^{-k\alpha} |2^k A_\rho \xi|^{-\frac{1}{2}} \rho^{(1-1/q)(n-1)}$$

where N and $A_\rho\xi$ are as in (2.6); C is independent of $k \in \mathbb{Z}$ and $\rho > 0$.

Proof We will only prove the case $n > 2$ since the proof for $n = 2$ is essentially the same (using Lemma 2.2 instead of Lemma 2.1).

For any fixed $\xi \in R^n$ we choose a rotation O such that $O(\xi) = |\xi|\mathbf{1} = |\xi|(1, 0, \dots, 0)$. Let $y' = (s, y_2', y_3', \dots, y_n')$. Then it is easy to see that $\hat{\sigma}_k(\xi)$ is equal to, up to a constant C,

$$\int_{I_k} b(t) t^{-1-\alpha} \int_{S^{n-1}} a\big(O^{-1}(y')\big)\, e^{-it|\xi|\langle \mathbf{1}, y'\rangle}\, d\sigma(y')\, dt,$$

where O^{-1} is the inverse of O. Now $a\big(O^{-1}(y')\big)$ is again a (q, ∞) atom with support in $B(\xi', \rho) \cap S^{n-1}$, since supp $a(y') \subseteq B(\mathbf{1}, \rho) \cap S^{n-1}$. Thus we have

$$\hat{\sigma}_k(\xi) = C \int_{I_k} b(t) t^{-1-\alpha} \int_R F_a(s) e^{-it|\xi| s}\, ds\, dt, \tag{3.5}$$

where $F_a(s)$ is the function defined in Lemma 2.1. By the Lemma 2.1, without loss of generality, we may assume that F_a is supported in $\big(-2r(\xi'), 2r(\xi')\big)$. Thus using (2.4) and (2.5) we have

$$\begin{aligned}|\hat{\sigma}_k(\xi)| &\le C\|b\|_\infty \int_{I_k} t^{-1-\alpha} \Big| \int_R F_a(s) e^{-it|\xi| s}\, ds \Big|\, dt \\ &\le C 2^{-k\alpha} \|b\|_\infty |2^k A_\rho \xi|^{N+1} \rho^{(1-1/q)(n-1)},\end{aligned}$$

which proves $(*)$ of the lemma.

Using Holder's inequality on (3.5), we have

$$\begin{aligned}\hat{\sigma}_k(\xi)| &\le C\|b\|_\infty 2^{-k\alpha} \int_{2^k|\xi|}^{2^{k+1}|\xi|} t^{-1} \Big| \int_R F_a(s) e^{-its}\, ds \Big|\, dt \\ &\le C\|b\|_\infty 2^{-k\alpha} (2^k|\xi|)^{-\frac{1}{2}} \Big\{ \int_R |\hat{F}_a(t)|^2\, dt \Big\}^{\frac{1}{2}} \\ &\le C\|b\|_\infty 2^{-k\alpha} (2^k|\xi|)^{-\frac{1}{2}} \Big\{ \int_R |F_a(t)|^2\, dt \Big\}^{\frac{1}{2}}.\end{aligned}$$

By (2.5), we know

$$\left\{ \int_R |F_a(t)|^2 \, dt \right\}^{\frac{1}{2}} \leq C\rho^{(1-1/q)(n-1)} |A_\rho \xi'|^{-\frac{1}{2}}.$$

Thus we have $(**)$ of the lemma.

Now we return to the proof of (3.1). Let $\{\Phi_j\}_{-\infty}^{\infty}$ be a smooth partition of unity in $(0, \infty)$ adapted to the intervals $(2^{j-1}, 2^{j+1})$. To be precise, we choose a radial function $\Phi \in C^\infty(R^n)$ satisfying $\operatorname{supp}(\Phi) \subseteq \{x, \frac{1}{2} < |x| \leq 2\}$, $0 \leq \Phi(x) \leq 1$ and $\Phi(x) > c > 0$ if $\frac{3}{5} \leq |x| \leq \frac{5}{3}$. We let $\Phi_j(x) = \Phi(2^j x)$ and require that Φ satisfies

$$\sum_{j=-\infty}^{\infty} \Phi_j(t)^2 = 1 \quad \text{for all } t.$$

It is easy to see $\operatorname{supp}(\Phi) \subseteq (2^{-j-1}, 2^{-j+1})$.

Define the multiplier operators S_j on $\mathcal{S}(R^n)$ by

$$(S_j f)^\wedge(\xi) = \hat{f}(\xi)\Phi_j(A_\rho \xi).$$

Following the proof of Lemma in [DR, p. 544], we decompose the operator $T_{a,\alpha}$ by

$$T_{a,\alpha}(f) = \sum_{j=-\infty}^{\infty} \tilde{T}_j f, \tag{3.6}$$

where

$$\tilde{T}_j f = \sum_k S_{j+k}\big(T_k(S_{j+k} f)\big).$$

By Littlewood-Paley theory, for any $p \in (1, \infty)$, we have

$$\|\tilde{T}_j f\|_{L^p(R^n)} \leq C \Big\| \Big(\sum_k |T_k(S_{j+k} f)|^2 \Big)^{\frac{1}{2}} \Big\|_{L^p(R^n)}. \tag{3.7}$$

Thus

$$\begin{aligned} \|\tilde{T}_j f\|_2^2 &\leq C \sum_k \int_{R^n} |T_k(S_{j+k} f)(y)|^2 \, dy \\ &= C \sum_k \int_{R^n} |\Phi_{j+k}(A_\rho \xi) \hat{\sigma}_k(\xi) \hat{f}(\xi)|^2 \, d\xi \\ &\leq C \sum_k \int_{D_{j+k}} |\hat{\sigma}_k(\xi) \hat{f}(\xi)|^2 \, d\xi \end{aligned}$$

where $D_j = \{\xi : 2^{-j-1} \leq |A_\rho \xi| \leq 2^{-j+1}\}$.

If $j \geq 0$, by $(*)$ of Lemma 3.5,

$$\begin{aligned}\|\tilde{T}_j f\|_2^2 &\leq C\|b\|_\infty \sum_k 2^{-2k\alpha} \rho^{2(1/q-1)(n-1)} \int_{D_{j+k}} |\hat{f}(\xi)|^2 (2^k |A_\rho \xi|)^{2N+2} \, d\xi \\ &\leq C\|b\|_\infty 2^{-(2N+2)j} \sum_k \rho^{2(1/q-1)(n-1)} \int_{D_{j+k}} |\hat{f}(\xi)|^2 2^{-2k\alpha} \, d\xi.\end{aligned}$$

Noting that $\alpha + (1/q-1)(n-1) = 0$ and on the domain D_{j+k}, $2^{-k\alpha} \cong 2^{j\alpha}|A_\rho \xi|^\alpha \leq 2^{j\alpha}\rho^\alpha|\xi|^\alpha$, we have

$$\|\tilde{T}_j f\|_2^2 \leq C\|b\|_\infty 2^{-2j(N-\alpha+1)} \sum_k \int_{D_k} |\hat{f}(\xi)|^2 \xi^{2\alpha} \, d\xi.$$

Therefore, for $j \geq 0$, we

$$\|\tilde{T}_j f\|_2 \leq C\|b\|_\infty 2^{-j(N+1-\alpha)} \|f\|_{L^2_\alpha(R^n)}. \tag{3.8}$$

Similarly, using $(**)$ in Lemma 3.1, we have for $j < 0$,

$$\|\tilde{T}_j f\|_2 \leq C\|b\|_\infty 2^{\frac{j}{2}} \|f\|_{L^2_\alpha(R^n)}. \tag{3.9}$$

Next, we estimate the L^p norm of $\tilde{T}_j f$. First we assume $p > 2$. Let $s = (\frac{p}{2})' = \frac{p}{p-2}$. By (3.7), we can take a non-negative $g \in L^s(R^n)$ with $\|g\|_s = 1$ such that

$$\|\tilde{T}_j f\|_p^2 \leq \sum_{k=-\infty}^{\infty} \int_{R^n} |T_k(S_{k+j} f)|^2 g \, dx. \tag{3.10}$$

Since $|T_k(S_{k+j} f)(x)|^2$ is bounded by

$$\begin{aligned}C\|b\|_\infty^2 \rho^{-(n-1)(1-1/q)} 2^{-k\alpha} &\int_{2^k \leq |y| \leq 2^{k+1}} |A(y')||y|^{-n} |S_{k+j} f(x-y)|^2 \, dy \\ &= C\|b\|_\infty^2 \rho^{-(n-1)(1-1/q)} 2^{-k\alpha} L_k(|S_{j+k} f|^2)(x),\end{aligned}$$

where $A(y') = \rho^{(n-1)(1-1/q)} a(y')$ is a $(1, \infty)$ atom, and

$$L_k f(x) = \int_{2^k \leq |y| \leq 2^{k+1}} |A(y')| \, |y|^{-n} f(x-y) \, dy, \tag{3.11}$$

we have that

$$\sum_k \int_{R^n} |T_k(S_{k+j} f)|^2 g \, dx \leq C\|b\|_\infty^2 \rho^{-2(n-1)(1-1/q)} \int_{R^n} \sum_k 2^{2k\alpha} |S_{k+j} f(x)|^2 N_A g(x) \, dx,$$

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where $N_A g(x) = \sup_k L_k^* g(x)$, and

$$L_k^* g(x) = \int_{2^k \le |y| \le 2^{k+1}} |y|^{-n} |A(y')| g(x+y)\, dy.$$

By the rotation method and the L^p boundedness of the Hardy-Littlewood maximal function, it is easy to see that

$$\|N_A g\|_{L^s} \le C \|g\|_{L^s} \le C.$$

Thus by Holder's inequality, we have

$$\|\tilde{T}_j f\|_{L^p(R^n)} \le C \|b\|_\infty \rho^{(n-1)(1-1/q)} \Big\| \Big(\sum_k |2^{-k\alpha} S_{k+j} f|^2 \Big)^{\frac{1}{2}} \Big\|_{L^p(R^n)}. \tag{3.12}$$

By checking the definition of S_k and the definition of the Triebel-Lizorkin spaces in [FJW], one can easily see that (or see the lemma in Appendix)

$$\Big\| \Big(\sum |(2^{-k}/\rho)^\alpha S_k f|^2 \Big)^{\frac{1}{2}} \Big\|_{L^p(R^n)} \cong \|f\|_{\dot{F}_p^{\alpha,2}(R^n)} \cong \|f\|_{L_\alpha^p(R^n)}.$$

Thus we obtain

$$\|\tilde{T}_j f\|_{L^p(R^n)} \le C \|b\|_\infty 2^{j\alpha} \|f\|_{L_\alpha^p(R^n)}. \tag{3.13}$$

(3.13) means that for any sufficiently large r, we have

$$\|\tilde{T}_j f\|_{L^r(R^n)} \le C \|b\|_\infty 2^{j\alpha} \|f\|_{L_\alpha^r(R^n)}. \tag{3.13$'$}$$

Now for any $p \ge 2$, let $r > p$. Using the Riesz-Thorin interpolation theorem between (3.13$'$) and (3.8), we have that, for any $j \ge 0$

$$\|\tilde{T}_j f\|_{L^p(R^n)} \le C \|b\|_\infty 2^{-j\{\theta(N+1-\alpha)-(1-\theta)\alpha\}} \|f\|_{L_\alpha^p(R^n)}.$$

where $\theta = \frac{2(r-p)}{p(r-2)}$. We can see that if $r \to \infty$, then θ goes to $\frac{2}{p}$ and $\{\theta(N+1-\alpha) - (1-\theta)\alpha\}$ goes to $\frac{2N}{p} + \frac{2}{p} - \alpha > 0$, because the choice of the N. Therefore we choose a sufficiently large r such that for any $j \ge 0$

$$\|\tilde{T}_j f\|_{L^p(R^n)} \le C \|b\|_\infty 2^{-j\delta} \|f\|_{L_\alpha^p(R^n)}. \tag{3.14}$$

where $\delta = \{\theta(N+1-\alpha) - (1-\theta)\alpha\} > 0$ and $\theta = \frac{2(r-p)}{p(r-2)}$.

Similarly interpolating between (3.9) and (3.13$'$), we have a $\theta > 0$ such that for $j < 0$

$$\|\tilde{T}_j f\|_{L^p(R^n)} \le C \|b\|_\infty 2^{\theta j} \|f\|_{L_\alpha^p(R^n)}. \tag{3.15}$$

By (3.6), (3.14) and (3.15) we now have

$$\|\tilde{T}_{a,\alpha}f\|_{L^p} \le \sum_{-\infty}^{\infty} \|\tilde{T}_j f\|_{L^p} \le C\|b\|_\infty \|f\|_{L^p_\alpha(R^n)} \sum_{j\ge 0}(2^{-j\delta}+2^{-j\theta}).$$

which proves (3.1) for $p \ge 2$.

If $p < 2$, then $p' \ge 2$. Remember

$$\|\tilde{T}_j f\|_{L^p(R^n)} \le C\|b\|_\infty \rho^{(n-1)(1/q-1)} \Big\| \Big(\sum_k 2^{-k\alpha} |L_k(|S_{j+k}f|)|^2\Big)^{\frac{1}{2}} \Big\|_{L^p(R^n)}.$$

Take a sequence of functions g_k, then

$$\begin{aligned}\Big|\int_{R^n} \sum_k 2^{-k\alpha} L_k(|S_{j+k}f|) g_k(x)\,dx\Big| \\ &= C\Big|\int_{R^n} \sum_k 2^{-k\alpha} |S_{j+k}f| L_k^* g_k(x)\,dx\Big| \\ &\le C\Big\| \Big(\sum_k 2^{-k\alpha}|S_{j+k}f|^2\Big)^{\frac{1}{2}}\Big\|_p \Big\| \Big(\sum_k |L_k^* g_k(x)|^2\Big)^{\frac{1}{2}}\Big\|_{p'}.\end{aligned}$$

Taking the supremum over $\{g_k\}$ with

$$\Big\| \Big(\sum_k |L_k^* g_k(x)|^2\Big)^{\frac{1}{2}}\Big\|_{p'} \le 1$$

we obtain, for $1 < p < 2$,

$$\|\tilde{T}_j f\|_{L^p(R^n)} \le C\|b\|_\infty \rho^{(n-1)(1/q-1)} \Big\| \Big(\sum_k 2^{-k\alpha}|S_{j+k}f|^2\Big)^{\frac{1}{2}}\Big\|_p.$$

Thus as in the proof of (3.13′), for any $r > 0$ we have

$$\|\tilde{T}_j f\|_{L^r(R^n)} \le C\|b\|_\infty 2^{j\alpha}\|f\|_{L^r_\alpha(R^n)}. \tag{3.16}$$

Choosing r sufficiently close to 1, similar to the case $p \ge 2$, we can complete the proof of (3.1) for $1 < p < 2$ by interpolating (3.8), (3.9) and (3.16).

Now, (3.2) can be obtained easily from (3.1). For any $\varepsilon > 0$, let $\tilde{b}(t) = b(t)\chi_{(\varepsilon,\infty)}(t)$. Then $\|\tilde{b}\|_\infty \le \|b\|_\infty$ for any $\varepsilon > 0$. Replacing b by $\tilde{b}$ in (3.1), we have

$$\|T_{a,\alpha,\varepsilon}(f)\|_{L^p(R^n)} \le C\|\tilde{b}\|_\infty \|f\|_{L^p_\alpha(R^n)}.$$

Corollary 1 *For the operator $T_{\Omega,\alpha}$ defined in Theorem 1, we have*

$$\|T_{\Omega,\alpha}(f)\|_{\dot{F}_p^{-\alpha,2}(R^n)} \le C\|f\|_{L^p} \quad \textit{for all } 1 < p < \infty.$$

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Proof Let T^* be the dual operator of $T = T_{\Omega,\alpha}$. It is easy to see that T^* has the same boundedness as T. For any $p \in (1,\infty)$, take its conjugate index p' and recall that the dual space of $\dot{F}_p^{-\alpha,2}(R^n)$ is $\dot{F}_{p'}^{\alpha,2}(R^n)$. Thus for $f \in L^p$ and any $g \in \dot{F}_{p'}^{\alpha,2}(R^n)$, we have

$$|\langle Tf, g\rangle| = |\langle T^* g, f\rangle| \le \|f\|_p \|T^* g\|_{p'} \le \|f\|_p \|g\|_{\dot{F}_{p'}^{\alpha,2}(R^n)}.$$

Corollary 2 *For any $\beta \in R$, we have*

$$\|T_{\Omega,\alpha}(f)\|_{\dot{F}_p^{-\beta,2}} \le C\|f\|_{\dot{F}_p^{\alpha-\beta,2}}.$$

Proof First we assume $\beta > 0$. Let R_β be the Riesz potential operator. Then we know that R_β is an isomorphism of $\dot{F}_p^{0,2}$ onto $\dot{F}_p^{-\beta,2}$. Namely $\|R_\beta f\|_{\dot{F}_p^{\alpha,2}} \cong \|f\|_{\dot{F}_p^{\alpha-\beta,2}}$. Thus we have

$$\|T_{\Omega,\alpha}(f)\|_{\dot{F}_p^{-\beta,2}} \cong \|T_{\Omega,\alpha}(R_\beta f)\|_{\dot{F}_p^{0,2}} \le C\|R_\beta f\|_{\dot{F}_p^{\alpha,2}} \le C\|f\|_{\dot{F}_p^{\alpha-\beta,2}}.$$

Using duality, we can obtain the corollary for $\beta < 0$.

Remark For $\gamma \ge 1$, let Δ_γ denote the collection of measurable functions $b(t)$ on R_+ satisfying

$$\|b\|_{\Delta_\gamma} = \Big(\sup_{R>0} \frac{1}{R}\int_0^R |b(t)|^\gamma\,dt\Big)^{\frac{1}{\gamma}} < \infty.$$

By checking the proof, in Theorem 1 we can replace the requirement b being bounded by a less restrictive one $b \in \Delta_2$.

4 Maximal Operators: Proof of Theorem 2

In this section we will study the truncated maximal operator

$$T^*_{\Omega,\alpha} f(x) = \sup_{\varepsilon>0} |T_{\Omega,\alpha,\varepsilon} f(x)|. \tag{4.1}$$

As we discussed in (1.9), to prove Theorem 2, it suffices to show the uniform boundedness of $T^*_{a_j,\alpha}(f)$ for all (q,∞) atoms a_j. Thus in this section, for the sake of simplicity, we may assume that Ω is a (q,∞) atom. For any such atom Ω, we define the maximal operator

$$M_{\Omega,\alpha} f(x) = \sup_k \int_{I_k} t^{-1-\alpha} \Big| \int_{S^{n-1}} \Omega(y') f(x - ty')\,d\sigma(y')\Big|\,dt, \tag{4.2}$$

where I_k is the interval $(2^k, 2^{k+1}]$. We have the following lemma.

Lemma 4.3 *Let $\alpha = m$, $m = 0, 1, 2, \ldots$, and Ω be a (q,∞) atom satisfying (1.5) with $q = \frac{n-1}{n-1+m}$. Then for $1 < p < \infty$,*

$$\|M_{\Omega,m} f\|_{L^p(R^n)} \le C\|f\|_{L^p_\alpha(R^n)}, \tag{4.3}$$

where the constant C is independent of the atom Ω.

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Proof Without loss of generality, we may assume supp $\Omega \subseteq B(\mathbf{1}, \rho) \cap S^{n-1}$. Now for any test function f, by (2.2), we have

$$M_{\Omega,m} f(x) = \sup_k \int_{I_k} t^{-1-m} \Big| \int_{S^{n-1}} \Omega(y') \big(f(x - ty') - f(x - t\mathbf{1}) \big) \, d\sigma(y') \Big| \, dt.$$

By Taylor's Theorem, for $y' \in$ supp Ω we have that

$$\Big| \int_{S^{n-1}} \Omega(y') \big(f(x - ty') - f(x - t\mathbf{1}) \big) \, d\sigma(y') \Big|$$

is less than or equal to

$$C \sum_{|\beta|=m} \int_0^1 \int_{S^{n-1}} |\Omega(y')| \, \big| (D^\beta f)(x - t\mathbf{1} + st\big(y' - \mathbf{1}\big)) \, \{t(\mathbf{1} - y')\}^\beta \big| \, d\sigma(y') \, ds$$

$$\leq C t^m \rho^m \sum_{|\beta|=m} \int_0^1 \int_{S^{n-1}} |\Omega(y')| \, \big| (D^\beta f)(x - t\mathbf{1} + st\big(y' - \mathbf{1}\big)) \, \big| \, d\sigma(y') \, ds$$

Therefore, $\|M_{\Omega,\alpha} f\|_{L^p(R^n)}$is dominated by

$$C \sum_{|\beta|=m} \int_0^1 \int_{S^{n-1}} \rho^m |\Omega(y')| \, \|\mu_{\gamma(s,y')}(D^\beta f)\|_{L^p(R^n)} \, d\sigma(y') \, ds,$$

where

$$\mu_{\gamma(s,y')} f(x) = \sup_k \int_{I_k} \big| f\big(x - \gamma(t, s, y')\big) \big| \, t^{-1} \, dt$$

and $\gamma(t, s, y') = t(1 + sy' - s\mathbf{1})$. It is known from [St, p. 477] that

$$\|\mu_{\gamma(s,y')} f\|_p \leq C \|f\|_p, \tag{4.4}$$

where C is independent of s and y'. Noting that $\int_{S^{n-1}} \rho^m |\Omega(y')| \, d\sigma(y') \leq C$ with C independent of Ω, by (4.2) we have

$$\|M_{\Omega,m} f\|_p \leq C \sum_{|\beta|=m} \|D^\beta f\|_p \cong C \|f\|_{L^p_m(R^n)}.$$

The lemma is proved.

We can obtain Theorem 2 by (1.9) and showing the following proposition.

Proposition 1 *Let $m = 0, 1, 2, \ldots,$ and Ω be a (q, ∞) atom satisfying (1.5) with $q = \frac{n-1}{n-1+m}$. For $1 < p < \infty$, we have*

$$\|T^*_{\Omega,m} f\|_{L^p(R^n)} \leq \|f\|_{L^p_m(R^n)}, \tag{4.5}$$

where the constant C is independent of Ω

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Proof Without loss of generality, we may assume that Ω is supported in $B(\mathbf{1}, \rho)$. Since

$$T_{\Omega,m} f = \sum_k T_k f = \sum_k \sigma_k * f,$$

for any $\varepsilon > 0$, there is a k such that $2^{k-1} < \varepsilon \le 2^k$. So it is easy to see that

$$T^*_{\Omega,m} f \le M_{\Omega,m} f + \sup_k \Big| \sum_{j=k}^{\infty} \sigma_j * f \Big|.$$

Write $J_k(f) = \sum_{j=k}^{\infty} \sigma_j * f$. By Lemma 4.3, to prove the proposition it suffices to show the boundedness of $J^*_k(f) = \sup_k |J_k(f)|$. Let δ be the Dirac delta function. Take a radial function $\varphi \in \mathcal{S}(R^n)$ such that $\varphi(\xi) = 1$ when $|\xi| \le 1$ and $\varphi(\xi) = 0$ if $|\xi| > 2$. Let $A_\rho \xi$ be defined as in Lemma 2.1. Let $\varphi_k(\xi) = \varphi(2^k |A_\rho \xi|)$ and $\hat{\Phi}_k(\xi) = \varphi_k(\xi)$. Now

$$\begin{aligned} J_k(f) &= (\delta - \Phi_k) * \sum_{j=k}^{\infty} \sigma_j * f + \Phi_k * (T_{\Omega,m} f) - \Phi_k * \sum_{-\infty}^{k-1} \sigma_j * f \\ &= I_{k,1}(f) + I_{k,2}(f) + I_{k,3}(f) \end{aligned}$$

clearly,

$$\| \sup_k |I_{k,2}(f)| \, \|_p \le C \| T_{\Omega,m} f \|_{L^p} \le C \|f\|_{L^p_m} \tag{4.6}$$

Next,

$$\begin{aligned} \sup_k |I_{k,3}(f)| &= \sup_k \Big| \sum_{j=1}^{\infty} \Phi_k * \sigma_{k-j} * f \Big| \\ &\le \sum_{j=1}^{\infty} \sup_k |\Phi_k * \sigma_{k-j} * f| \\ &= \sum_{j=1}^{\infty} \Delta_j(f). \end{aligned}$$

Each $\Delta_j(f)$ is bounded from L^p_m to L^p because of the boundedness of $M_{\Omega,m}$ in Lemma 4.3. By Plancherel's theorem

$$\|\Delta_j(f)\|_2^2 \le C \sum_{k=-\infty}^{\infty} \int_{R^n} |\hat{\Phi}_k(\xi) \hat{\sigma}_{k-j}(\xi) f(\xi)|^2 \, d\xi$$

Noting $\hat{\Phi}_k(\xi) = \varphi(2^k |A_\rho \xi|)$ and the choice for this φ, taking $N = m+1$ in Lemma 3.5, we have that $\|\Delta_j(f)\|_2^2$ is dominated by, up to a constant C,

$$\begin{aligned} &2^{-2(m+1)j} \rho^{2(1-1/q)(n-1)} \int_{R^n} |A_\rho \xi|^{2m} \sum_k |\hat{\Phi}_k(\xi)|^2 |2^k A_\rho \xi| \, |f(\xi)|^2 \, d\xi \\ &\qquad \le C 2^{-2(m+1)j} \sup_{\xi \ne 0} \sum_k |\hat{\Phi}_k(\xi)|^2 |2^k A_\rho \xi| \, \|f\|^2_{L^2_m(R^n)}, \end{aligned}$$

Since $|A_\rho\xi| \le \rho|\xi|$ and $m + (1 - 1/q)(n - 1) = 0$.

Noting the choice of $\hat{\Phi}_k$, we have $\|\Delta_j(f)\|_2 \le C2^{-(m+1)j}\|f\|_{L^2_m}$. This shows the (L^p_m, L^p) boundedness for $\sup_k |I_{k,3}(f)|$ by interpolating $\|\Delta_j\|_{L^p_\alpha \to L^p}$ and $\|\Delta_j\|_{L^2_\alpha \to L^2}$.

Finally,

$$\sup_k |I_{k,1}(f)| \le \sum_{j=1}^{\infty} G_j(f),$$

where

$$G_j(f) = \sup_k |(\delta - \Phi_k) * \sigma_{k+j} * f|.$$

Clearly by Lemma 4.3 we have $\|G_j(f)\|_{L^p} \le C\|f\|_{L^p_m}$.

On the other hand by Plancherel's formula and Lemma 3.5 we have

$$\|G_j(f)\|_2^2 \le \rho^{2(1-1/q)(n-1)} 2^{-j} \int_{R^n} 2^{-2k\alpha} \Big| \sum_k \big(1 - \varphi(2^k|A_\rho\xi|)\big) \Big| 2^k A_\rho \xi|^{-\frac{1}{2}} \hat{f}(\xi)|^2 \, d\xi$$

By the choice of ξ, we know that for any ξ such that $1 - \varphi(2^k|A_\rho\xi|) \ne 0$, it satisfies $2^{-k\alpha} \le C|A_\rho\xi|^\alpha$. Thus we have

$$\begin{aligned}\|G_j(f)\|_2 &\le C2^{-\frac{j}{2}}\|f\|_{L^2_m(R^n)} \sup_{\xi\ne 0} \sum_{2^k > |A_\rho\xi|^{-1}} 2^{-\frac{k}{2}}|A_\rho\xi|^{-\frac{1}{2}} \\ &\le C2^{-\frac{j}{2}}\|f\|_{L^2_m(R^n)}.\end{aligned}$$

Thus the boundedness of $\sup_k |I_{k,1}(f)|$ follows by the Riesz-Thorin interpolation. Proposition 1 is proved.

5 Boundedness on H^p Sobolev Spaces

In this section, we will study the operator $T_{\Omega,\alpha}$ on the space $F_p^{\alpha,2}$ for $0 < p \le 1$. It is known from [FJW] that $H^p = \dot{F}_p^{0,2}$, where H^p are the classical Hardy spaces. Thus, we denote, for $0 < p \le 1$, $H^p_\alpha = \dot{F}_p^{\alpha,2}$ and call H^p_α the H^p Sobolev spaces. The Triebel-Lizorkin space $\dot{F}_p^{\alpha,q}$ has a standard atomic decomposition obtained in [HPW]. Below we only review the case $q = 2$ (see also [Str]), which is what we need in this section.

***Definition 5.1* (see [HPW])** For $\alpha \ge 0$. A function $a(x)$ is said to be a $(p, 2, \alpha)$ *atom* if

(i) supp $a \subseteq Q$, Q is a cube in R^n.
(ii) $\|a\|_{L^2_\alpha} \le |Q|^{\frac{1}{2}-\frac{1}{p}}$.
(iii) For every polynomial P of degree at most $N = [n(1/p - 1) - \alpha]$, $\int_{R^n} a(x)P(x)\, dx = 0$.

In [HPW], we know that any $f \in H^p_\alpha$ has an atomic decomposition $f = \sum \lambda_j a_j$ with $(\sum |\lambda_j|^p)^{\frac{1}{p}} \cong \|f\|_{H^p_\alpha}$, where all $a_j(x)$ are $(p, 2, \alpha)$ atoms.

Now we are in a position to state the main result in this section.

Theorem 3 *Let* $0 < \alpha < \frac{n}{2}$ *and* $\Omega \in L^1(S^{n-1})$ *satisfy (1.5) for all* $m \le [\alpha]$. *We have a constant* $C > 0$ *such that*

$$\|T_{\Omega,\alpha} f\|_{H^p(R^n)} \le C\|f\|_{H^p_\alpha(R^n)},$$

for all $f \in C_c^\infty(R^n) \cap H^p_\alpha(R^n)$, *where* $\frac{n}{n+\alpha} < p \le 1$. *Thus* $T_{\Omega,\alpha}$ *is extended to be defined on* H^p_α *in the usual manner.*

Proof Let

$$K_{\Omega,\varepsilon}(x) = b(|x|)|x|^{-n-\alpha}\chi_{(\varepsilon,\infty)}(|x|)\Omega(x').$$

Then $K_\Omega = \text{p.v.}K_{\Omega,\varepsilon}$ defines a distribution in $\mathcal{D}'(\mathcal{D} = C_c^\infty)$ and $T_{\Omega,\alpha} f = K_\Omega * f$. It is known that $H^p(R^n)$ can be characterized by the Riesz transforms R_j, $j = 1, 2, \dots, n$, and their compositions (the higher Riesz transforms). For instance, if $\frac{n}{n+1} < p \le 1$, then

$$\|f\|_{H^p} \cong \|f\|_{L^p} + \sum_{j=1}^{\infty} \|R_j(f)\|_{L^p}.$$

Thus

$$\|T_{\Omega,\alpha}(f)\|_{H^p} \cong \|K_\Omega * f\|_{L^p} + \sum_{j=1}^{\infty} \|K_\Omega * R_j(f)\|_{L^p}.$$

It is also known (see [T] among many references) that the Riesz transforms are bounded on H^p_α. To prove the theorem, it suffices to show

$$\|T_{\Omega,\alpha}(f)\|_{L^p} \le C\|f\|_{H^p_\alpha}. \tag{5.1}$$

Checking the proof in [HPW], it is easy to see that for $f \in C_c^\infty$, one can choose an atomic decomposition $\sum \lambda_j a_j(x)$ such that for any multi-index β,

$$\lim_{m\to\infty} \sup_{x\in\mathcal{K}} \Big| D^\beta(\sum_{j=1}^{m} \lambda_j a_j(x) D^\beta f(x) \Big| = 0$$

for any compact set $\mathcal{K}$. In other words, $\sum \lambda_j a_j(x) = f(x)$ in the topology of $\mathcal{E}$ test space, where $\mathcal{E}$ is the set of all C^∞ functions. Now, following [St, p. 115], we write the distribution kernel $K_\Omega = K_\Omega^0 + K_\Omega^\infty$, where K_Ω^∞ is an L^1 function since $\alpha > 0$, and K_Ω^0 is a distribution in $\mathcal{D}'$ having compact support so that K_Ω^0 must be a distribution in $\mathcal{E}'$. Thus

$$T_{\Omega,\alpha}(f) = \sum \lambda_j T_{\Omega,\alpha}(a_j),$$

and

$$\|T_{\Omega,\alpha}(f)\|^p_{L^p(R^n)} \le \sum |\lambda_j|^p \|T_{\Omega,\alpha}(a_j)\|^p_{L^p(R^n)}.$$

We now only need to show that there exists a constant C independent of atoms, such that for any $(p, 2, \alpha)$ atom $a(x)$

$$\|T_{\Omega,\alpha}(a)\|_{L^p(R^n)} \le C. \tag{5.2}$$

Also without loss of generality, we may assume that the support of $a(x)$ is the ball $B = B(o, \rho)$.

We first show the case $p = 1$.

$$\|T_{\Omega,\alpha}(a)\|_{L^1} \leq \int_{|x|\leq 8\rho} |T_{\Omega,\alpha}a(x)|\,dx + \int_{|x|>8\rho} |T_{\Omega,\alpha}a(x)|\,dx$$
$$:= I_1 + I_2.$$

By Theorem 1 and (ii) in Definition 5.1, we have

$$I_1 \leq C\rho^{\frac{n}{2}}\|T_{\Omega,\alpha}(a)\|_{L^2} \leq C\rho^{\frac{n}{2}}\|a\|_{L^2_\alpha} \leq C. \tag{5.3}$$

By Fubini's Theorem,

$$\begin{aligned} I_2 &\leq C\int_B |a(y)| \int_{|x|>8\rho} |x-y|^{-n-\alpha}|\Omega(x-y)|\,dx\,dy \\ &\leq C\int_B |a(y)|\,dy \int_{|x|>4\rho} |x|^{-n-\alpha}|\Omega(x)|\,dx \\ &\leq C\|\Omega\|_{L^1(S^{n-1})}\rho^{-\alpha}\int_B |a(y)|\,dy. \end{aligned}$$

Noting $a(y)$ is a $(1, 2, \alpha)$ atom, we obtain $I_2 \leq C$ if we can prove the following lemma:

Lemma 5.5 *For any $(p, 2, \alpha)$ atom with support in $B = B(o, \rho)$ we have*

$$\int_B |a(x)|\,dx \leq C\rho^{n-\frac{n}{p}+\alpha}.$$

Proof Let χ_B be the characteristic function of B and let $\tilde{a}(x) = \big(\operatorname{sign} a(x)\big)\,\chi_B(x)$. Then by duality

$$\int_B |a(x)|\,dx = \int_{R^n} a(x)\tilde{a}(x)\,dx \leq \|a\|_{L^2_\alpha}\|\tilde{a}\|_{\dot{F}_2^{-\alpha,2}} \leq C\rho^{\frac{n}{2}-\frac{n}{p}}\|\tilde{a}\|_{\dot{F}_2^{-\alpha,2}}.$$

Let R_α be the Riesz potential. It is well known that (see [FJW])

$$\|\tilde{a}\|_{\dot{F}_2^{-\alpha,2}} \cong \|R_\alpha(\tilde{a})\|_{L^2} \leq \|\tilde{a}\|_{L^r}$$

with $r = \frac{2n}{n+2\alpha}$. Here $r > 1$, since $\alpha < \frac{n}{2}$. Now $\|\tilde{a}\|_{L^r} \leq C\rho^{\alpha+\frac{n}{2}}$. The lemma is proved.

Next we show (5.2) for the case $\frac{n}{n+\alpha} < p < 1$.

$$\|T_{\Omega,\alpha}(a)\|_{L^p}^p \leq \int_{|x|\leq 8\rho} |T_{\Omega,\alpha}a(x)|^p\,dx + \int_{|x|>8\rho} |T_{\Omega,\alpha}a(x)|^p\,dx$$
$$:= I_1 + I_2.$$

By Theorem 1 and (ii) in Definition 5.1, we have

$$I_1 \le C\rho^{n(1-\frac{p}{2})}\|T_{\Omega,\alpha}(a)\|_{L^2}^p \le C\rho^{n(1-\frac{p}{2})}\|a\|_{L^2_\alpha}^p \le C. \tag{5.4}$$

By Holder's inequality,

$$\begin{aligned} I_2 &\le C\sum_{j=3}^{\infty}(2^j\rho)^{n(1-p)}\Big(\int_B |a(y)| \int_{2^j\rho\le|x|<2^{j+1}\rho} |x-y|^{-n-\alpha}|\Omega(x-y)|\,dx\,dy\Big)^p \\ &\le C\sum_{j=2}^{\infty}(2^j\rho)^{n(1-p)}\Big(\int_B |a(y)|\,dy \int_{2^j\rho\le|x|<2^{j+1}\rho} |x|^{-n-\alpha}|\Omega(x)|\,dx\Big)^p \\ &\le C\|\Omega\|_{L^1(S^{n-1})}^p\sum_{j=2}^{\infty}(2^j\rho)^{n(1-p)}2^{-\alpha jp}\Big(\rho^{-\alpha}\int_B |a(y)|\,dy\Big)^p. \end{aligned}$$

Thus by Lemma 5.5,

$$I_2 \le C\|\Omega\|_{L^1(S^{n-1})}^p \sum_{j=2}^{\infty} 2^{j(n-np-\alpha p)} \le C,$$

because $n - np - \alpha p < 0$. The theorem is proved.

To enlarge the range of p in Theorem 3, we need to assume some "smoothness" on the function Ω. Here we will use the L^r-Dini condition that was used in [KW] and [DL]. For $\Omega \in L^r(S^{n-1})$, let $\omega_r(\delta)$ denote the integral modulus of continuity of order r of Ω,

$$\omega_r(\delta) = \sup_{|R|<\delta}\Big(\int_{S^{n-1}} |\Omega(Ry') - \Omega(y')|^r\,d\sigma(y')\Big)^{\frac{1}{r}},$$

where R is the rotation in R^n and $|R| = \|R - I\|$.

Theorem 4 *Let $0 < \alpha \le \frac{n}{2}$. Suppose that $\Omega \in L^1(S^{n-1})$ satisfies (1.5) for all $m \le [\alpha]$, and*

$$\int_0^1 t^{-1-\gamma}\omega_1(t)\,dt < \infty \quad \text{for some } 0 < \gamma \le 1.$$

Then there is a C such that

$$\|T_{\Omega,\alpha} f\|_{H^p} \le C\|f\|_{H^p_\alpha} \quad \text{for all } f \in H^p_\alpha \cap C_c^\infty(R^n),$$

where $p \in (\frac{n}{n+\alpha+\gamma}, \frac{n}{n+\alpha}]$.

Proof Similar to the proof of Theorem 3, it suffices to show that there is a constant $C > 0$ such that (5.2) holds for all $(p, 2, \alpha)$ atoms $a(x)$ with support in the ball $B = B(O, \rho)$. By checking the proof of Theorem 3, we only need to show $I_2 = \int_{|x|>8\rho} |T_{\Omega,\alpha}a(x)|^p\,dx \le \infty$.

By Holder's inequality and the cancellation condition on $a(y)$, we have

$$
\begin{aligned}
&I_2 \le \\
&C\sum_{j=3}^{\infty}(2^j\rho)^{n(1-p)}\Big(\int_{2^j\rho\le|x|<2^{j+1}\rho}\int_B |a(y)||x-y|^{-n-\alpha}|\Omega(x-y)-\Omega(x)|\,dy\,dx\Big)^p\\
&\quad + C\sum_{j=2}^{\infty}(2^j\rho)^{n(1-p)}\Big(\int_B \rho|a(y)|\,dy\int_{2^j\rho\le|x|<2^{j+1}\rho}|x|^{-n-\alpha-1}|\Omega(x)|\,dx\Big)^p\\
&= J_1 + J_2.
\end{aligned}
$$

It is easy to see that

$$J_2 \le C\|\Omega\|^p_{L^1(S^{n-1})}\sum_{j=2}^{\infty}(2^j\rho)^{n(1-p)}2^{-(\alpha+1)jp}\Big(\rho^{-\alpha}\int_B|a(y)|\,dy\Big)^p.$$

By Lemma 5.5, we have

$$J_2 \le C\|\Omega\|^p_{L^1(S^{n-1})}\sum_{j=2}^{\infty}2^{j(n-np-\alpha p-p)} \le C.$$

because $n - np - \alpha p - p < 0$. It remains to estimate J_1. By Fubini's Theorem,

$$J_1 \le C\sum_{j=3}^{\infty}(2^j\rho)^{n(1-p)}\Big(\int_B|a(y)|\int_{2^j\rho\le|x|<2^{j+1}\rho}|x|^{-n-\alpha}|\Omega(x-y)-\Omega(x)|\,dx\,dy\Big)^p.$$

Pick some β such that $\frac{\beta}{1-p} > 1$. By Holder' inequality, we have

$$J_1 \le C\Big(\int_B|a(y)|\sum_{j=3}^{\infty}(2^j\rho)^{n(1/p-1)}j^{\frac{\beta}{p}}\tilde{\Omega}_{\alpha,j}(y)\,dy\Big)^p$$

where

$$\tilde{\Omega}_{\alpha,j}(y) = \int_{2^j\rho\le|x|<2^{j+1}\rho}|x|^{-n-\alpha}|\Omega(x-y)-\Omega(x)|\,dx.$$

By the same argument as in Lemma 5.5, we have that

$$
\begin{aligned}
J_1 &\le C\rho^{n(p/2-1)}\Big\|\sum_{j=3}^{\infty}(2^j\rho)^{n(1/p-1)}j^{\frac{\beta}{p}}\tilde{\Omega}_{\alpha,j}\Big\|^p_{L^r(R^n)}\\
&\le C\rho^{n(p/2-1)}\Big(\sum_{j=3}^{\infty}(2^j\rho)^{n(1/p-1)}j^{\frac{\beta}{p}}\tilde{\Omega}_{\alpha,j}(y)\Big)^{\frac{p}{r}}
\end{aligned}
\tag{5.5}
$$

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where $r = \frac{2n}{n+2\alpha}$.

By [DL], we know that for $y \in B$,

$$\tilde{\Omega}_{\alpha,j}(y) = \int_{2^j\rho \le |x| < 2^{j+1}\rho} |x|^{-n-\alpha} |\Omega(x-y) - \Omega(x)|\, dx$$
$$\le C \int_{2^j\rho}^{2^{j+1}\rho} t^{-1-\alpha}\omega_1(|y|/t)\, dt.$$

Changing variable, we have

$$\tilde{\Omega}_{\alpha,j}(y) \le C2^{-j\alpha}\rho^{-\alpha}2^{-\gamma j} \int_{|y|/2^{j+1}\rho}^{|y|/2^j\rho} t^{-1-\gamma}\omega_1(t)\, dt.$$

Thus a simple computation shows that J_1 is bounded by

$$C\rho^{-\alpha p - np/2}\Big(\int_B \Big| \sum_{j=3}^{\infty} 2^{jn(1/p-1)} j^{\frac{\beta}{p}} 2^{-j(\alpha+\gamma)} \int_{|y|/2^{j+1}\rho}^{|y|/2^j\rho} t^{-1-\gamma}\omega(t)\, dt\Big|^r dy\Big)^{\frac{p}{r}}$$
$$\le C\rho^{-\alpha p - np/2}\Big(\int_0^1 t^{-1-\gamma}\omega(t)\, dt\Big)^p \Big(\int_B 1\, dy\Big)^{\frac{p}{r}} \le C$$

since $n(1/p-1) - \alpha - \gamma < 0$. The theorem is proved.

Theorem 3 and Theorem 4 can viewed as a supplement of the convolution case of Theorem 1 in [CDF].

In [CDF], we studied singular integrals with variable kernels. The function $\omega(x,z)$ defined on $R^n \times R^n$ is said to belong to $L^\infty(R^n) \times L^r(S^{n-1})$, $r \ge 1$, if it satisfies the following conditions:

(i) $\omega(x, \lambda z) = \omega(x, z)$ for all $\lambda > 0$ and all $x, z \in R^n$.

(ii) $\|\omega\|_{L^\infty \times L^r} := \sup_{x\in R^n} \big(\int_{S^{n-1}} |\omega(x,z')|^r\, d\sigma(z')\big)^{\frac{1}{r}} < \infty$,

where $z' = z/|z|$.

For $\alpha > 0$, we define the operator $T_{\omega,\alpha} f(x)$ with variable singular kernel by

$$T_{\omega,\alpha} f(x) = \text{p.v.} \int_{R^n} \omega(x, x-y)|x-y|^{-n-\alpha} f(y)\, dy,$$

where $f \in \mathcal{S}(R^n)$ and $\omega(x,y) \in L^\infty(R^n) \times L^1(S^{n-1})$ satisfies

$$\int_{S^{n-1}} \omega(x,y')Y_m(y')\, d\sigma(y') = 0 \qquad (5.6)$$

for all spherical harmonic polynomials Y_m with degree $\le [\alpha]$. We recall the following theorem by Calderon and Zygmund.

***Theorem C* [CZ3]** *If $\omega \in L^\infty(R^n) \times L^r(S^{n-1})$, $r > 2(n-1)/n$, satisfies*

$$\int_{S^{n-1}} \omega(x,y')\, d\sigma(y') = 0, \qquad (5.6')$$

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then there is a $C > 0$ *such that*

$$\|T_{\omega,0}f\|_{L^2(R^n)} \leq C\|f\|_{L^2(R^n)}$$

for all $f \in \mathcal{S}(R^n)$.

As an extension of the above theorem, we will establish the following theorem.

Theorem 5 *Let* $\alpha \geq 0$*. If* $\omega \in L^\infty(R^n) \times L^r(S^{n-1})$ *with* $r > \max\{1, \frac{2(n-1)}{n+2\alpha}\}$*. Additionally, assume that* ω *satisfies (5.6). Then there is a* $C > 0$ *such that*

$$\|T_{\omega,\alpha}f\|_{L^2(R^n)} \leq C\|f\|_{L^2_\alpha(R^n)}.$$

Proof Using the spherical harmonic development [CZ3] and (5.6), we have

$$\omega(x,\cdot) = \sum_{m=[\alpha]+1}^{\infty} \sum_{j=1}^{N(m)} a_{m,j}(x) Y_{m,j},$$

$$a_{m,j}(x) = \int_{S^{n-1}} \omega(x, y')\overline{Y_{m,j}(y')}\, d\sigma(y'),$$

where each $Y_{m,j}$ is a spherical harmonic polynomial with degree m and $N(m) \cong m^{n-2}$. In the following, for simplicity we denote $\sum_{m=[\alpha]+1}^{\infty}$ by $\sum_m$ and denote $\sum_{j=1}^{N(m)}$ by $\sum_j$. Let

$$T_{\alpha,m,j}f(x) = \int_{R^n} f(x-y)|y|^{-n-\alpha} Y_{m,j}(y')\, d\sigma(y').$$

By Holder's inequality, we now have

$$|T_{\omega,\alpha}f(x)|^2 \leq \Big(\sum_m \sum_j |a_{m,j}(x)|^2 m^{-\varepsilon(1+2\alpha)}\Big)\Big(\sum_m m^{\varepsilon(1+2\alpha)} \sum_j |T_{\alpha,m,j}f(x)|^2\Big),$$

where ε is less than and sufficiently close to 1. Now we can see that the series in the first parenthesis on the right side of the above inequality, for each x fixed, is equal to $\|\omega(x,\cdot)\|^2_{L^2_{-\beta}(S^{n-1})}$, where $L^2_{-\beta}(S^{n-1})$ is the Sobolev space on S^{n-1} with $\beta = \varepsilon(\frac{1}{2}+\alpha)$ for any $0 < \varepsilon < 1$. So choosing ε sufficiently close to 1, by the Sobolev imbedding theorem $L^r \subset L^2_{-\beta}$, (or use Proposition 4.4 in [Co2]),

$$\sup_{x\in R^n} \|\omega(x,\cdot)\|_{L^2_{-\beta}(S^{n-1})} \leq C\|\omega\|_{L^\infty(R^n)\times L^r(S^{n-1})}$$

with $r > \max\{1, \frac{2(n-1)}{n+2\alpha}\}$. So to prove the theorem it remains to show that for ε close to 1, we have

$$\sum_m m^{\varepsilon(1+2\alpha)} \sum_j \|T_{\alpha,m,j}f\|^2_{L^2(R^n)} \leq C\|f\|^2_{L^2_\alpha(R^n)}. \tag{5.7}$$

By Plancherel's theorem,

$$\|T_{\alpha,m,j}f\|_2^2 \cong \|\hat{f}I_{\alpha,m,j}\|_2^2 \tag{5.8}$$

where

$$I_{\alpha,m,j}(\xi) = \int_{R^n} |y|^{-n-\alpha} Y_{m,j}(y')e^{-2\pi i\langle y,\xi\rangle}\, dy.$$

Noting $m \geq [\alpha] + 1$, so by Theorem 3.10 on [SW, p. 158] and the Weber-Sonine integral formula (see [S]) we have

$$I_{\alpha,m,j}(\xi) \cong m^{-\frac{n}{2}-\alpha}|\xi|^{\alpha} Y_{m,j}(\xi).$$

Thus, noting $\sum_j |Y_{m,j}(\xi)|^2 \leq Cm^{n-2}$, we have

$$\sum_j \|\hat{f}I_{\alpha,m,j}\|_2^2 \leq Cm^{-n-2\alpha} \sum_j \int_{R^n} |\hat{f}(\xi)|^2|\xi|^{2\alpha}|Y_{m,j}(\xi)|^2\, d\xi.$$

By (5.8) and the above inequality, we obtain

$$\begin{aligned}\sum_j \|T_{\alpha,m,j}f\|_2^2 &\leq Cm^{-2-2\alpha} \int_{R^n} |\hat{f}(\xi)|^2|\xi|^{2\alpha}\, d\xi \\ &\leq Cm^{-2-2\alpha}\|f\|^2_{L^2_\alpha(R^n)}\end{aligned} \tag{5.9}$$

which implies (5.7). The theorem is proved.

By Theorem 5 and the proof of Theorem 3, we can easily obtain:

Theorem 6 *Let $0 < \alpha < \frac{n}{2}$ and $\omega \in L^\infty(R^n) \times L^r(S^{n-1})$, $r > \max\{1, \frac{2(n-1)}{(n+2\alpha)}\}$. If ω satisfies (5.6), then there is a $C > 0$ such that*

$$\|T_{\omega,\alpha}f\|_{H^p(R^n)} \leq C\|f\|_{H^p_\alpha(R^n)} \quad \text{for all } f \in C_c^\infty(R^n) \cap H^p_\alpha(R^n).$$

where $\frac{n}{n+\alpha} < p \leq 1$.

6 Appendix

6.1 Triebel-Lizorkin Spaces and Sobolev Spaces

Fix a radial function $\Phi \in C^\infty(R^n)$ satisfying $\operatorname{supp}(\Phi) \subseteq \{x : \frac{1}{2} < |x| \leq 2\}$, $0 \leq \Phi(x) \leq 1$, and $\Phi(x) > c > 0$ if $\frac{3}{5} \leq |x| \leq \frac{5}{3}$. Let $\Phi_j(x) = \Phi(2^j x)$. Define the function Ψ_j by $\hat{\Psi}_j(\xi) = \Phi_j(\xi)$, so that $\widehat{\Psi_j * f}(\xi) = \hat{f}(\xi)\Phi_j(\xi)$.

For $1 < p, q < \infty$ and $\alpha \in R$, the homogeneous Triebel-Lizorkin space $\dot{F}_p^{\alpha,q}(R^n)$ is the set of all distributions f satisfying

$$\|f\|_{\dot{F}_p^{\alpha,q}(R^n)} = \Big\| \Big(\sum_k |2^{-k\alpha}\Psi_k * f|^q \Big)^{\frac{1}{q}} \Big\|_{L^p(R^n)} < \infty \tag{6.1}$$

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The homogeneous Sobolev spaces L^p_α has several equivalent definitions. One of them is $L^p_\alpha = \dot{F}^{\alpha,2}_p$, namely $\|f\|_{\dot{F}^{\alpha,2}_p} = \|f\|_{L^p_\alpha}$. From [FJW] we know that for any $f \in L^2_\alpha(R^n)$,

$$\|f\|_{L^2_\alpha(R^n)} \cong \Big(\int_{R^n} |\hat{f}(\xi)|^2 |\xi|^{2\alpha}\, d\xi \Big)^{\frac{1}{2}}) \tag{6.2}$$

if α is a nonnegative integer, then for any $f \in L^p_\alpha$,

$$\|f\|_{L^p_\alpha(R^n)} \cong \sum_{|l|=\alpha} \|D^l f\|_{L^p(R^n)}. \tag{6.3}$$

It is also know (see [Tr] or [FJW]) that the choice of Φ in the definition of $\dot{F}^{\alpha,q}_p$ is quite flexible. For instance, for the above Φ and any fixed number γ between $\frac{1}{2}$ and 1, let $\Phi_{j,\gamma}(x) = \Phi(2^j \gamma x)$. Define the functions $\Psi_{j,\gamma}$ by $\hat{\Psi}_{j,\gamma}(\xi) = \Phi_{j,\gamma}(\xi)$. Then using $\Psi_{k,\gamma}$ instead of Ψ_k in (6.1), we obtain a Triebel-Lizorkin norm equivalent to the norm in (6.1). Also the ratio of these two norms is between two positive constants c_1 and c_2, that are independent of $\gamma \in [\frac{1}{2}, 1]$. Furthermore, for any $\rho > 0$, we have

$$c_1 \|f\|_{\dot{F}^{\alpha,q}_p} \le \Big\| \Big(\sum_k |(2^k\rho)^{-\alpha} \Psi_{k,\rho} * f|^q \Big)^{\frac{1}{q}} \Big\|_{L^p} \le c_2 \|f\|_{\dot{F}^{\alpha,q}_p} \tag{6.4}$$

where $c_2 \ge c_1$ are independent of $\rho > 0$.

In fact, for any $\rho > 0$, there is an integer m such that $\frac{1}{2} < \rho 2^m \le 1$. Let $\gamma = \rho 2^m$. Then by the definitions, it is easy to see that

$$\begin{aligned} \sum_k |(2^k\rho)^{-\alpha} \Psi_{k,\rho} * f(x)|^q &= C \sum_k |\gamma^{-\alpha} (2^{k-m})^{-\alpha} \Psi_{k-m,\gamma} * f(x)|^q \\ &= C \sum_k |2^{-\alpha k} \Psi_{k,\gamma} * f(x)|^q, \end{aligned}$$

where C is independent of $\rho > 0$. Substituting this in the middle term of (6.4) we obtain (6.4).

Let Φ and Φ_j be defined as above. Define the multiplier operators S_j by $\widehat{S_j f}(\xi) = \Phi_j(|A_\rho \xi|) \hat{f}(\xi)$, where $A_\rho = \operatorname{diag}(\rho^2, \rho, \dots, \rho)$, $0 < \rho \le 2$. We have the following easy lemma:

Lemma *For any fixed $\rho \in (0, 2]$, we have*

$$\Big\| \Big(\sum_k |(2^k\rho)^{-\alpha} S_k f|^2 \Big)^{\frac{1}{2}} \Big\|_{L^p} \le C \|f\|_{\dot{F}^{\alpha,2}_p},$$

where the constant C is independent of ρ.

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Proof For $x = (x_1, x_2, \dots, x_n)$, let f^ρ be the function defined by $f^\rho(x) = f(\rho x_1, x_2, \dots, x_n)$. Changing variables and using (6.2), it is easy to check

$$\Big\| \Big(\sum_k |(2^k\rho)^{-\alpha} S_k f|^2 \Big)^{\frac{1}{2}} \Big\|_{L^p} \le C \|\rho^{\frac{1}{p}} f^\rho\|_{\dot{F}_p^{\alpha,2}} = C\|\rho^{\frac{1}{p}} f^\rho\|_{L^p_\alpha} = C\|L_\rho f\|_{L^p_\alpha},$$

where L_ρ is the linear operator defined by $L_\rho f = \rho^{\frac{1}{p}} f^\rho$. When $\alpha = 0$, it is clear $\|L_\rho f\|_{L^p} = \|f\|_{L^p}$.

If $\alpha = 1, 2, \dots$, by (6.3), noting $0 < \rho \le 2$, a simple computation also shows

$$\|L_\rho f\|_{L^p_\alpha} \le C\|f\|_{L^p_\alpha}, \tag{6.5}$$

with C independent of ρ. We invoke an interpolation in the Sobolev spaces $(L^p_s, L^p_t)_{\theta,2} = \dot{B}^u_{p,2}$, $u = (1-\theta)s + \theta t$, $0 \le \theta \le 1$ [BL, p. 147], where $\dot{B}^u_{p,q}$ are the homogeneous Besov spaces. From [FJW], we know $\dot{B}^u_{p,2} = L^p_u$. Thus we obtain (6.5) for all $\alpha \ge 0$.

6.2 Lipschitz Spaces on Unit Sphere

Let $\beta > 0$ and k greater than β, the space $\Lambda^\beta(S^{n-1})$ is the set of all $g \in L^\infty(S^{n-1})$ with norm

$$\|g\|_{\Lambda^\beta} = \|g\|_{L^\infty(S^{n-1})} + \sup\Big\{ \Big\| \frac{d^k}{dr^k} g(r\cdot) \Big\|_{L^\infty(S^{n-1})} (1-r)^{k-\beta} : 0 \le r < 1 \Big\} < \infty.$$

It is shown in [Co2] that $(H^q)^* = \Lambda^\beta$ with $\beta = (n-1)(1/q - 1)$. Particularly, if $q = (n-1)(n-1-\alpha)$ then $(H^q)^* = \Lambda^\alpha$. For a test function $f \in \mathcal{S}(R^n)$, let $f_{x,t}(y') = f(x - ty')$. Then it is not difficult to see that there is a constant $C > 0$, that might depend on x but not on t, such that $\|f_{x,t}\|_{\Lambda^\alpha} < C$. For simplicity and illustration, we check this fact for $\alpha > 0$ and $n = 2$.

Clearly, we only need to check for $j = 1, 2$,

$$\sup\Big\{ \Big\| \frac{d^k}{dr^k} f_{x,t}(re^{2\pi i\theta}) \Big\|_\infty (1-r)^{k-\alpha} : r \in I_j \Big\} < C, \tag{6.6}$$

where $I_1 = [0, \frac{4}{5}]$ and $I_2 = [\frac{4}{5}, 1)$.

On I_1, we write $f_{x,t}$ by its Fourier expansion

$$f_{x,t}(re^{2\pi i\theta}) = \sum_{m=-\infty}^{\infty} r^{|m|} a_m(x,t) e^{2\pi i m\theta},$$

where

$$a_m(x,t) = \int_0^1 f(x - te^{2\pi i\tau}) e^{-2\pi m i\tau}\, d\tau.$$

Thus for all $e^{2\pi i\theta}$, $r \in I_1$ and all x,t we have

$$\Big| \frac{d^k}{dr^k} f_{x,t}(re^{2\pi i\theta}) \Big| (1-r)^{k-\alpha} \le C \sum |m|^k \Big(\frac{4}{5}\Big)^{|m|} < C.$$

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If $t \geq 5|x|$ and $r \in [\frac{4}{5}, 1)$, since f is a test function,

$$\Big| \frac{d^k}{dr^k} f_{x,t}(re^{2\pi i\theta}) \Big| (1-r)^{k-\alpha} = \Big| \frac{d^k}{dr^k} f(x - tre^{2\pi i\theta}) \Big| (1-r)^{k-\alpha}$$
$$\leq t^k |f^{(k)}(x - tre^{2\pi i\theta})|$$
$$\leq Ct^k \Big\{ \Big| \frac{4}{5} t \Big| - |x| \Big\}^{-k} \leq C.$$

If $t < 5|x|$ and $r \in [\frac{4}{5}, 1)$, then

$$\Big| \frac{d^k}{dr^k} f_{x,t}(re^{2\pi i\theta}) \Big| (1-r)^{k-\alpha} \leq C|x|^k \leq C,$$

uniformly for $\theta, r \in I_2$ and $t > 0$.

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