A METRICAL RESULT ON THE DISCREPANCY OF $(n\alpha)$

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In the following let Ω be the set of irrational numbers in the interval [0,1] and let λ be Lebesgue measure restricted to Ω. For any real number x, let $\{x\} = x - [x]$ be the fractional part of x. Let N be a natural number and let $\alpha \in \Omega$. Then

$$D_N(\alpha) := \sup_{0 \le x \le y \le 1} \left| \sum_{n=1}^{N} c_{[x,y)}(\{n\alpha\}) - N(y - x) \right|$$

is known as the *discrepancy* of the sequence $(n\alpha)_{n \ge 1}$ modulo 1; here $c_{[x,y)}$ denotes the characteristic function of the interval $[x, y)$.

In this paper we shall prove that

$$\sup_{N>1} \lambda\left(\left\{\alpha \in \Omega \mid \left| D_N(\alpha) - \frac{2}{\pi^2} \log N \log\log N \right| \ge K \log N\right\}\right) = O\left(K^{-1/3}\right).$$

The convergence of $D_N(\alpha)/(\log N \log\log N)$ in measure to $\frac{2}{\pi^2}$ was first proved in **[5]**. At that time no remainder term was available: neither the theory of $D_N(\alpha)$ nor the metrical theory of continued fractions were developed far enough. Nevertheless we can follow the idea of the proof there. By the way we shall prove some consequences of the metrical theory of continued fractions that may be of some interest for themselves, although even weaker estimates would be sufficient to prove our theorem.

1. Foundations. Any $\alpha \in \Omega$ has a unique continued fraction expansion $\alpha = [0; a_1(\alpha), \ldots]$. We denote the n-th convergent of α by $p_n(\alpha)/q_n(\alpha)$.

Let $\Phi : \mathbb{R} \to [0, 1]$ be defined by $\Phi(z) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{z} e^{-y^2/2} dy$. Denoting the constant $\frac{12 \log 2}{\pi^2}$ by τ and correcting some misprints in **[6]**, we have the following result.

PROPOSITION 1.1. *There are positive constants σ and K such that for all integers $n \ge 2$ we have*

$$\sup_{z \in \mathbb{R}} \left| \lambda\left(\left\{\alpha \in \Omega \mid \log q_n(\alpha) \le z\right\}\right) - \Phi\left(\frac{\tau z - n}{\sigma\tau\sqrt{n}}\right) \right| \le K \frac{\log n}{\sqrt{n}}.$$

The value of τ, which is not stated in the paper above follows from the well known relation

$$\lim_{n \to \infty} \frac{1}{n} \log q_n(\alpha) = \frac{\pi^2}{12 \log 2}$$

almost everywhere. In the following let σ be the constant stated in this theorem.

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Let $N \geq 1$ be an integer. For any $\alpha \in \Omega$ there exists a unique nonnegative integer $m = m_N(\alpha)$, such that $q_m(\alpha) \leq N < q_{m+1}(\alpha)$. We shall use this notation throughout the paper.

In the following we shall use the inequalities $1 - \Phi(z) = O\left(\frac{1}{z} e^{-z^2/2}\right)$ for $z > 0$ and $\Phi(z) = O\left(\frac{1}{|z|} e^{-z^2/2}\right)$ for $z < 0$.

PROPOSITION 1.2. *For any $x \leq 1$ and any integer $N \geq 3$ we have*

$$\lambda(\{\alpha \in \Omega \mid m_N(\alpha) \leq \tau(1-x)\log N\}) = 1 - \Phi\left(\frac{x\sqrt{\log N}}{\sigma\sqrt{\tau}}\right) + O\left(\frac{\log\log N}{\sqrt{\log N}}\right).$$

The O-constant is absolute.

Proof. Let $\gamma := \tau(1-x)\log N$ and $A(x) := \{\alpha \in \Omega \mid m_N(\alpha) \leq \tau(1-x)\log N\}$. Then $A(x) = \{\alpha \in \Omega \mid q_{[\gamma]+1}(\alpha) > N\}$ and therefore if $x \leq 1/2$

$$\lambda(A(x)) = 1 - \Phi\left(\frac{\tau\log N - 1 - [\gamma]}{\sigma\tau\sqrt{1+[\gamma]}}\right) + O\left(\frac{\log\log N}{\sqrt{\log N}}\right).$$

If $f(w) := \frac{\tau\log N - w}{\sigma\tau\sqrt{w}}$, then for $w \geq w' \geq \frac{\tau}{2}\log N$ we have

$$f(w) - f(w') = \left(\frac{\log N}{\sigma\sqrt{ww'}} + \frac{1}{\sigma r}\right)\left(\sqrt{w} - \sqrt{w'}\right) = O\left(\frac{w-w'}{\sqrt{\log N}}\right).$$

Since $|\Phi(z) - \Phi(z')| \leq |z - z'|$ (for all $z, z' \in \mathbb{R}$) we get

$$\lambda(A(x)) = 1 - \Phi\left(\frac{\tau\log N - \gamma}{\sigma\tau\sqrt{\gamma}}\right) + O\left(\frac{\log\log N}{\sqrt{\log N}}\right)$$

$$= 1 - \Phi\left(\frac{x\sqrt{\log N}}{\sigma\sqrt{\tau}\sqrt{1-x}}\right) + O\left(\frac{\log\log N}{\sqrt{\log N}}\right).$$

Let us prove the assertion first of all for $1/2 < x \leq 1$. Then

$$\lambda(A(x)) \leq \lambda(A(1/2)) = O\left(\frac{\log\log N}{\sqrt{\log N}}\right)$$

$$1 - \Phi\left(\frac{x\sqrt{\log N}}{\sigma\sqrt{\tau}}\right) = O\left(\frac{1}{\sqrt{\log N}}\right).$$

Assume now that $x < -1/2$. Then

$$\lambda(A(x)) \geq \lambda(A(-1/2)) = 1 + O\left(\frac{\log\log N}{\sqrt{\log N}}\right)$$

$$1 - \Phi\left(\frac{x\sqrt{\log N}}{\sigma\sqrt{\tau}}\right) = 1 + O\left(\frac{1}{\sqrt{\log N}}\right).$$

Next we assume that $1/2 \geq |x| \geq \sqrt{\frac{3}{2}}\sigma\sqrt{\tau}\sqrt{\frac{\log\log N}{\log N}}$. Then $\frac{|x|\sqrt{\log N}}{\sigma\sqrt{\tau(1-x)}} \geq \sqrt{\log\log N}$ and therefore

$$1-\Phi\left(\frac{x\sqrt{\log N}}{\sigma\sqrt{\tau(1-x)}}\right)=\frac{1-\operatorname{sgn}x}{2}+O\left(\frac{1}{\sqrt{\log N}}\right).$$

Analogously

$$1-\Phi\left(\frac{x\sqrt{\log N}}{\sigma\sqrt{\tau}}\right)=\frac{1-\operatorname{sgn}x}{2}+O\left(\frac{1}{\sqrt{\log N}}\right).$$

From this the assertion follows in this case, too.

Finally let $|x| < \sqrt{\frac{3}{2}}\sigma\sqrt{\tau}\sqrt{\frac{\log\log N}{\log N}}$. This results in

$$\frac{|x|\sqrt{\log N}}{\sigma\sqrt{\tau}}\left|\frac{1}{\sqrt{1-x}}-1\right| = O\left(x^2\sqrt{\log N}\right)+O\left(\frac{\log\log N}{\sqrt{\log N}}\right)$$

and therefore

$$\Phi\left(\frac{x\sqrt{\log N}}{\sigma\sqrt{\tau}}\right)=\Phi\left(\frac{x\sqrt{\log N}}{\sigma\sqrt{\tau}\sqrt{1-x}}\right)+O\left(\frac{\log\log N}{\sqrt{\log N}}\right).$$

COROLLARY 1.1. *For any integer $N \geq 3$ and any $x \geq 0$ we have*

$$\lambda(\{\alpha\in\Omega|\ |m_N(\alpha)-\tau\log N|\geq x\tau\log N\})=2\left(1-\Phi\left(\frac{x\sqrt{\log N}}{\sigma\sqrt{\tau}}\right)\right)+O\left(\frac{\log\log N}{\sqrt{\log N}}\right).$$

The O-constant is absolute.

Proof. This follows immediately from Proposition 1.2.

NOTATION. Let $\mu := \frac{\pi}{2\log 2}$. Let $G : \mathbb{R} \to [0, 1]$ be that distribution function which satisfies

$$e^{-\mu|t|\left(1+i\operatorname{sgn}t\frac{2}{\pi}\log|t|\right)}=\int_{\mathbb{R}} e^{itx}G'(x)dx \quad (t\in\mathbb{R}).$$

We note that

$$G'(x)=\frac{1}{2\pi}\int_{\mathbb{R}} e^{-itx-\mu|t|\left(1+i\operatorname{sgn}t\frac{2}{\pi}\log|t|\right)}dt \quad (x\in\mathbb{R}).$$

Furthermore for $\beta \in \{-1, 1\}$ and $x \in \mathbb{R}$ we put

$$p(x,\beta)=\frac{1}{2\pi}\int_{\mathbb{R}} e^{-itx}e^{-|t|\left(1-i\beta\frac{2}{\pi}\operatorname{sgn}t\log|t|\right)}dt$$

and we note that $G'(x) = \frac{1}{\mu}p(\frac{x}{\mu} - \frac{2}{\pi}\log\mu, -1)$. Correcting four misprints in [**4**] we get the next result.

PROPOSITION 1.3. *For $n \in \mathbb{Z}_+$ let $b_n := \Im \int_0^\infty e^{-t} t^n \left(i - \frac{1}{\pi} \log t\right)^n dt$, $c_n(\varphi)$ the coefficient of y^n in the power series expansion of $f(y) := e^{y^{-2}\left(e^{i\varphi y}(1-i\varphi y)-1-\frac{\varphi^2 y^2}{2}\right)+i\varphi y}$ and $a_n = \Re \int_0^\infty e^{-\varphi^2/2} c_n(\varphi) d\varphi$. Then for any positive integer N and $x \to \infty$* we have

$$p\left(x + \frac{2}{\pi} \log x, -1\right) = \frac{1}{\pi x} \sum_{n=0}^{N} \frac{2^n b_n}{n!} x^{-n} + O\left(x^{-N-2}\right) \tag{1}$$

$$p\left(\frac{4}{\pi} \log x, 1\right) = \frac{x}{2\sqrt{e}} e^{-2x^2/(\pi e)} \left(1 + \sqrt{\frac{2}{\pi}} \sum_{n=1}^{N} \left(\frac{\pi e}{2}\right)^n a_n x^{-n} + O\left(x^{-N-1}\right)\right). \tag{2}$$

LEMMA 1.1. *For $x \to \infty$ we have*

(1) $p(x, -1) = \dfrac{2}{\pi x^2} + \dfrac{8 \log x}{\pi^2 x^3} + O\left(x^{-3}\right)$,

(2) $p\left(\dfrac{4}{\pi} \log x, 1\right) = \dfrac{x}{2\sqrt{e}} e^{-2x^2/(e\pi)} \left(1 + O\left(x^{-2}\right)\right)$.

Proof. (1) We have $b_0 = 0$ and $b_1 = 1$. Therefore $p\left(x + \frac{2}{\pi} \log x, -1\right) = \frac{2}{\pi x^2} + O\left(x^{-3}\right)$. For every $y \in \mathbb{R}$ there is exactly one $x(y) > 0$ with $x(y) + \frac{2}{\pi} \log x(y) = y$. We have $\lim_{x \to \infty} x(y) = \infty$ and therefore, if y is large enough, $2x(y) \geq y \geq x(y)$. This implies $\log x(y) = \log y + O(1)$ and therefore $x(y) + \frac{2}{\pi} \log y = y + O(1)$; from this we get $\frac{x(y)}{y} = 1 - \frac{2}{\pi y} \log y + O\left(\frac{1}{y}\right)$. This implies that

$$p(y, 1) = \frac{2}{\pi x(y)^2} + O\left(y^{-3}\right) = \frac{2}{\pi y^2} \left(1 + \frac{4 \log y}{\pi y} + O\left(\frac{1}{y}\right)\right) + O\left(y^{-3}\right).$$

(2) Let $g(y) = -y^{-2}\left(e^{i\varphi y}(1 - i\varphi y) - 1 - \frac{\varphi^2 y^2}{2}\right) + i\varphi y$. Then $g(y) = i\varphi y\left(1 - \varphi^2/3\right) + O\left(y^2\right)$ for $y \to 0$. Therefore $f(y) = e^{g(y)} = 1 + i\varphi y\left(1 - \varphi^2/3\right) + O\left(y^2\right)$ and $c_1(\varphi) = i\varphi\left(1 - \varphi^2/3\right)$. This implies $a_1 = 0$.

PROPOSITION 1.4. *For $x \to \infty$ we have*

(1) $G(x) = 1 - \dfrac{1}{x \log 2} - \dfrac{\log x}{x^2 \log^2 2} + O\left(x^{-2}\right)$,

(2) $G(-x) = \sqrt{2}\left(1 - \Phi\left(\sqrt{\dfrac{2}{e \log 2}} 2^{x/2}\right)\right)(1 + O(2^{-x}))$.

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Proof. (1) $\int_u^\infty \frac{\log x}{x^3}dx = \frac{\log u}{2u^2} + \frac{1}{4u^2}$ implies

$$G(x) = 1 - \int_x^\infty G'(y)dy = 1 - \frac{1}{\mu}\int_x^\infty p\left(\frac{y}{\mu} - \frac{2}{\pi}\log\mu, -1\right)dy$$

$$= 1 - \int_{\frac{x}{\mu}-\frac{2}{\pi}\log\mu}^\infty p(y,-1)dy$$

$$= 1 - \frac{2}{\pi\left(\frac{x}{\mu}-\frac{2}{\pi}\log\mu\right)} - \frac{8}{\pi^2}\frac{\log\left(\frac{x}{\mu}-\frac{2}{\pi}\log\mu\right)}{2\left(\frac{x}{\mu}-\frac{2}{\pi}\log\mu\right)^2} + O(x^{-3}).$$

This implies the assertion.

(2) For $y \to \infty$ we have $p(y,1) = \frac{e^{\pi y/4}}{2\sqrt{e}}e^{-\frac{2}{e\pi}e^{\pi y/2}}\left(1 + O\left(e^{-\pi y/2}\right)\right)$ and therefore

$$\int_x^\infty p(y,1)dy = \frac{2}{\pi\sqrt{e}}\int_{e^{\pi y/4}}^\infty e^{-2u^2/(e\pi)}\left(1+O\left(u^{-2}\right)\right)du$$

$$= \frac{1}{\sqrt{\pi}}\int_{\frac{2}{\sqrt{e\pi}}e^{\pi x/4}}^\infty e^{-v^2/2}\left(1+O\left(v^{-2}\right)\right)dv$$

$$= \frac{1}{\sqrt{\pi}}\int_{\frac{2}{\sqrt{e\pi}}e^{\pi x/4}}^\infty e^{-v^2/2}dv\left(1+O\left(e^{-\pi x/2}\right)\right)$$

$$= \frac{1}{\sqrt{\pi}}\left(\sqrt{2\pi} - \int_{-\infty}^{\frac{2}{\sqrt{e\pi}}e^{\pi x/4}} e^{-v^2/2}dv\right)\left(1+O\left(e^{-\pi x/2}\right)\right)$$

$$= \sqrt{2}\left(1 - \Phi\left(\frac{2}{\sqrt{e\pi}}e^{\pi x/4}\right)\right)\left(1+O\left(e^{-\pi x/2}\right)\right).$$

For all $x \in \mathbb{R}$ we have $p(-x,-1) = p(x,1)$ and therefore

$$G(-x) = \int_{-\infty}^{-x} G'(y)dy = \frac{1}{\mu}\int_{-\infty}^{-x} p\left(\frac{y}{\mu} - \frac{2}{\pi}\log\mu, -1\right)dy$$

$$= \frac{1}{\mu}\int_x^\infty p\left(\frac{y}{\mu} + \frac{2}{\pi}\log\mu, 1\right)dy$$

$$= \int_{\frac{x}{\mu}+\frac{2}{\pi}\log 2}^\infty p(y,1)dy.$$

In the following we denote. Euler's constant by γ.

PROPOSITION 1.5. [1]. *There is a constant* $K>0$ *such that for all integers* $N\geq 2$ and all $x>0$.

$$\left|\lambda\left(\left\{\alpha\in\Omega\,\Big|\sum_{k=1}^{N}a_k(\alpha)<x\right\}\right)-G\left(\frac{x}{N}-\frac{\log N-\gamma}{\log 2}\right)\right|\leq K\frac{\log^2 N}{N}.$$

Proof. This result was proved in [1] only for the case to which λ is replaced by the Gaussian measure $P(A)=\frac{1}{\log 2}\int_A\frac{dx}{1+x}$. L. Heinrich kindly pointed out to me that the same result is valid for any measure μ that is absolutely continuous with respect to P and whose density function is strictly positive on [0, 1] and continuous in the sense of Lipschitz. We give a sketch of the proof (suggested by L. Heinrich). From Theorem 2 in [2, p.8] it results that there exist constants $C>0$ and $t\in(0,1)$ such that, for all $m,q\in\mathbb{N}$,

$$|P(\{\alpha\in\Omega|a_m(\alpha)=q\})-\mu(\{\alpha\in\Omega|a_m(\alpha)=q\})|\leq Ct^mP(\{\alpha\in\Omega\mid a_m(\alpha)=q\}).$$

Denoting by E_μ the expectation value with respect of μ we get easily from the formula above

$$|\sum_{k=1}^{N}E_\mu(e^{ita_k}-1)-NE_P(e^{ita_1}-1)|\leq C_1|E_P(e^{ita_1}-1)|.$$

It follows that [2, Lemma 4] remains valid if we replace in it E_P by E_μ and T_0 by $\frac{C*}{\log^2 N}$. This last statement follows from [3, Satz 4]; this theorem is again stated only in the case $\mu=P$, but following the lines in [2] immediately after Corollary 2 one can prove that it is valid even in the general case.

PROPOSITION 1.6. *There is a constant* $K>0$ *such that for all* $x>0$ *and all integers* $N\geq 3$

$$|\lambda\left(\left\{\alpha\in\Omega|\sum_{k=1}^{m_N(\alpha)}a_k(\alpha)\leq x\right\}\right)-G\left(\frac{x}{\tau\log N}-\frac{\log(\tau\log N)-\gamma}{\log 2}\right)|\leq K\frac{(\log\log N)^{3/2}}{\sqrt{\log N}}.$$

Proof. For $N\geq 3$, $x>0$ and $0\leq\varepsilon\leq 1/2$ we define

$$f_N(x,\varepsilon)=\frac{x}{\tau(1+\varepsilon)\log N}-\frac{\log(\tau(1+\varepsilon)\log N)-\gamma}{\log 2}.$$

Then

$$f_N(x,\varepsilon)-f_N(x,0)=O\left(\varepsilon\left(\frac{x}{\log N}+1\right)\right).$$

This implies that there is η such that

$$G(f_N(x,\varepsilon))-G(f_N(x,0))=O\left(\varepsilon\left(\frac{x}{\log N}+1\right)G'(f_N(x,\eta))\right)$$

and $0\leq\eta\leq\varepsilon$.

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Now $f_N(x, \varepsilon) \geq 1$ implies $f_N(x, \eta) \geq 1$ and therfore by Proposition 1.3 a simple calculation yields

$$G(f_N(x,\varepsilon)) - G(f_N(x,0)) = O\left(\varepsilon\left(\frac{x}{\log N}+1\right)\frac{1}{f_N^2(x,\varepsilon)}\right) = O(\varepsilon \log\log N).$$

If $f_N(x,\varepsilon) < 1$ we get $\frac{x}{\log N} + 1 = O(\log\log N)$ and therefore again

$$G(f_N(x,\varepsilon)) - G(f_N(x,0)) = O(\varepsilon \log\log N).$$

Assume now that $\varepsilon > 0$. Proposition 1.2 and Proposition 1.5 imply

$$\begin{aligned}\lambda\left(\left\{\alpha\in\Omega \mid \sum_{k=1}^{m_N(\alpha)} a_k(\alpha) \leq x\right\}\right) &\geq \lambda\left(\left\{\alpha\in\Omega \mid \sum_{k\leq\tau(1+\varepsilon)\log N} a_k(\alpha)\leq x\right\}\right)\\ &\quad - \lambda(\{\alpha\in\Omega \mid m_N(\alpha)\geq \tau(1+\varepsilon)\log N\})\\ &= G(f_N(x,\varepsilon)) + O\left(\frac{\log^2\log N}{\log N}\right) - \Phi\left(\frac{-\varepsilon\sqrt{\log N}}{\sigma\sqrt{\tau}}\right) + O\left(\frac{\log\log N}{\sqrt{\log N}}\right)\\ &= G(f_N(x,0)) + O\left(\epsilon\log\log N + \frac{1}{\epsilon\sqrt{\log N}}e^{-\epsilon^2\log N/(2\sigma^2\tau)} + \frac{\log\log N}{\sqrt{\log N}}\right).\end{aligned}$$

Now we put $\varepsilon = \sigma\sqrt{2\tau\frac{\log\log N}{\log N}}$. Then we get

$$\lambda\left\{\left(\alpha\in\Omega \mid \sum_{k=1}^{m_N(\alpha)} a_k(\alpha)\leq x\right\}\right) \geq G(f_N(x,0)) + O\left(\frac{(\log\log N)^{3/2}}{\sqrt{\log N}}\right).$$

Analogously the converse inequality can be proved.

From these results we conclude some Lemmas which will be used later.

LEMMA 1.2. *There exists a constant* $c > 0$ *with the following property*: if $0 < \mu < \frac{\tau}{2}$, $N \geq 2$ *is an integer*, $M := \left[\left(\frac{\tau}{2} - \mu\right)\log N\right]$, $\nu : \Omega \to \mathbb{Z}_+$, $\nu(\alpha) = \max\{m_N(\alpha) - M, 0\}$ *and* $1 \leq w \leq \log N$, *then*

$$\begin{aligned}&\lambda\left(\left\{\alpha\in\Omega \mid \textit{the denominator of } [0; a_{\nu(\alpha)+1}(\alpha),\ldots,a_{m_N(\alpha)}(\alpha)] \textit{ is} \geq \sqrt{\tfrac{N}{w}}\right\}\right)\\ &\qquad\leq c\left(\frac{1}{\mu\sqrt{\log N}}e^{-\mu^2\log N/(32\sigma^2\tau^3)} + \frac{\log\log N}{\sqrt{\log N}}\right).\end{aligned}$$

Proof. The assertion is trivial if $\mu \leq 2\tau\frac{\log\log N}{\log N}$. Otherwise $\mu\log N - \tau\log\log N \geq \frac{\mu}{2}\log N$. Let A be the set mentioned in the Lemma and let

$$B := \left\{\alpha\in\Omega| \mid m_N(\alpha) - \tau\log N| \geq \frac{\mu}{4}\log N\right\}.$$

Then Corollary 1.1 implies

$$\lambda(B) = O\left(\frac{1}{\mu\sqrt{\log N}} e^{-\mu^2 \log N/(32\sigma^2\tau^3)} + \frac{\log\log N}{\sqrt{\log N}}\right).$$

For $\alpha \in A \backslash B$ we have

$$m_N(\alpha) \leq \left(\tau + \frac{\mu}{4}\right) \log N$$
$$\nu(\alpha) \geq \left(\tau - \frac{\mu}{4}\right) \log N - \left(\frac{\tau}{2} - \mu\right) \log N$$

and

$$m_N(\alpha) - \nu(\alpha) \leq \frac{\tau - \mu}{2} \log N.$$

Proposition 1.1 implies

$$\lambda(A\backslash B) = O\left(1 - \Phi\left(\frac{\frac{\tau}{2}\log\frac{N}{w} - \frac{\tau-\mu}{2}\log N}{\sigma\tau\sqrt{\frac{\tau-\mu}{2}\log N}}\right) + \frac{\log\log N}{\sqrt{\log N}}\right)$$
$$= O\left(1 - \Phi\left(\frac{\mu \log N - \tau \log w}{\sigma\tau\sqrt{2(\tau-\mu)\log N}}\right) + \frac{\log\log N}{\sqrt{\log N}}\right)$$
$$= O\left(1 - \Phi\left(\frac{\mu \log N - \tau \log\log N}{\sigma\tau\sqrt{2(\tau-\mu)\log N}}\right) + \frac{\log\log N}{\sqrt{\log N}}\right)$$
$$= O\left(1 - \Phi\left(\frac{\mu\sqrt{\log N}}{2\sigma\tau\sqrt{2(\tau-\mu)}}\right) + \frac{\log\log N}{\sqrt{\log N}}\right)$$
$$= O\left(\frac{1}{\mu\sqrt{\log N}} e^{-\mu^2 \log N/(8\sigma^2\tau^3)} + \frac{\log\log N}{\sqrt{\log N}}\right).$$

LEMMA 1.3. *There exists a constant $c > 0$ with the following property: if $0 < \mu < \frac{\tau}{8}$, $N \geq 2$ is an integer, $M := \left[\left(\frac{\tau}{2} - \mu\right) \log N\right]$,*

$$\nu : \Omega \to \mathbb{Z}_+, \nu(\alpha) = \max\{m_N(\alpha) - M, 0\}$$

and $T > 0$ then

$$\lambda\left(\left\{\alpha \in \Omega \mid \sum_{k=\nu(\alpha)}^{m_N(\alpha)-1} a_{k+1}(\alpha) \leq T\right\}\right)$$
$$\leq c\left(G\left(\frac{2T}{M} - \frac{\log\frac{M}{2} - \gamma}{\log 2}\right) + \frac{1}{\mu\sqrt{\log N}} e^{-\mu^2 \log N/(128\sigma^2\tau^3)} + \frac{\log\log N}{\sqrt{\log N}}\right).$$

Proof. It is similar to the proof of Lemma 1.2 (and even simpler).

The following Lemma should be well known.

LEMMA 1.4. *There is a constant $c > 0$ such that for all $k \in \mathbb{Z}_+$, $x > 0$ and $N \in \mathbb{N}$ we have*

$$\lambda\left(\left\{\alpha \in \Omega \mid \sum_{s=1}^{N} a_{s+k}^2(\alpha) \geq x\right\}\right) \leq c\frac{N}{\sqrt{x}}.$$

LEMMA 1.5. *There is a constant $c > 0$ with the following property: if $0 < \mu < \frac{\tau}{4}$, N is an integer ≥ 2, $M := \left[\left(\frac{\tau}{2} - \mu\right) \log N\right]$, $R > 0$ and*

$$\nu : \Omega \to \mathbb{Z}_+, \nu(\alpha) = \max\{m_N(\alpha) - M, 0\},$$

then

$$\lambda\left(\left\{\alpha \in \Omega \mid \sum_{k=\nu(\alpha)}^{m_N(\alpha)-1} a_{k+1}^2(\alpha) \geq R\right\}\right) \leq c\left(\frac{\log N}{\sqrt{R}} + \frac{1}{\mu\sqrt{\log N}} e^{-\mu^2 \log N/(8\sigma^2\tau^3)} + \frac{\log\log N}{\sqrt{\log N}}\right).$$

Proof. It follows the same idea as the proof of Lemma 1.2 (and is even simpler).

2. An application of the inequality of Tschebyscheff. Let $\alpha \in \Omega$ and let k be a nonnegative integer. It is well known that there is exactly one sequence $(c_s(k,\alpha))_{s\geq 0}$ of integers, such that $k = \sum_{s=0}^{\infty} c_s(k,\alpha)q_s(\alpha)$, where $0 \leq c_s(k,\alpha) \leq a_{s+1}(\alpha)$, $c_0(k,\alpha) < a_1(\alpha)$ and $s \geq 1$, $c_s(k,\alpha) = a_{s+1}(\alpha)$ implies $c_{s-1}(k,\alpha) = 0$ (the Ostrowski-expansion of k with respect to α). This can be formulated in a simpler way: there is exactly one sequence $(c_s(k,\alpha))_{s\geq 0}$ of integers, such that for all $t \geq 0$, $0 \leq k - \sum_{s=t}^{\infty} c_s(k,\alpha)q_s < q_t$. This variant can be generalized as follows.

LEMMA 2.1. *Let $(a_i)_{i\geq 1}$ be a sequence of positive integers and let $(q_i)_{i\geq -1}$ be a sequence of real numbers such that $q_{-1} = 0 < q_0$ and for $i \geq 0$, $q_{i+1} = a_{i+1}q_i + q_{i-1}$. Let $z \geq 0$ be a real number. Then there exists exactly one sequence $(c_i)_{i\geq 0}$ of integers such that, for all $t \geq 0$, $0 \leq z - \sum_{k=t}^{\infty} c_k q_k < q_t$. This sequence has the following properties.*

(1) *For all $k > 0$ we have $0 \leq c_k \leq a_{k+1}$,*
(2) *$c_0 < a_1$; also $k \geq 1$* and *$c_k = a_{k+1}$* imply *$c_{k-1} = 0$.*

Proof. This was already used in [**5**, p. 195].

We denote the "digits" c_k of z by $c_k(z)$. If the q_i are the denominators of the convergents of $\alpha \in \Omega$, we denote them by $c_k(z,\alpha)$. The following lemma is well known.

LEMMA 2.2. *Let $\alpha \in \Omega$. For $i \geq 0$, let $\left(\frac{p_{n,i}}{q_{n,i}}\right)_{n\geq 0}$ be the sequence of convergents of $\alpha_{i+1} := [0, a_{i+1}, \ldots]$. If $n \geq i$, then*

(1) $p_{n-i,i} = (-1)^{i+1}(q_n(\alpha)p_i(\alpha) - p_n(\alpha)q_i(\alpha))$,
(2) $q_{n-i,i} = (-1)^i(q_n(\alpha)p_{i-1}(\alpha) - p_n(\alpha)q_{i-1}(\alpha))$.

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Let $(a_i)_{i\geq 1}$ be a fixed sequence of positive integers. Subsequently c, d, i and j are non-negative integers with the following properties.

(a) $i = 0 \Rightarrow c = 1$,
(b) $i = 1 \Rightarrow c < a_1$,
(c) $i \geq 1 \Rightarrow 0 \leq c \leq a_i$,
(d) $0 \leq d \leq a_{j+1}$,
(e) $i \leq j$.

Under these conditions we put

$$L_{i,j}(c,d) := \big\{(x_k)_{i\leq k<j} \in \mathbb{Z}_+^{j-1} \mid \big(i=j \wedge d=a_{j+1}\big) \Rightarrow c=0, \\ (i<j \wedge x_i = a_{i+1}) \Rightarrow c=0, i\leq k<j \Rightarrow x_k \leq a_{k+1}, \\ (i<k<j \wedge x_k = a_{k+1}) \Rightarrow x_{k-1} = 0\big\}.$$

We note that if $c > 0$ then $L_{i,i(c,a_{i+1})} = \emptyset$, while in the case $d = a_{i+1} \Rightarrow c = 0$ we have $L_{i,i}(c,d) = \{\emptyset\}$.

LEMMA 2.3. *Let $\alpha = [0; a_1, \ldots] \in \Omega$ be the continued fraction expansion of α with convergents $\frac{p_n}{q_n}$. Assume that* (a)–(e) *above are satisfied.*

(1) *If $c > 0$ and $d < a_{j+1}$ we have* $|L_{i,j}(c,d)| = (-1)^i\big(q_j p_{i-1} - p_j q_{i-1}\big)$.
(2) *If $d < a_{j+1}$ we have*

$$|L_{i,j}(0,d)| = (-1)^{i+1}\big(q_j(p_i - p_{i-1}) - p_j(q_i - q_{i-1})\big).$$

(3) *$i < j$ implies* $|L_{i,j}\big(c, a_{j+1}\big)| = |L_{i,j-1}(c,0)|$.

Proof. We repeat that any nonnegative integer $m < q_j$ has a unique expansion $m = \sum_{k=0}^{j-1} x_k q_k$, where $0 \leq x_k \leq a_{k+1}, x_0 < a_1$ and, for $k \geq 1$, $x_k = a_{k+1}$ implies $x_{k-1} = 0$.

$$(1)\ |L_{0,j}(1,d)| = \left|\left\{\sum_{k=0}^{j-1} x_k q_k \middle| x_k \in \mathbb{Z}, 0 \leq x_k \leq a_{k+1}, x_0 < a_1, x_k = a_{k+1} \Rightarrow x_{k-1} = 0\right\}\right| \\ = \big|\{0, \ldots, q_j - 1\}\big| = q_j.$$

Applying this result to $\alpha_{i+1} = [0; a_{i+1}, \ldots,]$ instead of to α and using Lemma 2.2(2) we get the assertion.

(2) The result is valid if $i = j$. Assume that $i < j$. Then

$$L_{i,j}(0,d) = L_{i,j}(1,d) \cup \big(\{a_{i+1}\} \times L_{i+1,j}(1,d)\big)$$

and therefore $|L_{i,j}(0,d)| = |L_{i,j}(1,d)| + |L_{i+1,j}(1,d)|$.

(3) follows from $L_{i,j}\big(c, a_{j+1}\big) = L_{i,j-1}(c,0) \times \{0\}$.

Let $I = \{i_1, \ldots, i_t\} \subseteq \{0, \ldots, m-1\}$, where $i_j < i_{j+1}$ for $1 \leq j < t$. Let $i_0 = -1$ and $i_{t+1} = m$. Assume that, for any $i \in I$, $0 \leq c_i \leq a_{i+1}$ is given. For $0 \leq k < q_m$, let $k = \sum_{j=0}^{m-1} c_j(k) q_j$ be the Ostrowski-expansion of k. We put $c_{i_0} = 1$ and $c_{i_{t+1}} = 0$. Then

$$|\{x \in \mathbb{Z}_+ | k < q_m, i \in I \Rightarrow c_i(k) = c_i\}| = \prod_{j=0}^{t} |L_{i_{j+1}, i_{j+1}}(c_{i_j}, c_{i_{j+1}})|.$$

For the rest of this paper we define $B_2(x) = \{x\}^2 - \{x\} + \frac{1}{6}$; B_2 is the second Bernoulli-polynomial.

LEMMA 2.4. *Let $\alpha = [0; a_1, \ldots] \in \Omega$ be the continued fraction expansion of α with convergents $\frac{p_n}{q_n}$ and let $m \in \mathbb{Z}_+$. For $0 \le s \le t < m$ let*

$$U_{s,t} = \frac{1}{q_m} \sum_{k=0}^{q_m - 1} a_{s+1} B_2\left(\frac{c_s(k)}{a_{s+1}}\right) a_{t+1} B_2\left(\frac{c_t(k)}{a_{t+1}}\right), \; V_s = \frac{1}{q_m} \sum_{k=0}^{q_m - 1} a_{s+1} B_2\left(\frac{c_s(k)}{a_{s+1}}\right).$$

(1) $V_s = \frac{1}{6} a_{s+1} + \frac{1}{6}(-1)^s q_s (a_{s+1}^2 - 1)\left(p_s - \frac{p_m}{q_m} q_s\right)$.

(2) *For $s < t$,* $U_{s,t} - V_s V_t = \frac{(-1)^{s+t+1}}{36} q_s^2 \left(p_t - \frac{p_m}{q_m} q_t\right)^2 (a_{s+1}^2 - 1)(a_{t+1}^2 - 1)$.

(3) $U_{s,s} - V_s^2 = \frac{(-1)^{s+1}}{30} q_s \left(p_s - \frac{p_m}{q_m} q_s\right)\left(a_{s+1}^3 - \frac{1}{a_{s+1}}\right) - \frac{q_s^2}{36}\left(p_s - \frac{p_m}{q_m} q_s\right)^2 (a_{s+1}^2 - 1)^2$.

Proof. We carry out the proofs, which are tedious but trivial in principle, up to those points from which subsequently it is clear how one has to proceed.

(1) Note that for any $q \in \mathbb{N}$, $\frac{1}{q} \sum_{k=1}^{q-1} B_2\left(\frac{k}{q}\right) = \frac{1}{6}\left(\frac{1}{q} - 1\right)$. This and Lemma 2.3 imply

$$\sum_{k=0}^{q_m - 1} B_2\left(\frac{c_s(k)}{a_{s+1}}\right) = \sum_{c=0}^{a_{s+1}} B_2\left(\frac{c}{a_{s+1}}\right) |L_{0,s}(1, c)| . |L_{s+1,m}(c, 0)|$$

$$= \frac{1}{6} q_s (-1)^s (q_m (p_{s+1} - p_s) - p_m (q_{s+1} - q_s))$$

$$+ \sum_{c=1}^{a_{s+1} - 1} B_2\left(\frac{c}{a_{s+1}}\right) q_s (-1)^{s+1} (q_m p_s - p_m q_s) + \frac{1}{6} q_{s-1} (-1)^{s+1} (q_m p_s - p_m q_s).$$

(2) First of all we prove that

$$\sum_{c=0}^{a_{s+1}} B_2\left(\frac{c}{a_{s+1}}\right) |L_{0,s}(1, c)| . |L_{s+1,t}(c, a_{t+1})| = \sum_{k=0}^{q_{t-1} - 1} B_2\left(\frac{c_s(k)}{a_{s+1}}\right).$$

If $s < t - 1$ this formula follows from Lemma 2.3(3). If $s = t - 1$, then the left hand side equals $\frac{1}{6} |L_{0,s}(1, 0)| = \frac{q_s}{6}$. Because of $0 \le k < q_{t-1} \Rightarrow c_s(k) = 0$, the right hand side is again equal to $\frac{q_s}{6}$.

From this it follows that

$$\frac{q_m^2}{a_{s+1}a_{t+1}}U_{s,t} = \sum_{c=0}^{a_{s+1}}\sum_{d=0}^{a_{t+1}} B_2\left(\frac{c}{a_{s+1}}\right)B_2\left(\frac{d}{a_{t+1}}\right)|L_{0,s}(1,c)|.|L_{s+1,t}(c,d)|.|L_{t+1,m}(d,0)|$$

$$= \frac{(-1)^{t+1}}{6}(q_mp_t - p_mq_t)\sum_{c=0}^{a_{s+1}} B_2\left(\frac{c}{a_{s+1}}\right)|L_{0,s}(1,c)|.|L_{s+1,t}(c,a_{t+1})|$$

$$+(-1)^{t+1}(q_mp_t - p_mq_t)\sum_{c=0}^{a_{s+1}}\sum_{d=1}^{a_{t+1}-1} B_2\left(\frac{c}{a_{s+1}}\right)B_2\left(\frac{d}{a_{t+1}}\right)|L_{0,s}(1,c)|.|L_{s+1,t}(c,0)|$$

$$+\frac{(-1)^t}{6}(q_m(p_{t+1}-p_t) - p_m(q_{t+1}-q_t))\sum_{c=0}^{a_{s+1}} B_2\left(\frac{c}{a_{s+1}}\right)|L_{0,s}(1,c)|.|L_{s+1,t}(c,0)|$$

$$= (-1)^{t+1}(q_mp_t - p_mq_t)\left(\frac{1}{6}\sum_{k=0}^{q_{t-1}-1} B_2\left(\frac{c_s(k)}{a_{s+1}}\right) + \sum_{d=1}^{a_{t+1}-1} B_2\left(\frac{d}{a_{t+1}}\right)\sum_{k=0}^{q_t-1} B_2\left(\frac{c_s(k)}{a_{s+1}}\right)\right)$$

$$+\frac{(-1)^t}{6}(q_m(p_{t+1}-p_t) - p_m(q_{t+1}-q_t))\sum_{k=0}^{q_t-1} B_2\left(\frac{c_s(k)}{a_{s+1}}\right).$$

(3) If $q \in \mathbb{N}$ we have

$$\sum_{k=0}^{q-1} B_2\left(\frac{k}{q}\right)^2 = \frac{q}{180} + \frac{1}{18q} - \frac{1}{30q^3}.$$

Therefore

$$\frac{q_m}{a_{s+1}^2}U_{s,s} = \sum_{k=0}^{q_m-1} B_2\left(\frac{c_s(k)}{a_{s+1}}\right)^2$$

$$= \sum_{c=0}^{a_{s+1}} B_2\left(\frac{c}{a_{s+1}}\right)^2 |L_{0,s}(1,c)|.|L_{s+1,m}(c,0)|$$

$$= \frac{(-1)^s q_s}{36}(q_m(p_{s+1}-p_s) - p_m(q_{s+1}-q_s))$$

$$+(-1)^{s+1}q_s(q_mp_s - p_mq_s)\sum_{c=1}^{a_{s+1}-1} B_2\left(\frac{c}{a_{s+1}}\right)^2 + \frac{(-1)^{s+1}q_{s-1}}{36}(q_mp_s - p_mq_s).$$

PROPOSITION 2.1. *Let $\alpha = [0; a_1, \ldots]$ be the continued fraction expansion of $\alpha \in \Omega$ wth convergents $\frac{p_n}{q_n}$ Let $m > v$ be nonnegative integers and let $\mu > 0$. Then*

(1) $\left|\left\{k \in \mathbb{Z}_+ | k < q_m, \left|\sum_{s=v}^{m-1} a_{s+1}B_2\left(\frac{c_s(k)}{a_{s+1}}\right)\right| \geq \frac{m-v}{2} + \mu\left(\sum_{s=v}^{m-1} a_{s+1}^2\right)^{1/2}\right\}\right| \leq \frac{q_m}{2\mu^2}$,

(2) $\left|\left\{k \in \mathbb{Z}_+ | k < q_m, \sum_{s=0}^{v-1} c_s(k)q_s \in [0,\mu] \cup [q_v - \mu, q_m)\right\}\right| \leq 4\frac{([\mu]+1)q_m}{q_v}$.

Proof. (1) Let A be the set occuring in the Proposition. For $0 \le s \le t < m$ let

$$V_s := \frac{1}{q_m} \sum_{k=0}^{q_m-1} a_{s+1} B_2\left(\frac{c_s(k)}{a_{s+1}}\right)$$

$$U_{s,t} := \frac{1}{q_m} \sum_{k=0}^{q_m-1} a_{s+1} a_{t+1} B_2\left(\frac{c_s(k)}{a_{s+1}}\right) B_2\left(\frac{c_t(k)}{a_{t+1}}\right)$$

and for $0 \le k < q_m$

$$Z_s(k) := a_{s+1} B_2\left(\frac{c_s(k)}{a_{s+1}}\right) - V_s.$$

Using $\sum_{k=s+1}^{\infty} \frac{1}{q_t^2} \le \frac{4}{q_{s+1}^2}$ (an inequality which is used repeatedly in the following text) we get with the help of Lemma 2.4(2)

$$\begin{aligned}
\left|\sum_{s=v}^{m-1} \sum_{t=s+1}^{m-1} (U_{s,t} - V_s V_t)\right| &\le \frac{1}{36} \sum_{s=v}^{m-1} a_{s+1}^2 q_s^2 \sum_{t=s+1}^{m-1} a_{t+1}^2 \left(p_t - \frac{p_m}{q_m} q_t\right)^2 \\
&\le \frac{1}{36} \sum_{s=v}^{m-1} a_{s+1}^2 q_s^2 \sum_{t=s+1}^{m-1} \frac{a_{t+1}^2}{q_{t+1}^2} \\
&\le \frac{1}{36} \sum_{s=v}^{m-1} q_{s+1}^2 \sum_{t=s+1}^{m-1} \frac{1}{q_t^2} \le \frac{1}{9} \sum_{s=v}^{m-1} 1 = \frac{m-v}{9}.
\end{aligned}$$

For $v \le s \le m$ let $\beta_s := [a_s; a_{s+1}, \dots, a_m]$. Then for $s < m$, $|p_s - \frac{p_m}{q_m} q_s| = \frac{1}{q_s \beta_{s+1} + q_{s-1}}$ and therefore, by Lemma 2.4(1), we have

$$\begin{aligned}
\left|\sum_{s=v}^{m-1} V_s\right| &\le \frac{1}{6} \sum_{s=v}^{m-1} a_{s+1} \left(1 - a_{s+1} q_s \left|p_s - \frac{p_m}{q_m} q_s\right|\right) + \frac{1}{6} \sum_{s=v}^{m-1} \left|p_s - \frac{p_m}{q_m} q_s\right| q_s \\
&\le \frac{1}{6} \sum_{s=v}^{m-1} a_{s+1} \left(1 - \frac{a_{s+1} q_s}{q_s \beta_{s+1} + q_{s-1}}\right) + \frac{1}{6} \sum_{s=v}^{m-1} \frac{q_s}{q_{s+1}} \\
&\le \frac{1}{6} \sum_{s=v}^{m-1} \frac{a_{s+1}(q_s + q_{s-1})}{q_s \beta_{s+1} + q_{s-1}} + \frac{m-v}{6} \\
&\le \frac{1}{3} \sum_{s=v}^{m-1} \frac{a_{s+1} q_s}{q_{s+1}} + \frac{m-v}{6} \le \frac{m-v}{2}.
\end{aligned}$$

At the end, if $0 \le s < m$ we get, by Lemma 2.4(3),

$$|U_{s,s} - V_s^2| = |\frac{1}{30} q_s|p_s - \frac{p_m}{q_m} q_s| \left(a_{s+1}^3 - \frac{1}{a_{s+1}}\right) - \frac{q_s^2}{36}\left(p_s - \frac{p_m}{q_m} q_s\right)^2 \left(a_{s+1}^2 - 1\right)^2|$$

$$= (a_{s+1}^2 - 1) q_s|p_s - \frac{p_m}{q_m} q_s| \left(\frac{a_{s+1}^2 + 1}{30 a_{s+1}} - \frac{1}{36}(a_{s+1}^2 - 1) q_s|p_s - \frac{p_m}{q_m} q_s|\right)$$

$$\le \frac{a_{s+1}^3 q_s}{30 q_{s+1}} \le \frac{a_{s+1}^2}{30}.$$

Now let

$$B := \left\{ k \in \mathbb{Z}_+ | k < q_m, |\sum_{s=v}^{m-1} Z_s(k) \ge \mu \left(\sum_{s=v}^{m-1} a_{s+1}^2 \right)^{1/2} \right\}.$$

Then

$$\frac{\mu^2}{q_m} |B| \sum_{s=v}^{m-1} a_{s+1}^2 \le \frac{1}{q_m} \sum_{k=0}^{q_m - 1} \left(\sum_{s=v}^{m-1} Z_s(k) \right)^2$$

$$= \sum_{s=v}^{m-1} (U_{s,s} - V_s^2) + 2 \sum_{s=v}^{m-1} \sum_{t=s+1}^{m-1} (U_{s,t} - V_s V_t)$$

$$\le \sum_{s=v}^{m-1} |U_{s,s} - V_s^2| + 2 \frac{m-v}{9}$$

$$\le \left(\frac{2}{9} + \frac{1}{30} \right) \sum_{s=v}^{m-1} a_{s+1}^2$$

$$\le \frac{1}{2} \sum_{s=v}^{m-1} a_{s+1}^2;$$

this implies that $|B| \le \frac{q_m}{2\mu^2}$.

It is therefore enough to prove $A \subseteq B$. Let $k \in A$. Then we get

$$|\sum_{s=v}^{m-1} Z_s(k)| \ge |\sum_{s=v}^{m-1} a_{s+1} B_s \left(\frac{c_s(k)}{a_{s+1}} \right)| - |\sum_{s=v}^{m-1} V_s|$$

$$\ge \frac{m-v}{2} + \mu \left(\sum_{s=v}^{m-1} a_{s+1}^2 \right)^{1/2} - \frac{m-v}{2}$$

$$= \mu \left(\sum_{s=v}^{m-1} a_{s+1}^2 \right)^{1/2}.$$

Therefore $k \in B$.

(2) Let $A := \left\{k \in \mathbb{Z}_+ | k < q_m, \sum_{s=0}^{\nu-1} c_s(k)q_s \in [0, \mu]\right\}$.

Then

$$\begin{aligned}
|A| &= \sum_{0\le t\le \mu} |L_{\nu,m}(c_{\nu-1}(t), 0)| \\
&= \sum_{\substack{0\le t\le\mu \\ c_{\nu-1}(t)=0}} (-1)^{\nu+1}(q_m(p_\nu - p_{\nu-1}) - p_m(q_\nu - q_{\nu-1})) \\
&\quad + \sum_{\substack{0\le t\le\mu \\ c_{\nu-1}(t)>0}} (-1)^{\nu}(q_m p_{\nu-1} - p_m q_{\nu-1}) \\
&= \sum_{0\le t\le\mu} (-1)^{\nu}(q_m p_{\nu-1} - p_m q_{\nu-1}) + \sum_{\substack{0\le t\le\mu \\ c_{\nu-1}(t)=0}} (-1)^{\nu+1}(q_m p_\nu - p_m q_\nu) \\
&\le \frac{2([\mu]+1)q_m}{q_\nu}.
\end{aligned}$$

Let

$$B := \left\{k \in \mathbb{Z}_+ \mid k < q_m, \sum_{s=0}^{\nu-1} c_s(k)q_s \in [q_\nu - \mu, q_\nu)\right\}.$$

Then we get similarly

$$\begin{aligned}
|B| &= \sum_{q_\nu-\mu\le t<q_\nu} |L_{\nu,m}(c_{\nu-1}(t), 0)| \\
&= \sum_{q_\nu-\mu\le t<q_\nu} (-1)^{\nu}(q_m p_{\nu-1} - p_m q_{\nu-1}) + \sum_{\substack{q_\nu-\mu\le t<q_\nu \\ c_{\nu-1}(t)=0}} (-1)^{\nu+1}(q_m p_\nu - p_m q_\nu) \\
&\le \frac{2[\mu]q_m}{q_\nu}.
\end{aligned}$$

Besides Proposition 2.1 we need the following (apparent) generalization.

COROLLARY 2.1. *Let $\alpha = [0; a_1, \ldots] \in \Omega$ be the continued fraction expansion of α. Let $(q_i)_{i\ge -1}$ be a sequence of real numbers such that $q_{-1} = 0 < q_0$ and, for $i \ge 0$, $q_{i+1} = q_{i+1}q_i + q_{i-1}$. Let $Z \subseteq [0, q_m)$, $d := \inf\{|x-y| | x, y \in Z, x \ne y\} > 0$, $m > \nu$ be nonnegative integers and let μ be any positive real number. Then*

(1) $\left|\left\{z \in Z \middle| \left|\sum_{s=\nu}^{m-1} a_{s+1}B_2\left(\frac{c_s(z)}{a_{s+1}}\right)\right| \ge \frac{m-\nu}{2} + \mu\left(\sum_{s=\nu}^{m-1} a_{s+1}^2\right)^{1/2}\right\}\right| \le \frac{q_m}{2\mu^2 q_0}\left(1 + \frac{q_0}{d}\right)$,

(2) $\left|\left\{z \in Z \middle| z - \sum_{s=\nu}^{m-1} c_s(z)q_s \in [0, \mu] \cup [q_\nu - \mu, q_\nu)\right\}\right| \le 4\left(\frac{q_0}{d} + 1\right)\left(\left[\frac{\mu}{q_0}\right] + 2\right)\frac{q_m}{q_\nu}$.

Proof. Let $k(z) := \left[\frac{z}{q_0}\right] q_0$ (for $z \in Z$),

$$A := \left\{ z \in Z \,\Big|\, \Big|\sum_{s=v}^{m-1} a_{s+1} B_2\left(\frac{c_s(z)}{a_{s+1}}\right)\Big| \geq \frac{m-v}{2} + \mu\left(\sum_{s=v}^{m-1} a_{s+1}^2\right)^{1/2} \right\},$$

$$A' = \left\{ z \in Z \mid z - \sum_{s=v}^{m-1} c_s(z) q_s \in [0, \mu] \cup [q_v - \mu, q_v) \right\},$$

$$B := \left\{ k \in q_0\mathbb{Z}_+ \,\Big|\, k < q_m, \Big|\sum_{s=v}^{m-1} a_{s+1} B_2\left(\frac{c_s(k)}{a_{s+1}}\right) \geq \frac{m-v}{2} + \mu\left(\sum_{s=v}^{m-1} a_{s+1}^2\right)^{1/2} \right\},$$

$$B'(\mu) := \left\{ k \in q_0\mathbb{Z}_+ \mid k < q_m, k - \sum_{s=v}^{m-1} c_s(k) q_s \in [0, \mu] \cup [q_v - \mu, q_v) \right\}.$$

Then, by Proposition 2.1 applied to $\frac{q_m}{q_0}$, we get $|B| \leq \frac{q_m}{2\mu^2 q_0}$ and $|B'(\mu)| \leq 4\left(\left[\frac{\mu}{q_0}\right] + 1\right)\frac{q_m}{q_v}$.

Let us prove that for $0 \leq s < m$, $c_s(z) = c_s(k(z))$. From the inequalities $\sum_{s=v}^{m-1} c_s(k(z)) q_s \leq k(z) \leq z$ we get, for $t \geq 0$,

$$\begin{aligned} 0 &\leq z - \sum_{s=t}^{m-1} c_s(k(z)) q_s \\ &< k(z) - \sum_{s=t}^{m-1} c_s(k(z)) q_s + q_0 \\ &\leq q_t - q_0 + q_0 = q_t. \end{aligned}$$

This implies the assertion. Therefore if $z \in A$, then $k(z) \in B$. We get

$$|A| = \sum_{z \in A} 1 \leq \sum_{k \in B} \sum_{\substack{z \in A \\ k \leq z < k + q_0}} 1 \leq \sum_{k \in B}\left(\frac{q_0}{d} + 1\right) \leq \left(\frac{q_0}{d} + 1\right)\frac{q_m}{2\mu^2 q_0}.$$

Assume now that $z \in A'$. Then

$$\begin{aligned} \sum_{s=0}^{v-1} c_s(k))q_s &= k(z) - \sum_{s=v}^{m-1} c_s(z) q_s \\ &= k(z) - z + z - \sum_{s=v}^{m-1} c_s(z) q_s \\ &\in [0, \mu] \cup [q_v - \mu - q_0, q_v) \end{aligned}$$

and therefore $k(z) \in B'(\mu + q_0)$. We get

$$\begin{aligned} |A'| = \sum_{z \in A'} 1 &\leq \sum_{k \in B'(\mu + q_0)} \sum_{\substack{z \in A' \\ k \leq z < k + q_0}} 1 \\ &\leq 4\left(\frac{q_0}{d} + 1\right)\left(\left[\frac{\mu + q_0}{q_0}\right] + 1\right)\frac{q_m}{q_v} \\ &= 4\left(\frac{q_0}{d} + 1\right)\left(\left[\frac{\mu}{q_0}\right] + 2\right)\frac{q_m}{q_v}. \end{aligned}$$

3. The Main Lemma. We use the following property, which is an immediate consequence of the mixing property of the random variables $a_i(\alpha)$.

Let $B \subseteq \Omega$ be measurable; for $\alpha \in \Omega$ let $\alpha_n = [0; a_{n+1}(\alpha), \ldots]$ and $b \in \mathbb{N}^n$. Then

$$\lambda\left(\left\{\alpha \in \Omega | \alpha_n \in B, 1 \le i \le n \Rightarrow a_i(\alpha) = b_i\right\}\right) \le 2\lambda(B)\lambda\left(\left\{\alpha \in \Omega | 1 \le i \le n \Rightarrow a_i(\alpha) = b_i\right\}\right).$$

LEMMA 3.1. *There is a positive real number $c > 0$ with the following property: if N is a positive and m a nonnegative integer, $(b_i)_{1 \le i \le m}$ is a sequence of natural numbers, q_m is the denominator of $[0; b_1, \ldots, b_m]$,*

$$A := \left\{\alpha \in \Omega \mid 1 \le i \le m \Rightarrow a_i(\alpha) = b_i\right\}$$

and

$$B := \left\{\alpha \in A \mid 0 \le i < m \Rightarrow c_i(N, \alpha) = 0\right\},$$

then $\lambda(B) \le c\left(\frac{q_m}{N} + \frac{1}{q_m}\right)\lambda(A)$.

Proof. For $0 \le n \le m$, let $\frac{p_n}{q_n}$ be the convergent of $[0; b_1, \ldots, b_m]$. Let

$$K := \left\{(k, l) \mid 0 \le k \le l, lq_m + kq_{m-1} = N\right\}.$$

We prove that, for some constant c_1, $\sum_{(k,l)\in K} \frac{1}{l} \le c_1\left(\frac{q_m}{N} + \frac{1}{q_m}\right)$.

Because of $\gcd(q_{m-1}, q_m) = 1$, there are integers k_0, l_0 with $l_0 q_m + k_0 q_{m-1} = N$. For any $(k, l) \in K$, there is a $j \in \mathbb{Z}$ with $l = l_0 + jq_{m-1}$. Now $0 \le k \le l$ implies $lq_m \le N \le l(q_m + q_{m-1})$ and therefore $\frac{N}{q_m+q_{m-1}} \le l_0 + jq_{m-1} \le \frac{N}{q_m}$. The number of these integers j is at most $1 + \frac{N}{q_m(q_m+q_{m-1})}$. Therefore we get

$$\sum_{(k,l)\in K} \frac{1}{l} \le \sum_{\frac{N}{q_m+q_{m-1}} \le l \le \frac{N}{q_m}} \frac{1}{l} \le \frac{q_m + q_{m-1}}{N}\left(1 + \frac{N}{q_m(q_m + q_{m-1})}\right) \le \frac{2q_m}{N} + \frac{1}{q_m}.$$

For $0 \le k \le l$ we put $B_{k,l} := \{\alpha \in A || k - \alpha_m l| \le 4\}$. We prove that $B \subseteq \cup_{(k,l)\in K} B_{k,l}$.

Let $\alpha \in B$ and $t \ge m$. Lemma 2.2 implies that

$$q_t(\alpha) = q_{m-1}p_{t-m,m}(\alpha) + q_m q_{t-m,m}(\alpha).$$

Therefore

$$N = \sum_{t=m}^{\infty} c_t(N, \alpha) q_t(\alpha)$$

$$= q_{m-1} \sum_{t=m}^{\infty} c_t(N, \alpha) p_{t-m,m}(\alpha) + q_m \sum_{t=m}^{\infty} c_t(N, \alpha) q_{t-m,m}(\alpha).$$

If we put $k := \sum_{t=m}^{\infty} c_t(N,\alpha)p_{t-m,m}(\alpha)$ and $l = \sum_{t=m}^{\infty} c_t(N,\alpha)q_{t-m,m}(\alpha)$, we get $0 \le k \le l$ and therefore $(k,l) \in K$. Furthermore

$$|k - \alpha_m l| \le \sum_{t=m}^{\infty} c_t(N,\alpha)|p_{t-m,m}(\alpha) - \alpha_m q_{t-m,m}(\alpha)|$$

$$\le \sum_{t=m}^{\infty} a_{t+1}(\alpha)\frac{1}{q_{t-m+1,m}(\alpha)}$$

$$\le \sum_{t=m}^{\infty} \frac{1}{q_{t-m,m}(\alpha)} = \sum_{t=0}^{\infty} \frac{1}{q_{t,m}(\alpha)} \le 4.$$

Therefore $\alpha \in B_{k,l}$. We get

$$\lambda(B) \le \sum_{(k,l)\in K} \lambda(B_{k,l}) \le 2\lambda(A) \sum_{(k,l)\in K} \frac{8}{l} \le 16c_1\lambda(A)\left(\frac{q_m}{N} + \frac{1}{q_m}\right).$$

PROPOSITION 3.1. *There is a positive real number c with the following property: if N is a positive integer,* $0 < \mu \le \frac{\tau}{4}$, $M = \left[\left(\frac{\tau}{2} - \mu\right)\log N\right] \ge 2$, $\varepsilon > 0$ *and*

$$A = \left\{\alpha \in \Omega \,\middle|\, \left|\sum_{s=0}^{M-1} a_{s+1}(\alpha)B_2\left(\frac{c_s(N,\alpha)}{a_{s+1}(\alpha)}\right)\right| \ge \varepsilon \sum_{s=0}^{M-1} a_{s+1}(\alpha)\right\},$$

then

$$\lambda(A) \le c\left(\frac{1}{\mu\sqrt{\log N}}e^{-\mu^2 \log N/(4\sigma^2\tau^3)} + \frac{1}{(\varepsilon \log M)^{2/3}}\right).$$

Proof. Let $a := \sqrt{\frac{e \log 2}{2}}$. The assertion is trivial if $\varepsilon \log M \le 2 \log 2$ or if $\log N - \gamma \le 4 \log (a \log M)$. We may assume the opposite and we put $v := (\varepsilon \log M)^{1/3}$. Then

$$\frac{\varepsilon}{\log 2}(\log M - \gamma - 2\log(a \log M)) \ge \frac{\varepsilon \log M}{2 \log 2} \ge 1.$$

Therefore there exists an $R > 0$ such that

$$\frac{1}{2} + \frac{v\sqrt{R}}{M} = \frac{\varepsilon}{\log 2}(\log M - \gamma - 2\log(a \log M)).$$

This R satisfies the inequality $\frac{v\sqrt{R}}{M} \ge \frac{\varepsilon \log M}{4 \log 2}$.

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We put

$$B := \left\{\alpha \in \Omega \mid q_M(\alpha) \geq \sqrt{N}\right\}$$

$$C := \left\{\alpha \in \Omega \mid \sum_{s=0}^{M-1} a_{s+1}^2(\alpha) \geq R\right\}$$

$$D := \left\{\alpha \in \Omega \mid \sum_{s=0}^{M-1} a_{s+1}(\alpha) \leq \frac{M}{2\varepsilon} + \frac{\nu\sqrt{R}}{\varepsilon}\right\}$$

$$\mathcal{B} := \left\{b \in \mathbb{N}^M \mid \bar{q}_M < \sqrt{N}, \sum_{s=0}^{M-1} b_{s+1}^2 < R, \sum_{s=0}^{M-1} b_{s+1} > \frac{M}{2\varepsilon} + \frac{\nu\sqrt{R}}{\varepsilon}\right\}$$

(where $\bar{q}_M$ denotes the denominator of $[0; b_1, \ldots, b_M]$). For $b \in \mathcal{B}$ let

$$E_b' := \left\{\alpha \in \Omega \mid 1 \leq s \leq M \Rightarrow a_s(\alpha) = b_s\right\},$$

$$E_b := \left\{\alpha \in E_b' \middle| \left|\sum_{s=0}^{M-1} b_{s+1} B_2\left(\frac{c_s(N,\alpha)}{b_{s+1}}\right)\right| \geq \varepsilon \sum_{s=0}^{M-1} b_{s+1}\right\}.$$

By Proposition 1.1 we have

$$\lambda(B) = 1 - \Phi\left(\frac{\frac{\tau}{2}\log N - M}{\sigma\tau\sqrt{M}}\right) + O\left(\frac{\log M}{\sqrt{M}}\right)$$

$$\leq 1 - \Phi\left(\frac{\mu\sqrt{\log N}}{\sqrt{2}\sigma\tau^{3/2}}\right) + O\left(\frac{\log M}{\sqrt{M}}\right)$$

$$= O\left(\frac{1}{\mu\sqrt{\log N}} e^{-\mu^2 \log N/(4\sigma^2\tau^3)} + \frac{\log M}{\sqrt{M}}\right).$$

Lemma 1.2 implies

$$\lambda(C) = O\left(\frac{M}{\sqrt{R}}\right) = O\left(\frac{\nu}{\varepsilon \log M}\right) = O\left((\varepsilon \log M)^{-2/3}\right).$$

By Proposition 1.5 and Proposition 1.4(2) we get

$$\lambda(D) = G\left(\frac{1}{2\varepsilon} + \frac{\mu\sqrt{R}}{\varepsilon M} - \frac{\log M - \gamma}{\log 2}\right) + O\left(\frac{\log^2 M}{M}\right)$$

$$= G\left(-\frac{2}{\log 2}\log(a \log M)\right) + O\left(\frac{\log^2 M}{M}\right)$$

$$= O\left(1 - \Phi(\log M) + \frac{\log^2 M}{M}\right) = O\left(\frac{\log^2 M}{M}\right).$$

Assume now that $b \in \mathcal{B}$ and $\alpha \in E_b$. Then

$$|\sum_{s=0}^{M-1} b_{s+1} B_2\left(\frac{c_s(N,\alpha)}{b_{s+1}}\right)| \geq \frac{M}{2} + \nu\sqrt{R}$$
$$\geq \frac{M}{2} + \nu\left(\sum_{s=0}^{M-1} b_{s+1}^2\right)^{1/2}.$$

Denoting by

$$V := \left\{k \in \mathbb{Z}_+ | k < \bar{q}_M, |\sum_{s=0}^{M-1} b_{s+1} B_2\left(\frac{c_s(N,\alpha)}{b_{s+1}}\right)| \geq \frac{M}{2} + \nu\left(\sum_{s=0}^{M-1} b_{s+1}^2\right)^{1/2}\right\}$$

we get, with the help of Proposition 2.1(1), $|V| = O(\frac{\bar{q}_M}{\nu^2})$. Since

$$N = \sum_{s=M}^{m_N(\alpha)} c_s(N,\alpha) q_s(\alpha) + \sum_{s=0}^{M-1} c_s(N,\alpha)\bar{q}_s$$

we have $k := \sum_{s=0}^{M-1} c_s(N,\alpha)\bar{q}_s \in V$ and $c_s(N-k,\alpha) = 0$ for $0 \leq s < M$. Now $\bar{q}_M < \sqrt{N}$ and Lemma 3.1 imply

$$\lambda(E_b) = O\left(\lambda(E_b') \sum_{k\in V}\left(\frac{\bar{q}_M}{N-k} + \frac{1}{\bar{q}_M}\right)\right)$$
$$= O\left(\lambda(E_b')|V|\left(\frac{\bar{q}_M}{N-\bar{q_M}} + \frac{1}{\bar{q_M}}\right)\right)$$
$$= O\left(\lambda(E_b')|V|\frac{1}{\bar{q}_M}\right) = O\left(\frac{\lambda(E_b')}{\nu^2}\right).$$

From this it follows that $\lambda\left(\bigcup_{b\in\mathcal{B}} E_b\right) = O(\frac{1}{\nu^2}) = O\left((\varepsilon \log M)^{-2/3}\right)$. Finally we have $A \subseteq B \cup C \cup D \cup \bigcup_{b\in\mathcal{B}} E_b$ and this proves the assertion of the proposition.

We need an estimation for

$$\lambda\left(\left\{\alpha \in \Omega || \sum_{s=0}^{m_n(\alpha)} a_{s+1}(\alpha) B_2\left(\frac{c_s(N,\alpha)}{a_{s+1}(\alpha)}\right)| \geq \varepsilon \sum_{s=0}^{m_N(\alpha)} a_{s+1}(\alpha)\right\}\right)$$

from above. To get such a result we have to estimate

$$\lambda\left(\left\{\alpha \in \Omega || \sum_{s=M}^{m_N(\alpha)} a_{s+1}(\alpha) B_2\left(\frac{c_s(N,\alpha)}{a_{s+1}(\alpha)}\right)| \geq \varepsilon \sum_{s=M}^{m_N(\alpha)} a_{s+1}(\alpha)\right\}\right)$$

from above. To manage this task is more difficult (although similar in principle) because of the fact that the upper summation limit depends now on α.

Let k and l be nonnegative integers. Then

$$\sum_{t=0}^{\infty} \lambda\left(\left\{\alpha \in \Omega \mid q_{t-1}(\alpha) = k, q_t(\alpha) = l\right\}\right) \leq \frac{4}{l^2}.$$

See [5, p. 207]

LEMMA 3.2. *There is a positive real number c with the following property: if N is a positive and m a nonnegative integer,* $b \in \mathbb{N}^m$, *q denotes the denominator of* $[0; b_1, \dots, b_m]$, $\nu : \Omega \to \mathbb{Z}_+$ is measurable,

$$B = \left\{\alpha \in \Omega \mid 1 \leq i \leq m \Rightarrow a_i(\alpha) = b_i\right\}$$

and

$$A = \left\{\alpha \in \Omega \mid 1 \leq i \leq m \Rightarrow a_{\nu(\alpha)+i}(\alpha) = b_i, q_{m+\nu(\alpha)}(\alpha) = N\right\},$$

then $\lambda(A) \leq c\left(\frac{q^2}{N^2} + \frac{1}{N}\right)\lambda(B)$.

Proof. For $0 \leq k \leq l$, let

$$A_{k,l} := \left\{\alpha \in \Omega \mid 1 \leq i \leq m \Rightarrow a_{\nu(\alpha)+i}(\alpha) = b_i, q_{\nu(\alpha)-1}(\alpha) = k, q_{\nu(\alpha)}(\alpha) = l\right\}.$$

Let p be the numerator of $[0; b_1, ..., b_m]$. Now $\alpha \in A$ implies $N = qq_{\nu(\alpha)}(\alpha) + pq_{\nu(\alpha)-1}(\alpha)$ and therefore we get $A \subseteq \bigcup_{\substack{0 \leq k \leq l \\ kp+lq=N}} A_{k,l}$.

Now

$$A_{k,l} \subseteq \left\{\alpha \in \Omega \mid \exists t(t \in \mathbb{Z}_+ \wedge 1 \leq i \leq m \Rightarrow a_{t+i}(\alpha) = b_i \wedge k = q_{t-1}(\alpha) \wedge l = q_t(\alpha))\right\}.$$

Therefore

$$\begin{aligned}\lambda(A) &\leq \sum_{\substack{0 \leq k \leq l \\ kp+lq=N}} \sum_{t=0}^{\infty} \lambda\left(\left\{\alpha \in \Omega \mid 1 \leq i \leq m \Rightarrow a_{t+i}(\alpha) = b_i \wedge k = q_{t-1}(\alpha) \wedge l = q_t(\alpha)\right\}\right) \\ &\leq 2\lambda(B) \sum_{\substack{0 \leq k \leq l \\ kp+lq=N}} \sum_{t=0}^{\infty} \lambda\left(\left\{\alpha \in \Omega \mid k = q_{t-1}(\alpha), l = q_t(\alpha)\right\}\right) \leq 8\lambda(B) \sum_{\substack{0 \leq k \leq l \\ kp+lq}} \frac{1}{l^2}.\end{aligned}$$

Let (k, l) be a solution of $kp + lq = N$ with the side condition $0 \leq k \leq l$. Then $lq \leq N \leq l(p + q)$ and therefore $\frac{N}{p+q} \leq l \leq \frac{N}{q}$. The number of the solutions of this equation with this side condition is therefore at most $1 + \frac{1}{p}\left(\frac{N}{q} - \frac{N}{p+q}\right) \leq 1 + \frac{N}{q^2}$. This implies that

$$\sum_{\substack{0 \leq k \leq l \\ kp+lq=N}} \frac{1}{l^2} \leq \frac{(p+q)^2}{N^2}\left(1 + \frac{N}{q^2}\right) \leq 4\frac{q^2}{N^2}\left(1 + \frac{N}{q^2}\right) = \frac{4q^2}{N^2} + \frac{4}{N}.$$

LEMMA 3.3. *Let* $N > w, M, b_1, \dots, b_M$ *and* $K < M$ *be natural numbers. Let* $\nu : \Omega \to \mathbb{Z}_+$ *be measurable. For* $0 \leq i \leq M$ *let* $\frac{\bar{p}_i}{\bar{q}_i}$ *be the i-th convergent to* $[0; b_1, \dots, b_M]$. *We put* $q'_{-1} = 0, q'_0 = \frac{N}{w\bar{q}_M}$ *and, for* $0 \leq i < M$, $q'_{i+1} = b_{i+1}q'_{i-1}$. *Let* $\bar{q}_M^2 \leq \frac{N}{w+1}$,

$$A := \left\{\alpha \in \Omega \mid 1 \leq i \leq M \Rightarrow a_{\nu(\alpha)+i}(\alpha) = b_i, N < q_{M+\nu(\alpha)+1}(\alpha), \frac{N}{w+1} < q_{M+\nu(\alpha)}(\alpha) \leq \frac{N}{w}\right\}$$

and

$$B := \{\alpha \in \Omega \mid 1 \le i \le M \Rightarrow a_i(\alpha) = b_i\}.$$

For $\frac{N}{w+1} < j \le \frac{N}{w}$ *let* $z_j = \frac{N}{wj}(N - wj)$. *Then there exists, for every* $\alpha \in A$, *exactly one sequence* $(c'_i(\alpha))_{0\le i<M} \in \mathbb{Z}^M$ *such that, for all* $v \in \mathbb{Z}_+$ *which satisfy* $0 \le v < M$, $0 \le zq_{M+v(\alpha)}(\alpha) - \sum_{s=v}^{M-1} c'_s(\alpha)q'_s < q'_v$ *is valid. Furthermore there is a positive and absolute constant* c *such that*

$$\lambda(\{\alpha \in A \mid \exists u(K \le u < M \wedge c'_u(\alpha) \ne c_{u+v(\alpha)}(N, \alpha))\}) \le c\frac{\lambda(B)}{\bar{q}_K}.$$

Proof. For $0 \le i \le M$ we have $\bar{q}_i = \frac{q'_i}{q'_0}$ and therefore $q'_i = \frac{N}{w\bar{q}_M}\bar{q}_i$. In particular $q'_M = \frac{N}{w}$. Now $\alpha \in A$ implies $N < (w+1)q_{M+v(\alpha)}(\alpha)$ and therefore $\frac{N - wq_{M+v(\alpha)}(\alpha)}{q_{m+v(\alpha)}(\alpha)} < 1$. This results in $z_{q_{M+v(\alpha)}(\alpha)} < q'_M$. The first assertion therefore follows from Lemma 2.1.

If $\bar{q}_K < 5$, the result follows from the assertion preceding Lemma 2. We may therefore assume that $\bar{q}_K \ge 5$. The case $A = \emptyset$ is trivial. Otherwise we have, for $\alpha \in A$,

$$q'_0 = \frac{N}{w\bar{q}_M} \ge \frac{N}{w(\bar{q}_M q_{v(\alpha)}(\alpha) + \bar{p}_M q_{v(\alpha)-1}(\alpha))} = \frac{N}{wq_{M+v(\alpha)}(\alpha)} \ge 1.$$

Let

$$C := \left\{q_{M+v(\alpha)}(\alpha) \mid \alpha \in A, \exists u\left(K \le u < M \wedge z_{q_{M+v(\alpha)}}(\alpha) - \sum_{s=u}^{M-1} c'_s(\alpha)q'_s \in \left[0, \frac{5q'_u}{\bar{q}_u^2}\right] \cup \left[q'_u - \frac{5q'_u}{\bar{q}_u^2}, q'_u\right)\right)\right\}.$$

We prove that for every $\alpha \in A$, for which there is a u, with $K \le u < M$ and with $c'_u(\alpha) \ne c_{u+v(\alpha)}(N, \alpha)$, we have $q_{M+v(\alpha)}(\alpha) \in C$. As long as this assertion is not proved we write q_s instead of $q_s(\alpha)$ and v for $v(\alpha)$.

Let $\alpha \in A$ and assume that $v \le j \le M + v$. Then $q_j = \bar{q}_{j-v}q_v + \bar{p}_{j-v}q_{v-1}$. This gives us

$$\begin{aligned}\left|\frac{Nq_j}{wq_M + v} - q'_{j-v}\right| &= \frac{N}{w}\left|\frac{\bar{q}_{j-v}q_v + \bar{p}_{j-v}q_{v-1}}{\bar{q}_M q_v + \bar{p}_M q_{v-1}} - \frac{\bar{q}_{j-v}}{\bar{q}_M}\right| \\ &= \frac{N}{w}q_{v-1}\frac{|\bar{p}_{j-v}\bar{q}_M - \bar{p}_M\bar{q}_{j-v}|}{\bar{q}_M(\bar{q}_M q_v + \bar{p}_M q_{v-1})} \\ &= \frac{Nq_{v-1}}{wq_{M+v}}\left|\bar{p}_{j-v} - \frac{\bar{p}_M}{\bar{q}_M}\bar{q}_{j-v}\right| \\ &\le \frac{Nq_{v-1}}{wq_{M+v}\bar{q}_{j-v+1}}.\end{aligned}$$

From $\frac{N}{w+1} < q_{M+\nu} \leq \frac{N}{w}$ it follows that $c_{M+\nu}(N,\alpha) = w$. Assume now that u is chosen maximal such that $c'_u(\alpha) \neq c_{u+\nu}(N,\alpha)$. Then

$$z_{qM+\nu} - \sum_{s=u}^{M-1} c'_s(\alpha)q'_s - \frac{N}{wq_{M+\nu}}\left(N - \sum_{s=u+\nu}^{M+\nu} c_s(N,\alpha)q_s\right) =$$

$$-N + \frac{Nc_{u+\nu}(N,\alpha)q_{u+\nu}}{wq_{M+\nu}} - c'_u(\alpha)q'_u + \sum_{s=u+\nu+1}^{M+\nu-1} c_s(N,\alpha)\left(\frac{Nq_s}{wq_{M+\nu}} - q'_{s-\nu}\right)$$

$$+ \frac{Nc_{M+\nu}(N,\alpha)q_{M+\nu}}{wq_{M+\nu}}.$$

This implies

$$\left|\frac{N}{wq_{M+\nu}}\left(N - \sum_{s=u+\nu}^{M+\nu} c_s(N,\alpha)q_s\right) - \left(z_{qM+\nu} - \sum_{s=u}^{M-1} c'_s(\alpha)q'_s\right) - \left(c'_u(\alpha) - c_{u+\nu}(N,\alpha)\right)\frac{Nq_{u+\nu}}{wq_{M+\nu}}\right|$$

$$= \left|c'_u(\alpha)q'_u - \frac{Nc_{u+\nu}(N,\alpha)q_{u+\nu}}{wq_{M+\nu}} - \sum_{s=u+\nu+1}^{M+\nu-1} c_s(N,\alpha)\left(\frac{Nq_s}{wq_{M+\nu}} - q'_{s-\nu}\right)\right.$$

$$\left. - \left(c'_u(\alpha - c_{u+\nu}(N,\alpha))\frac{Nq_{u+\nu}}{wq_{M+\nu}}\right)\right|$$

$$= \left|c'_u(\alpha)\left(q'_u - \frac{Nq_{u+\nu}}{wq_{M+\nu}}\right) + \sum_{s=u+\nu+1}^{M+\nu-1} c_s(N,\alpha)\left(q'_{s-\nu} - \frac{Nq_s}{wq_{M+\nu}}\right)\right|$$

$$\leq \sum_{s=u+\nu}^{M+\nu-1} b_{s-\nu+1}\left|q'_{s-\nu} - \frac{Nq_s}{wq_{M+\nu}}\right|$$

$$\leq \frac{Nq_{\nu-1}}{wq_{M+\nu}}\sum_{s=u}^{M-1}\frac{b_{s+1}}{\bar{q}_{s+1}} \leq \frac{Nq_{\nu-1}}{wq_{M+\nu}}\sum_{s=u}^{M-1}\frac{1}{\bar{q}_s}$$

$$\leq \frac{4Nq_{\nu-1}}{wq_{M+\nu}\bar{q}_u} \leq \frac{4q_{\nu-1}q'_u}{q_\nu\bar{q}_u^2} < 4\frac{q'_u}{\bar{q}_u^2}.$$

First of all we get

$$|c_u(\alpha) - c_{u+\nu}(N,\alpha)|\frac{Nq_{u+\nu}}{wq_{M+\nu}} < \max\left\{\frac{Nq_{u+\nu}}{wq_{M+\nu}}, q'_u\right\} + \frac{4Nq_{\nu-1}}{wq_{M+\nu}\bar{q}_u}$$

$$\leq \frac{Nq_{u+\nu}}{wq_{M+\nu}} + \frac{5Nq_{\nu-1}}{wq_{M+\nu}\bar{q}_u}$$

and therefore

$$|c'_u(\alpha) - c_{u+v}(N,\alpha)| < 1 + \frac{5q_{v-1}}{q_{u+v}\bar{q}_u} \le 1 + \frac{5}{\bar{q}_u} \le 2.$$

This results in $c'_u(\alpha) - c_{u+v}(N,\alpha) = \pm 1$. Assume first that $c'_u(\alpha) - c_{u+v}(N,\alpha) = 1$. Then

$$zq_{M+v} - \sum_{s=u}^{M-1} c'_s(\alpha)q'_s + \frac{Nq_{u+v}}{wq_{M+v}} \le \frac{4q'_u}{\bar{q}_u^2} + \frac{Nq_{u+v}}{wq_{M+v}} wq_{M+v}$$

and therefore $zq_{M+v} - \sum_{s=u}^{M-1} c'_s(\alpha)q'_s \le \frac{4q'_u}{\bar{q}_u^2}$. If $c'_u(\alpha) - c_{u+v}(N,\alpha) = -1$, we get

$$\begin{aligned} zq_{M+v} - \sum_{s=u}^{M-1} c'_s(\alpha)q'_s &\ge \frac{Nq_{u+v}}{wq_{M+v}} - \frac{4q'_u}{\bar{q}_u^2} \\ &\ge q'_u - \frac{Nq_{v-1}}{wq_{M+v}\bar{q}_u} - \frac{4q'_u}{\bar{q}_u^2} \ge q'_u - \frac{5q'_u}{\bar{q}_u^2}. \end{aligned}$$

Hence $q_{M+v} \in C$ is proved. Next observe that $\frac{N}{w+1} < j \le \frac{N}{w}$ implies $z_j - z_{j+1} = \frac{Nw}{N+w} \ge \frac{1}{2}$. Now $q'_0 \ge 1$ and Corollary 2.1(2) imply

$$|C| \le \sum_{u=K}^{M-1} 4(2q'_0+1)\left(\frac{5q'_u}{\bar{q}_u^2 q'_0} + 2\right)\frac{N}{wq'_u} \le \frac{12N}{w} q'_0 \sum_{u=K}^{M-1}\left(\frac{5}{\bar{q}_u}+2\right)\frac{1}{q'_u} \le \frac{36N}{w}\frac{4q'_0}{q'_K} = \frac{144N}{w\bar{q}_K}.$$

For $q \in C$ we get $\bar{q}_M^2 \le \frac{N}{w+1} < q$. Let c be chosen as in Lemma 3.2. Then this implies

$$\begin{aligned} &\lambda\big(\{\alpha \in A \mid \exists u(K \le u < M \wedge c'_u(\alpha) \ne c_{u+v(\alpha)}(N,\alpha))\}\big) \\ &\le \sum_{q\in C} \lambda\big(\{\alpha \in \Omega \mid 1 \le i \le M \Rightarrow a_{i+v(\alpha)}(\alpha) = b_i, q_{M+v(\alpha)}(\alpha) = q\}\big) \\ &\le c\lambda(B)\sum_{q\in C}\left(\frac{\bar{q}_M^2}{q^2} + \frac{1}{q}\right) \le 2c\lambda(B)\sum_{q\in C}\frac{1}{q} \le 2c\frac{w+1}{N}\lambda(B)|C| \le 2.144c\frac{w+1}{w}\frac{\lambda(B)}{\bar{q}_K} \le 4.144c\frac{\lambda(B)}{\bar{q}_K}. \end{aligned}$$

LEMMA 3.4. *There is a $c > 0$ with the following property: if $w < N, K < M$ are positive integers, ε, κ, R are positive reals,*

$$\nu : \Omega \to \mathbb{Z}_+, \nu(\alpha) = \max\{m_N(\alpha) - M, 0\},$$

$$\mathcal{B} = \left\{ b \in \mathbb{N}^M \mid \bar{q}_M(b)^2 \le \frac{N}{w+1}, \sum_{i=K}^{M-1} b_{i+1}^2 \le R, \sum_{i=K}^{M-1} b_{i+1} > \frac{M-K}{2\varepsilon} + \frac{\kappa\sqrt{R}}{\varepsilon} \right\},$$

(*where $\bar{q}_M$ denotes the denominator of* $[0; b_1, \dots, b_M]$), *for* $b \in \mathcal{B}$,

$$E_b = \{\alpha \in \Omega \mid 1 \le i \le M \Rightarrow a_{i+\nu(\alpha)}(\alpha) = b_i,$$

$$\Big|\sum_{i=K+\nu(\alpha)}^{m_N(\alpha)-1} a_{i+1}(\alpha) B_2\left(\frac{c_i(\nu,\alpha)}{a_{i+1}(\alpha)}\right)\Big| \ge \varepsilon \sum_{i=K+\nu(\alpha)}^{m_N(\alpha)-1} a_{i+1}(\alpha), \frac{N}{w+1} < q_{m_N(\alpha)}(\alpha) \le \frac{N}{w}\},$$

then $\sum_{b\in\mathcal{B}} \lambda(E_b) \le c\left(\left(\frac{1+\sqrt{5}}{2}\right)^{-K} + \frac{1}{\kappa^2}\right)$.

Proof. Let us define for $b \in \mathcal{B}$,

$$A_b := \{\alpha \in \Omega \mid 1 \le i \le M \Rightarrow a_i(\alpha) = b_i\},$$

$$B_b := \left\{\alpha \in \Omega \mid 1 \le i \le M \Rightarrow a_{i+\nu(\alpha)}(\alpha) = b_i, \frac{N}{w+1} < q_{m_N(\alpha)}(\alpha) \le \frac{N}{w}\right\},$$

$$q_{-1}(b)' = 0,\ q_0(b)' = \frac{N}{w\bar{q}_M(b)}$$

and, for $0 \le i < M$, let $q'_{i+1}(b) = b_{i+1} q'_i(b) + q'_{i-1}(b)$. Note that $q'_M(b) = \frac{N}{w}$ and that (even in the case $m_N(\alpha) < M$)$N < q_{M+\nu(\alpha)+1}(\alpha)$. For $\frac{N}{w+1} < j \le \frac{N}{w}$, let $z_j = \frac{N}{wj}(N - wj)$. From Lemma 3.3 it follows that for any $\alpha \in B_b$ there is exactly one sequence $(c'_i(\alpha))_{0\le i<M} \in \mathbb{Z}^M$, such that $0 \le \nu < M$ implies $0 \le z q_{m_N(\alpha)}(\alpha) - \sum_{s=\nu}^{M-1} c'_s(\alpha) q'_s(b) < q'_\nu(b)$. Let

$$V_b := \left\{ q_{m_N(\alpha)}(\alpha) \in \left(\frac{N}{w+1}, \frac{N}{w}\right] \mid \alpha \in B_b, \Big|\sum_{i=K}^{M-1} b_{i+1} B_2\left(\frac{c'_s(\alpha)}{b_{i+1}}\right)\Big| \ge \frac{M-K}{2} + \kappa\left(\sum_{i=K}^{M-1} b_{i+1}^2\right)^{1/2} \right\}.$$

Corollary 2.1(1) implies $|V_b| \le \frac{q'_M(b)}{\kappa^2} = \frac{N}{w\kappa^2}$. For $\nu \in V_b$ we have $\bar{q}_M^2(b) \le \frac{N}{w+1} \le \nu$.

Let $\alpha \in E_b$. Then

$$\Big|\sum_{i=K}^{M-1} b_{i+1} B_2\left(\frac{c_{i+\nu(\alpha)}(N,\alpha)}{b_{i+1}}\right)\Big| \ge \varepsilon \sum_{i=K}^{M-1} b_{i+1} > \frac{M-K}{2} + \kappa\sqrt{R} \ge \frac{M-K}{2} + \kappa\left(\sum_{i=K}^{M-1} b_{i+1}^2\right)^{1/2}.$$

If for every i with $1 \le i < M$, we have $c_{i+\nu(\alpha)}(N,\alpha) = c'_i(\alpha)$, then $q_{m_N(\alpha)}(\alpha) \in V_b$.

Therefore Lemma 3.2 and Lemma 3.3 imply

$$\begin{aligned}
\lambda(E_b) &\le \lambda\big(\{\alpha \in B_b \mid \exists i(K \le i < M \wedge c_{i+\nu(\alpha)}(N,\alpha) \ne c'_i(\alpha))\}\big) + \lambda\big(\{\alpha \in B_b \mid q_{m_N(\alpha)}(\alpha) \in V_b\}\big)\\
&\le \lambda(A_b)\left(\frac{1}{\bar{q}_K(b)} + \sum_{\nu\in V_b}\left(\frac{\bar{q}_M^2(b)}{\nu^2} + \frac{1}{\nu}\right)\right)\\
&= O\left(\lambda(A_b)\left(\left(\frac{1+\sqrt{5}}{2}\right)^{-K} + \sum_{\nu\in V_b}\frac{1}{\nu}\right)\right)\\
&= O\left(\lambda(A_b)\left(\left(\frac{1+\sqrt{5}}{2}\right)^{-K} + \frac{w+1}{N}\frac{N}{w\kappa^2}\right)\right).
\end{aligned}$$

This implies the assertion of the Lemma.

LEMMA 3.5. *There is a positive number c with the following property: if* $N > 2$ *is an integer,* $\varepsilon > 0, 0 < \mu < \frac{\tau}{8}, M = \left[\left(\frac{\tau}{2} - \mu\right) \log N\right], K = [\mu \log N]$ *and* $\nu : \Omega \Rightarrow \mathbb{Z}_+,$ *is given by* $\nu(\alpha) = \max\{m_N(\alpha) - M, 0\}$, *then*

$$\lambda\left(\left\{\alpha \in \Omega \Big| \Big| \sum_{s=\nu(\alpha)+K}^{m_N(\alpha)-1} a_{s+1}(\alpha) B_2\left(\frac{c_s(N,\alpha)}{a_{s+1}(\alpha)}\right)\Big| \geq \sum_{s=\nu(\alpha)+K}^{m_N(\alpha)-1} a_{s+1}(\alpha)\right\}\right)$$

$$\leq c\left(\left(\frac{1+\sqrt{5}}{2}\right)^{-K/2} + (\varepsilon \log\log N)^{-1/3} + \mu^{-1/2}(\log N)^{-1/4} e^{-\mu^2 \log N/(64\sigma^2\tau^3)}\right).$$

Proof. Let $a := \sqrt{\frac{e \log 2}{2}}$. If

$$\frac{1}{2\log 2}\left(\log \frac{M-K}{2} - \gamma\right) \leq \frac{1}{\varepsilon} + \frac{2}{\log 2} \log\left(a \log\left(M-K\right)\right),$$

then $\varepsilon \log\log N = O(1)$ and the assertion is trivial. Therefore we may assume the contrary.

Let A be the set occurring in the Lemma and let $\kappa = (\varepsilon \log M)^{1/3}$. There is an $R > 0$ such that

$$\frac{1}{\varepsilon} + \frac{2\kappa\sqrt{R}}{\varepsilon(M-K)} - \frac{1}{\log 2}\left(\log \frac{M-K}{2} - \gamma\right) = -\frac{2}{\log 2} \log\left(a \log\left(M-K\right)\right).$$

We have $\frac{2\kappa\sqrt{R}}{\varepsilon(M-K)} \geq \frac{1}{2\log 2} \log \frac{M-K}{2}$ and therefore $\frac{M-K}{\kappa\sqrt{R}} + O\left(\frac{1}{\varepsilon \log(M-K)}\right)$, which results in $\frac{\log N}{\sqrt{R}} = O\left(\frac{\kappa}{\varepsilon \log M}\right)$. Let w be an integer with $2 \leq w \leq \log\log N$ and let $A'_w := \Big\{\alpha \in A \mid \frac{N}{w+1} < q_{m_N(\alpha)}(\alpha) \leq \frac{N}{w}$, the denominator of $\left[0; a_{\nu(\alpha)+1}(\alpha), \ldots, a_{m_N(\alpha)}(\alpha)\right]$ is

$$\text{less than } \sqrt{\frac{N}{w+1}}, \sum_{s=\nu(\alpha)+K}^{m_N(\alpha)-1} a_{s+1}^2(\alpha) < R, \sum_{s=\nu(\alpha)+K}^{m_N(\alpha)-1} a_{s+1}(\alpha) > \frac{M-K}{2\varepsilon} + \frac{\kappa\sqrt{R}}{\varepsilon}\Big\}.$$

For $b \in \mathbb{N}^M$, let $\bar{q}_M(b)$ be the denominator of $[0; b_1, \ldots, b_M]$. We put

$$\mathcal{B} := \left\{b \in \mathbb{N}^M \mid \bar{q}_M^2(b) \leq \frac{N}{w+1}, \sum_{s=K}^{M-1} b_{s+1}^2 \leq R, \sum_{s=K}^{M-1} b_{s+1} > \frac{M-K}{2\varepsilon} + \frac{\kappa\sqrt{R}}{\varepsilon}\right\}.$$

Then

$$A'_w \subseteq \bigcup_{b \in \mathcal{B}} \left\{\alpha \in \Omega \mid 1 \leq i \leq M \Rightarrow a_{\nu(\alpha)+i}(\alpha) = b_i, \frac{N}{w+1} < q_{m_N(\alpha)}(\alpha) \leq \frac{N}{w},\right.$$

$$\left.\Big| \sum_{s=\nu(\alpha)+K}^{m_N(\alpha)-1} a_{s+1}(\alpha) B_2\left(\frac{c_s(N,\alpha)}{a_{s+1}(\alpha)}\right)\Big| \geq \varepsilon \sum_{s=\nu(\alpha)+K}^{m_N(\alpha)-1} a_{s+1}(\alpha)\right\}$$

and therefore Lemma 3.4 implies $\lambda(A'_w) = O\left(\left(\frac{1+\sqrt{5}}{2}\right)^{-K} + \frac{1}{\kappa^2}\right)$.

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Let $A_w := \left\{\alpha \in A \mid \frac{N}{w+1} < q_{m_N(\alpha)}(\alpha) \le \frac{N}{w}\right\}$. From Lemma 1.2, Lemma 1.5 and Lemma 1.3 we get, replacing in the last two Lemmas μ by 2μ, $\lambda(A_w) \le \lambda(A'_w) + O\left(\frac{1}{\mu\sqrt{\log N}} e^{-\mu^2 \log N/(32\sigma^2\tau^3)} + \frac{\log\log N}{\sqrt{\log N}} + \frac{\log N}{\sqrt{R}} + G\left(\frac{1}{\varepsilon} + \frac{2\kappa\sqrt{R}}{\varepsilon(M-K)} - \frac{\log\frac{M-K}{2} - \gamma}{\log 2}\right)\right)$. The last summand is equal to $G\left(-2\log(a\log(M-K))\frac{1}{\log 2}\right) = O\left(1 - \Phi(\log(M-K)) = O\left(\frac{1}{\log M}\right)\right.$.

Therefore

$$\lambda(A_w) = O\left(\left(\frac{1+\sqrt{5}}{2}\right)^{-K} + \frac{1}{(\varepsilon\log M)^{2/3}} + \frac{1}{\mu\sqrt{\log N}} e^{-\mu^2\log N/(32\sigma^2\tau^3)}\right).$$

For any $k \in \mathbb{N}$,

$$\sum_{n=0}^{\infty} \lambda(\{\alpha \in \Omega | q_n(\alpha) = k\}) \le \lambda\left(\left\{\alpha \in \Omega | \exists j, 0 \le j \le k, |\alpha - \frac{j}{k}| < \frac{1}{k^2}\right\}\right) \le \frac{2}{k}$$

and therefore

$$\lambda\left(\left\{\alpha \in \Omega | q_{m_N(\alpha)}(\alpha) \le \frac{N}{w}\right\}\right) \le \sum_{1\le k\le N/w} \sum_{n=0}^{\infty} \lambda(\{\alpha \in \Omega | q_n(\alpha) = k, q_{n+1}(\alpha) > N\})$$

$$\le \sum_{1\le k\le N/w} \sum_{n=0}^{\infty} \lambda\left(\left\{\alpha \in \Omega \mid q_n(\alpha) = k, a_{n+1}(\alpha) \ge \frac{N}{w} - 1\right\}\right)$$

$$\le 2 \sum_{1\le k\le N/w} \sum_{n=0}^{\infty} \lambda(\{\alpha \in \Omega \mid q_n(\alpha) = k\}) \lambda\left(\left\{\alpha \in \Omega \mid a_1(\alpha) \ge \frac{N}{k} - 1\right\}\right)$$

$$\le 4 \sum_{1\le k\le N/w} \frac{k}{N-k}\frac{1}{k} \le \frac{4}{w-1}.$$

We get, for all w_0 which satisfy $2 \le w_0 \le \log\log N$,

$$\lambda(A) \le \sum_{1\le w<w_0} \lambda(A_w) + \frac{4}{w_0 - 1}$$

$$= O\left(w_0\left(\frac{1+\sqrt{5}}{2}\right)^{-K} + w_0(\varepsilon\log\log N)^{-2/3} + \frac{w_0}{\mu\sqrt{\log N}} e^{-\mu^2\log N/(32\sigma^2\tau^3)} + \frac{1}{w_0}\right).$$

Putting

$$w_0 := \left[\min\left\{\left(\frac{1+\sqrt{5}}{2}\right)^{K/2}, (\varepsilon\log\log N)^{1/3}, \mu^{1/2}(\log N)^{1/4} e^{\mu^2\log N.(64\sigma^2\tau^3)}\right\}\right]$$

we get the desired result.

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LEMMA 3.6. *There is a constant* $c > 0$ *with the following property: if* $\varepsilon > 0, 0 < \mu < \frac{\varepsilon\tau}{4}$, $\mu < \frac{\tau}{2}$, $N \geq 16$ *is an integer*, $M = \left[\left(\frac{\tau}{2} - \mu\right) \log N\right], K = [\mu \log N]$ *and* $\nu : \Omega \to \mathbb{Z}_+$, $\nu(\alpha) = \max\{m_N(\alpha) - M, 0\}$, *then*

$$\lambda\left(\left\{\alpha \in \Omega \,\Big|\, \Big|\sum_{s=M}^{\nu(\alpha)+K-1} a_{s+1}(\alpha) B_2\left(\frac{c_s(N,\alpha)}{a_{s+1}(\alpha)}\right)\Big| \geq \varepsilon \sum_{s=0}^{m_N(\alpha)} a_{s+1}(\alpha)\right\}\right) \leq \frac{c}{\varepsilon \log\log N}.$$

Proof. The assertion is trivial if $\varepsilon \log\log N \leq 1$. We denote by A the set occurring in the Lemma. Put

$$B := \left\{\alpha \in \Omega \mid \frac{1}{6}\sum_{s=M}^{\nu(\alpha)+K-1} a_{s+1}(\alpha) \geq \varepsilon \sum_{s=0}^{m_N(\alpha)} a_{s+1}(\alpha)\right\}$$

$$C := \{\alpha \in \Omega \,|\, |m_N(\alpha) - \tau \log N| \geq \varepsilon \log N\},$$

$$D := \left\{\alpha \in \Omega \mid \sum_{s=0}^{m_N(\alpha)} a_{s+1}(\alpha) \leq \frac{\tau - \varepsilon}{\log 2} \log N \log(\tau \log N)\right\}.$$

Corollary 1.1 implies $\lambda(C) = O\left(\left(\frac{1}{\varepsilon} + \log\log N\right)\frac{1}{\sqrt{\log N}}\right) = O\left(\frac{1}{\varepsilon \log\log N}\right)$. Proposition 1.4 implies

$$\begin{aligned}
\lambda(D) &= G\left(\frac{\tau-\varepsilon}{\tau \log 2}\log(\tau\log N) - \frac{\log(\tau \log N) - \gamma}{\log 2}\right) + O\left(\frac{(\log\log N)^{3/2}}{\sqrt{\log N}}\right)\\
&= O\left(G\left(\frac{-\varepsilon}{\tau \log 2}\log(\tau \log N) + \frac{\gamma}{\log 2}\right) + \frac{(\log\log N)^{3/2}}{\sqrt{\log N}}\right)\\
&= O\left(1 - \Phi\left(\sqrt{\frac{2}{\varepsilon \log 2}} e^{-\gamma/2} e^{\varepsilon \log(\tau \log N)/(2\tau)} + \frac{(\log\log N)^{3/2}}{\sqrt{\log N}}\right)\right)\\
&= O\left(e^{-\varepsilon \log(\tau \log N)/(2\tau)} + \frac{(\log\log N)^{3/2}}{\sqrt{\log N}}\right)\\
&= O\left(\frac{1}{\varepsilon \log\log N}\right).
\end{aligned}$$

For N large enough we get

$$\begin{aligned}
\frac{\varepsilon\tau}{\varepsilon+\mu}\log(\tau\log N) - \log(3(\varepsilon+\mu)\log N) + \gamma &\geq \frac{\varepsilon\tau}{\varepsilon+\mu}\log(\tau\log N) - \log(\tau\log N)\\
&\geq \frac{4\tau}{4+\tau}\log(\tau\log N) - \log(\tau\log N)\\
&\geq \log(\tau\log N).
\end{aligned}$$

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Assume now that $\alpha \in B\backslash(C \cup D)$. Then on the one hand

$$\sum_{s=M}^{\nu(\alpha)+K-1} a_{s+1}(\alpha) \geq 6\varepsilon \frac{\tau-\varepsilon}{\log 2} \log N \log(\tau \log N) \geq \frac{3\varepsilon\tau}{\log 2} \log N \log(\tau \log N)$$

and on the other hand

$$\nu(\alpha)+K-M \leq (\tau+\varepsilon)\log N + K - 2M \leq (\varepsilon+3\mu)\log N + 2 \leq 3(\varepsilon+\mu)\log N.$$

Proposition 1.3 implies that

$$\lambda(B\backslash(C\cup D)) = O\left(\lambda\left(\left\{\alpha \in \Omega \mid \sum_{1\leq s\leq 3(\varepsilon+\mu)\log N} a_s(\alpha) \geq \frac{3\varepsilon\tau}{\log 2}\log N \log(\tau\log N)\right\}\right)\right)$$

$$= O\left(1 - G\left(\frac{\varepsilon\tau}{(\varepsilon+\mu)\log 2}\log(\tau\log N) - \frac{1}{\log 2}(\log(3(\varepsilon+\mu)\log N) - \gamma)\right) + \frac{\log^2 \log N}{\varepsilon \log N}\right)$$

$$= O\left(1 - G\left(\frac{\log(\tau\log N)}{\log 2}\right) + \frac{\log^2\log N}{\varepsilon\log N}\right) = O\left(\frac{1}{\varepsilon\log\log N}\right).$$

$A \subseteq B$ implies the assertion.

We are now able to prove the main Lemma.

MAIN LEMMA. *There is a constant $c > 0$ such that for all $\varepsilon > 0$ and all integers $N \geq 2$*

$$\lambda\left(\left\{\alpha\in\Omega \| \sum_{s=0}^{m_N(\alpha)} a_{s+1}(\alpha)B_2\left(\frac{c_s(N,\alpha)}{a_{s+1}(\alpha)}\right)| \geq \varepsilon \sum_{s=0}^{m_N(\alpha)} a_{s+1}(\alpha)\right\}\right) \leq \frac{c}{(\varepsilon\log\log N)^{1/3}}.$$

Proof. Let $\mu := \frac{2\varepsilon}{\sqrt{\log N}}\log\log N$, $M := \left[\left(\frac{\tau}{2}-\mu\right)\log N\right]$, $K := [\mu\log N]$, $\nu : \Omega \to \mathbb{Z}_+$, $\nu(\alpha) = \max\{m_N(\alpha) - M, 0\}$ and let A be the set occurring in the Lemma. Furthermore let

$$A_1 := \left\{\alpha\in\Omega \| \sum_{s=0}^{M-1} a_{s+1}(\alpha)B_2\left(\frac{c_s(N,\alpha)}{a_{s+1}(\alpha)}\right)| \geq \frac{\varepsilon}{4}\sum_{s=0}^{m_N(\alpha)} a_{s+1}(\alpha)\right\},$$

$$A_2 := \left\{\alpha\in\Omega \| \sum_{s=M}^{\nu(\alpha)+K-1} a_{s+1}(\alpha)B_2\left(\frac{c_s(N,\alpha)}{a_{s+1}(\alpha)}\right)| \geq \frac{\varepsilon}{4}\sum_{s=0}^{m_N}(\alpha)a_{s+1}(\alpha)\right\},$$

$$A_3 := \left\{\alpha\in\Omega \| \sum_{s=\nu(\alpha)+K}^{m_N(\alpha)-1} a_{s+1}(\alpha)B_2\left(\frac{c_s(N,\alpha)}{a_{s+1}(\alpha)}\right)| \geq \frac{\varepsilon}{4}\sum_{s=0}^{m_N(\alpha)} a_{s+1}(\alpha)\right\},$$

$$A_4 := \left\{\alpha\in\Omega | a_{m_N(\alpha)+1}(\alpha)|B_2\left(\frac{c_{m_N(\alpha)}(N,\alpha)}{a_{m_N(\alpha)+1}(\alpha)}\right)| \geq \frac{\varepsilon}{4}\sum_{s=0}^{m_N(\alpha)} a_{s+1}(\alpha)\right\}.$$

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First of all we estimate $\lambda(A_4)$ from above. Let

$$B := \{\alpha \in \Omega \mid |m_N(\alpha) - \tau \log N| \geq \mu \log N\},$$

$$C := \left\{\alpha \in \Omega \mid a_{m_N(\alpha)+1}(\alpha) \geq \frac{\varepsilon}{4} m_N(\alpha)\right\}.$$

Note that $A_4 \subseteq C$ and that

$$\lambda(B) = O\left(\frac{1}{\mu\sqrt{\log N}}\right) = O\left(\frac{1}{\varepsilon \log\log N}\right).$$

Now $\alpha \in C \backslash B$ implies $a_{m_N(\alpha)+1}(\alpha) \geq \frac{\varepsilon(\tau-\mu)}{4} \log N$. Therefore

$$\lambda(C\backslash B) = O\left(\sum_{|k-\tau\log N|<\mu\log N} \lambda\left(\left\{\alpha \in \Omega | a_{k+1}(\alpha) \geq \frac{\varepsilon(\tau-\mu)}{4}\log N\right\}\right)\right) = O\left(\frac{\mu}{\varepsilon}\right) = O\left(\frac{\log\log N}{\sqrt{\log N}}\right).$$

This results in $\lambda(A_4) = O\left(\frac{1}{\varepsilon \log\log N}\right)$.

Since $A \subseteq \bigcup_{i=1}^{4} A_i$, we get from Proposition 3.1, Lemma 3.6 and Lemma 3.5

$$\lambda(A) = O\left(\frac{1}{\varepsilon\log\log N} + \frac{1}{(\varepsilon \log\log N)^{2/3}}\right) + O\left(\left(\frac{1+\sqrt{5}}{2}\right)^{-K/2} + \frac{1}{(\varepsilon\log\log N)^{1/3}} + \frac{1}{(\varepsilon \log\log N)^{1/2}}\right).$$

Hence $\frac{K}{2} \geq \varepsilon\sqrt{\log N}\log\log N - 1 \geq \frac{\varepsilon}{2}\sqrt{\log N}$ immediately implies the assertion.

4. The proof of the main theorem. Let $\alpha \in \Omega$ with continued fraction expansion $[0; a_1, a_2, \ldots]$ and convergents $\frac{p_n}{q_n}$. Let us put

$$\omega_N^+(\alpha) = \sup_{0\leq x\leq 1}\left(\sum_{n=1}^{N} c_{[0,x)}(\{n\alpha\}) - Nx\right),$$

$$\omega_N^-(\alpha) = \sup_{0\leq x\leq 1}\left(Nx - \sum_{n=1}^{N} c_{[0,x)}(\{n\alpha\})\right).$$

Denoting by m the integer $m_N(\alpha)$, we obtain the following result.

PROPOSITION 4.1. (See [7].)

$$\omega_N^+(\alpha) = \sum_{2|j\leq m}\left(a_{j+1}\{q_jN\alpha\}(1-\{q_jN\alpha\}) + \{q_jN\alpha\}(\{q_{j+1}\alpha N\} - \{q_{j-1}N\alpha\})\right) + O(1),$$

$$\omega_N^-(\alpha) = \sum_{2\nmid j\leq m}\left(a_{j+1}\{q_jN\alpha\}(1-\{q_jN\alpha\}) + \{q_jN\alpha\}(\{q_{j+1}\alpha N\} - \{q_{j-1}N\alpha\})\right) + O(1).$$

The O-constant is absolute.

Proof. This is essentially Corollary 2 in §1 of [7].

COROLLARY 4.1. $D_N(\alpha) = \sum_{j=0}^{m} a_{j+1}\{q_jN\alpha\}(1 - \{q_jN\alpha\}) + O(1)$. *The O-constant is absolute.*

Proof. This follows from $D_N(\alpha) = \omega_N^+(\alpha) + \omega_N^-(\alpha)$.

COROLLARY 4.2. $D_N(\alpha) = \sum_{j=0}^{m_N(\alpha)} c_j(N,\alpha)\left(1 - \frac{c_j(N,\alpha)}{a_{j+1}(\alpha)}\right) + O(\log N)$. *The O-constant is absolute.*

Proof. We put, for $i, j \geq 0$,

$$s_{i,j} := q_{\min(i,j)}(\alpha q_{\max(i,j)} - p_{\max(i,j)}),$$

$A_j := \sum_{i=0}^{m} c_i(N,\alpha)s_{i,j}$ and $P := \{j \mid 0 \leq j \leq m,\ A_j > 0\}$. Then $\{q_jN\alpha\} = A_j + 1 - c_P(j)$. (Use the proof of Corollary 3 of §1 in [7] and note the slightly different notion of A_j there.) Furthermore it is easily seen that

$$A_j = c_j(N,\alpha)s_{j,j} + O\left(\frac{1}{a_{j+1}}\right) = \frac{(-1)^j c_j(N,\alpha)}{a_{+1}} + O\left(\frac{1}{a_{j+1}}\right)$$

and that $2c_P(j) - 1 \neq (-1)^j$ implies $c_j(N,\alpha) = 0$. Using $m_N(\alpha) = O(\log N)$ we get the assertion.

COROLLARY 4.3. *There is a constant $c > 0$ such that, for all integers $N > 2$ and all $\varepsilon > 0$,*

$$\lambda\left(\left\{\alpha \in \Omega \mid |D_N(\alpha) - \frac{1}{6}\sum_{k=0}^{m_N(\alpha)} a_{k+1}(\alpha)| \geq \varepsilon \sum_{k=0}^{m_N(\alpha)} a_{k+1}(\alpha)\right\}\right) \leq \frac{c}{(\varepsilon \log\log N)^{1/3}}.$$

Proof. Let A be the set occurring in the Corollary. Corollary 4.2 implies the existence of a constant $K > 0$ such that, for all $N > 2$ and all $\alpha \in \Omega$,

$$|D_N(\alpha) + \sum_{k=0}^{m_N(\alpha)} a_{k+1}(\alpha)B_2\left(\frac{c_k(N,\alpha)}{a_{k+1}(\alpha)}\right) - \frac{1}{6}\sum_{k=0}^{m_N(\alpha)} a_{K+1}(\alpha)| \leq K \log N.$$

Let

$$B := \left\{\alpha \in \Omega \mid |\sum_{k=0}^{m_N(\alpha)} a_{k+1}(\alpha)B_2\left(\frac{c_k(N,\alpha)}{a_{k+1}(\alpha)}\right)| \geq \frac{\varepsilon}{2}\sum_{k=0}^{m_N(\alpha)} a_{k+1}(\alpha)\right\},$$

$$C := \left\{\alpha \in \Omega \mid \frac{2K}{\varepsilon}\log N \geq \sum_{k=0}^{m_N(\alpha)} a_{k+1}(\alpha)\right\}.$$

We have $A \subseteq B \cup C$. If $\frac{4K}{\varepsilon\tau} \geq \frac{\log(\tau \log N) - \gamma}{\log 2}$, the assertion is trivial. Otherwise

$$\frac{2K}{\varepsilon\tau} - \frac{\log(\tau\log N) - \gamma}{\log 2} \leq -\frac{\log(\tau\log N) - \gamma}{2\log 2}$$

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and therefore

$$\lambda(C) = O\left(G\left(\frac{2K}{\varepsilon\tau} - \frac{\log(\tau \log N) - \gamma}{\log 2}\right) + \frac{(\log\log N)^{3/2}}{\sqrt{\log N}}\right)$$

$$= O\left(G\left(-\frac{\log(\tau \log N) - \gamma}{2 \log 2}\right) + \frac{(\log\log N)^{3/2}}{\sqrt{\log N}}\right)$$

$$= O\left(\frac{1}{\log/\log N}\right).$$

From the Main Lemma in §3 we get the result.

THEOREM. *There is a constant $c > 0$ such that, for all $K > 0$ and every integer $N > 2$,*

$$\lambda\left(\left\{\alpha \in \Omega \,|\, |D_N(\alpha) - \frac{2}{\pi^2} \log N \log\log N \,| \geq K \log N\right\}\right) \leq \frac{c}{K^{1/3}}.$$

Proof. Let ε be positive,

$$A := \left\{\alpha \in \Omega \mid D_N(\alpha) \geq \left(\frac{2}{\pi^2} + \varepsilon\right) \log N \log\log N\right\},$$

$$g_N(\alpha) = \sum_{k=0}^{m_N(\alpha)} a_{k+1}(\alpha), \quad \eta = \frac{\varepsilon \log 2}{1 + \varepsilon \log 2}\left(\frac{1}{\tau} - \frac{1}{6}\right),$$

$$B := \left\{\alpha \in \Omega \,|\, |D_N(\alpha) - \frac{1}{6} g_N(\alpha)| \geq \eta g_N(\alpha)\right\}.$$

Corollary 4.3 implies

$$\lambda(B) = O\left(\frac{1}{(\eta \log\log N)^{1/3}}\right) = O\left(\frac{1}{(\varepsilon \log\log N)^{1/3}}\right).$$

Furthermore $\alpha \in A \backslash B$ implies that $D_N(\alpha) \geq \left(\frac{\tau}{6 \log 2} + \varepsilon\right) \log N \log\log N$ and we have $g_N(\alpha) \geq 6 D_N(\alpha) - 6\eta g_N(\alpha)$. This results in

$$g_N(\alpha) \geq \frac{6}{1 + 6\eta} D_N(\alpha) \geq \frac{\frac{\tau}{\log 2} + 6\varepsilon}{1 + 6\eta} \log N \log\log N = \frac{\tau}{\log 2}(1 + \varepsilon \log 2) \log N \log\log N.$$

Now Proposition 1.6 implies

$$\lambda(A \backslash B) = 1 - G\left(\frac{(1+\varepsilon)\log\log N}{\log 2} - \frac{\log(\tau \log N) - \gamma}{\log 2}\right) + O\left(\frac{(\log\log N)^{3/2}}{\sqrt{\log N}}\right)$$

$$= 1 - G\left(\varepsilon \log\log N - \frac{\log \tau - \gamma}{\log 2}\right) + O\left(\frac{(\log\log N)^{3/2}}{\sqrt{\log N}}\right)$$

$$= O\left(\frac{1}{\varepsilon \log\log N} + \frac{(\log\log N)^{3/2}}{\sqrt{\log N}}\right).$$

Choosing $\varepsilon = \frac{K}{\log\log N}$, we get $\lambda(A) = O(K^{-1/3})$.

Similarly

$$\lambda\left(\left\{\alpha \in \Omega \mid D_N(\alpha) \leq \left(\frac{2}{\pi^2} - \varepsilon\right) \log N \log\log N\right\}\right) = O(K^{-1/3}).$$

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