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AN EXTENSION OF THE BANACH–STONE THEOREM

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Abstract

We establish an extension of the Banach–Stone theorem to a class of isomorphisms more general than isometries in a noncompact framework. Some applications are given. In particular, we give a canonical representation of some (not necessarily linear) operators between products of function spaces. Our results are established for an abstract class of function spaces included in the space of all continuous and bounded functions defined on a complete metric space.

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1. Introduction

Let K be a compact Hausdorff topological space and $C(K)$ be the Banach space of all continuous real functions on K endowed with the sup-norm

$$\|\varphi\|_\infty := \sup_{k\in K} |\varphi(k)|.$$

The problem in what is known as the classical Banach–Stone theorem traces its origin back to the book *Théorie des Opérations Linéaires* (1932) of Banach [9]. In this book, Banach considered the problem of when two spaces of type $C(K)$ are isometric. He solved this problem for the case of compact metric spaces K, also giving a description of such isometries. In 1937, Stone [26] extended this result to general compact spaces K.

THEOREM 1.1 (Banach (1932); Stone (1937)). *Let K and L be compact spaces. Then $C(K)$ is isometrically isomorphic to $C(L)$ if and only if K and L are homeomorphic. More precisely, let $T : C(K) \to C(L)$ be an isomorphism. Then* (1) $\Leftrightarrow$ (2).

(1) *The isomorphism T is isometric for the norm $\|\cdot\|_\infty$.*
(2) *There exist a homeomorphism $\pi : L \to K$ and a continuous function $\epsilon : L \to \{\pm 1\}$ such that, for all $k \in K$ and for all $\varphi \in C(K)$, $T\varphi(k) = \epsilon(k)\varphi \circ \pi(k)$.*

The Banach–Stone theorem was investigated by several authors in diverse directions making way to several advances and publications. The list of contributions is long; we go back to the article of Garrido and Jaramillo [19] for a history of contributions and a complete list of references.

Throughout this paper, the space X is assumed to be a complete metric space. The space $(C_b(X), \|\cdot\|_\infty)$ denotes the Banach space of all bounded continuous real-valued functions on X. The following property and axioms, which we shall use in our hypotheses, are further discussed in Section 2. These axioms are verified for classical and known function spaces (see Proposition 2.5) like the space $C_b(X)$, the space $C_b^u(X)$ of all bounded and uniformly continuous functions or the space $\mathrm{Lip}_b^\alpha(X)$, $0 < \alpha \leq 1$, of all α-Holder and bounded functions. All these spaces are endowed with their natural norms of Banach spaces.

The property P^β. Let (X, d) be a complete metric space and $(A, \|\cdot\|)$ be a Banach space included in $C_b(X)$. We say that the space A has the property P^F (respectively, P^G) if, for every sequence $(x_n)_n \subset X$, the following assertions are equivalent:

(i) the sequence $(x_n)_n$ converges in (X, d);
(ii) the associated sequence of the Dirac masses $(\delta_{x_n})_n$ converges in $(A^*, \|\cdot\|^*)$ (respectively, in (A^*, w^*)), where $\|\cdot\|^*$ denotes the dual norm and w^* the weak-star topology.

By the property P^β, we mean P^F if $\beta = F$ or P^G if $\beta = G$.

Axioms. Let (X, d) be a complete metric space and A be a class of function spaces included in $C_b(X)$. We say that the space A satisfies the axioms (A_1)–(A_4^β) if the space A satisfies the following axioms:

(A_1) the space $(A, \|\cdot\|)$ is a Banach space such that $\|\cdot\| \geq \|\cdot\|_\infty$;
(A_2) the space A contain the constants;
(A_3) for each $n \in \mathbb{N}$, the set of all natural numbers, there exists a positive constant M_n such that for each $x \in X$ there exists a function $h_n : X \to [0, 1]$ such that $h_n \in A$, $\|h_n\| \leq M_n$, $h_n(x) = 1$ and $\mathrm{diam}(\mathrm{supp}(h_n)) < 1/(n + 1)$. This axiom implies in particular that the space A separates the points of X;
(A_4^β) the space A has the property P^β (with $\beta = F$ or $\beta = G$).

Let $f : X \to \mathbb{R} \cup \{+\infty\}$ be a lower semicontinuous and bounded from below function with nonempty domain, that is,

$$\mathrm{dom}(f) := \{x \in X : f(x) < +\infty\} \neq \emptyset.$$

We denote by $\overline{\mathrm{dom}(f)}$ the closure of $\mathrm{dom}(f)$ in X.

The main result of this paper (Theorem 1.2 below) extends the classical Banach–Stone theorem (in the case of a complete metric space X) in the following directions:

(i) the Banach–Stone theorem remains true for a class of isomorphisms more general than isometries (see more details in Sections 4.1 and 4.2);

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(ii) the theorem is true for an abstract class of function spaces which includes classical spaces;

(iii) the theorem remains true for complete metric spaces X that are not necessarily compact.

THEOREM 1.2. *Let X and Y be two complete metric spaces and $A \subset C_b(X)$ and $B \subset C_b(Y)$ be two Banach spaces satisfying the axioms (A_1)–(A_4^β) with the same β. Let $T : A \to B$ be an isomorphism and let $f : X \to \mathbb{R} \cup \{+\infty\}$ and $g : Y \to \mathbb{R} \cup \{+\infty\}$ be lower semicontinuous and bounded from below functions with nonempty domains. Then* (1) $\Leftrightarrow$ (2).

(1) *For all $\varphi \in A$,*

$$\sup_{y\in Y}\{|T\varphi(y)| - g(y)\} = \sup_{x\in X}\{|\varphi(x)| - f(x)\}.$$

(2) *There exist a homeomorphism $\pi : \overline{\mathrm{dom}(g)} \to \overline{\mathrm{dom}(f)}$ and a continuous function $\varepsilon : \overline{\mathrm{dom}(g)} \to \{\pm 1\}$ such that, for all $y \in \overline{\mathrm{dom}(g)}$ and all $\varphi \in A$,*

$$T\varphi(y) = \varepsilon(y)\varphi \circ \pi(y)$$

and

$$g(y) = f \circ \pi(y).$$

We obtain immediately (Corollary 1.3) the representation of the isometries for the sup-norm $\|\cdot\|_\infty$ when we take $f \equiv 0$ on X and $g \equiv 0$ on Y, but the general case also has some interest, as we will detail in Section 4.

COROLLARY 1.3. *Let X and Y be two complete metric spaces. Then the space $C_b(X)$ is isometrically isomorphic to $C_b(Y)$ if and only if X and Y are homeomorphic. More generally, let $A \subset C_b(X)$ and $B \subset C_b(Y)$ be two Banach spaces satisfying the axioms (A_1)–(A_4^β) with the same β. Let $T : A \to B$ be an isomorphism. Then* (1) $\Leftrightarrow$ (2).

(1) *The isomorphism T is isometric for the norm $\|\cdot\|_\infty$.*

(2) *There exist a homeomorphism $\pi : Y \to X$ and a continuous function $\varepsilon : Y \to \{\pm 1\}$ such that, for all $y \in Y$ and all $\varphi \in A$, we have $T\varphi(y) = \varepsilon(y)\varphi \circ \pi(y)$.*

Let us mention here that one of the consequences of Theorem 1.2 is the study of operators between product spaces $A \times A' \subset C_b(X) \times C_b(X, Z)$ and $B \times B' \subset C_b(Y) \times C_b(Y, W)$ for the norm $\|\cdot\|_{\infty,1}$ (where $C_b(X, Z)$ denotes the space of all bounded and continuous functions from X into a Banach space Z). The norm $\|\cdot\|_{\infty,1}$ is defined on $C_b(X) \times C_b(X, Z)$ by

$$\|(\varphi, \psi)\|_{\infty,1} := \sup_{x\in X}\{|\varphi(x)| + \|\psi(x)\|_Z\}.$$

A nontrivial class of operators $H : A \times A' \to B \times B'$, not necessarily linear, which preserve the norm $\|\cdot\|_{\infty,1}$, will be characterized by a canonical form in Theorem 4.9 and these corollaries (see Section 4.4).

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The proof of Theorem 1.2 will be given in Section 3. It is based on the differentiability of some convex functions generalizing the norm $\|\cdot\|_\infty$ and a duality result introduced in [6] together with the Deville–Godefroy–Zizler variational principle (see [14, 16]). Note that the original proof of the Banach–Stone theorem in the compact metric case, given by Banach in [9], is based on the Gâteaux differentiability of the norm $\|\cdot\|_\infty$.

This paper is organized as follow. In Section 2 we introduce the axioms which we shall use in this article and we give examples satisfying them. We also give some preliminary results which will permit us to give the proof of our main theorem. In Section 3 we give the proof of the main result (Theorem 1.2). In Section 4 we give various applications of Theorem 1.2.

2. Preliminary results

To prove our main result, we need to introduce some notions and to establish certain lemmas.

2.1. The Dirac masses and the property P^β. Let (X, d) be a complete metric space and $(A, \|\cdot\|)$ be a Banach space included in $C_b(X)$. By A^*, we denote the topological dual of A. By δ, we denote the Dirac map and by δ_x the Dirac mass associated to the point $x \in X$:

$$\begin{aligned}\delta : X &\to A^*\\ x &\mapsto [\delta_x : \varphi \mapsto \varphi(x)].\end{aligned}$$

Suppose that X and Y are complete metric spaces, $A \subset C_b(X)$ and $B \subset C_b(Y)$ are Banach spaces, $T : A \to B$ is an isomorphism and T^* its adjoint. With the aim of proving the Banach–Stone theorem, it is classical to look for a way to correspond the set of the Dirac masses $\delta(X)$ to the set $\delta(Y)$ via a homeomorphism h:

$$\begin{tikzcd} \delta(Y) \arrow[r, "h^{-1}"] \arrow[d, hook, "i"] & \delta(X) \arrow[d, hook, "i"] \\ B^* \arrow[r, "T^*"'] & A^* \end{tikzcd}$$

In the compact framework (that is, when $X = K$ and $Y = L$ are compact) and when T is isometric, the classical idea to correspond $\delta(X)$ to $\delta(Y)$ consists in the fact that the set of all extreme points of the dual unit ball of $C(K)^*$ is exactly the set $\pm\delta(K) := \{\pm\delta_k : k \in K\}$ (that is, the Arens–Kelley theorem (1947); see [19, Theorem 4]) and the fact that an isometry (here the isometry is T^*) sends necessarily extreme points to extreme points. Unfortunately, the Arens–Kelley theorem is not true if X is not compact or if A is an abstract class of functions. Indeed, on one hand, the set of extreme points of the unit ball of the dual space $(C_b(X))^*$ is $\pm\delta(\beta X)$ (where βX denotes the Stone–Cech compactification of X), which contains strictly the set $\pm\delta(X)$ when X is not compact. On the other hand, we do not know explicitly the extreme points of the dual unit ball of

A^* in the abstract cases. So, our purpose in this paper is to characterize the set $\pm\delta(X)$ in another way. We shall use the fact that the Dirac masses δ_x for $x \in X$ are in general the derivative of some conjugate function $f^\times$ (of a lower semicontinuous function f) defined on the space A. This class of conjugate functions generalizes the norm $\|\cdot\|_\infty$ (see [6, Theorem 2.8] and Section 3).

Once the correspondence between $\delta(X)$ and $\delta(Y)$ via a homeomorphism h has been determined, we try then to establish a homeomorphism between X and Y. The classical scheme of the correspondence would be the following one:

$$X \xrightarrow{\delta} \delta(X) \xrightarrow{h} \delta(Y) \xrightarrow{\delta^{-1}} Y.$$

It is well known that the Dirac map δ gives a homeomorphism between a compact space K and its image $\delta(K) \subset B_{C(K)^*}$ (the dual unit ball of $C(K)^*$) when $(\delta(K), w^*)$ is endowed with the weak-star topology. If (X, d) is a metric space, it is also possible in certain spaces like the space $C_b(X)$ or the space $C_b^u(X)$ to obtain that δ is a sequential homeomorphism from (X, d) onto $(\delta(X), w^*)$. There also exist spaces such as $\mathrm{Lip}_b^\alpha(X)$, $0 < \alpha \leq 1$, for which the map δ is a homeomorphism from (X, d) onto $(\delta(X), \|\cdot\|^*)$ (where $\|\cdot\|^*$ denotes the dual norm of $\mathrm{Lip}_b^\alpha(X)$). Note that δ cannot be a homeomorphism from (X, d) onto $(\delta(X), \|\cdot\|_\infty^*)$ in the dual spaces $(C_b(X))^*$ or $(C_b^u(X))^*$, since we always have $\|\delta_x - \delta_{x'}\|_\infty^* = 2$ if $x \neq x'$. Thus, the map δ enjoys certain properties which are connected to the nature of the function spaces A in question. This motivates the Definition 2.1 already mentioned in the Introduction, as well as the equivalent proposition which follows it. Some examples are given in Proposition 2.5.

A bornology on a Banach space A, denoted by β, will be any nonempty family of bounded sets whose union is all of A. If β is a bornology on A and χ is a real-valued function on A, we say that χ is β-differentiable at $a \in A$ with β-derivative, $\chi'(a) = p \in A^*$, if

$$\lim_{t \longrightarrow 0^+} t^{-1}(\chi(a + th) - \chi(a) - \langle p, th \rangle) = 0$$

uniformly for h in the elements of β. We denote by τ_β the topology on A^* of uniform convergence on the elements of β. When β is the class of all bounded subsets (respectively, all singletons) of A, the β-differentiability coincides with the usual Fréchet differentiability (respectively, Gâteaux differentiability) and τ_β coincides with the norm (respectively, weak*) topology on A^*. By G, we denote the Gâteaux bornology consisting of all singletons and by F we denote the Fréchet bornology consisting of all bounded sets.

Definition 2.1. *(The property P^β)* Let (X, d) be a complete metric space and $(A, \|\cdot\|)$ be a Banach space included in $C_b(X)$. We say that A has the property P^F (respectively, P^G) if, for each sequence $(x_n)_n \subset X$, the following two assertions are equivalent:

(i) the sequence $(x_n)_n$ converges in (X, d);

(ii) the associated sequence of the Dirac masses $(\delta_{x_n})_n$ converges in $(A^*, \|\cdot\|^*)$ (respectively, in (A^*, w^*)), where $\|\cdot\|^*$ denotes the dual norm and w^* the weak-star topology.

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The crucial property P^β ($\beta = F$ or G) is related to the geometry of the Banach space A and is connected to the β-differentiability of the sup-norm $\|\cdot\|_\infty$; for more details, see [6]. In this paper we deal only with the Gâteaux bornology $\beta = G$ or the Fréchet bornology $\beta = F$. In this case, the above definition can be formulated easily as follows.

Proposition 2.2. *Let (X, d) be a complete metric space and $(A, \|\cdot\|)$ be a Banach space included in $C_b(X)$ which separates the points of X.*

(1) *The space A has the property P^F if and only if the map*

$$\begin{aligned} \delta : (X, d) &\to (\delta(X), \|\cdot\|^*) \\ x &\mapsto \delta_x \end{aligned}$$

is a homeomorphism.

(2) *The space A has the property P^G if and only if the map*

$$\begin{aligned} \delta : (X, d) &\to (\delta(X), w^*) \\ x &\mapsto \delta_x \end{aligned}$$

is a sequential homeomorphism.

Remark 2.3. The map $\delta : X \to A^*$ is a nonlinear analogue of the canonical embedding $i : Z \to Z^{**}$, where Z is a Banach space and Z^{**} its bidual. This map permits us to linearize the metric space X in A^*. For more information in this direction, we refer to the paper of Godefroy and Kalton [20] when A is the set of all Lipschitz maps on X that vanish at some fixed point.

2.2. Axioms and examples. We give now the general axioms that the space A has to satisfy in our results. These axioms are satisfied by various and classical spaces of functions. We give below some examples. Let us mention here that the axioms (A_1) and (A_3) are related to the variational principle of Deville *et al.* [14] and Deville and Revalski [16] (see Theorem 2.7 below) and the axiom (A_4^β) was introduced and studied in [6] and is a part of the hypothesis of [6, Theorem 2.8]. The theorems [6, Theorem 2.8] and Theorem 2.7 will be used in a crucial way in the proof of our main result (Theorem 1.2).

Axioms 2.4. Let (X, d) be a complete metric space and A be a class of function spaces included in $C_b(X)$. We say that the space A satisfies the axioms (A_1)–(A_4^β) if A satisfies the following axioms.

(A_1) The space $(A, \|\cdot\|)$ is a Banach space such that $\|\cdot\| \geq \|\cdot\|_\infty$.

(A_2) The space A contains the constants.

(A_3) For each $n \in \mathbb{N}^*$, there exists a positive constant M_n such that for each $x \in X$ there exists a function $h_n : X \to [0, 1]$ such that $h_n \in A$, $\|h_n\| \leq M_n$, $h_n(x) = 1$ and $\operatorname{diam}(\operatorname{supp}(h_n)) < 1/n$. This axiom implies in particular that the space A separates the points of X.

(A_4^β) The space A has the property P^β ($\beta = F$ or $\beta = G$).

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Recall that by the space $C_b^u(X)$ we denote the Banach space of all bounded uniformly continuous functions on X and by $\mathrm{Lip}_b^\alpha(X)$ the Banach space of all α-Hölder and bounded functions on X $(0<\alpha\le 1)$. When X is a Banach space, we denote by $C_b^k(X)$ the Banach space of all k-times continuously Fréchet differentiable functions f such that $f, f', \ldots, f^{(k)}$ are uniformly bounded and by $C_b^{1,\alpha}(X)$ $(0<\alpha\le 1)$ the Banach space of all Fréchet differentiable functions f on X such that f and f' are uniformly bounded on X and f' is α-Hölder. Finally, by $C_b^{1,u}(X)$ we denote the Banach space of all Fréchet differentiable functions f on X such that f and f' are uniformly bounded on X and f' is uniformly continuous. All these spaces are provided with their natural norm $\|\cdot\|$ of Banach spaces that satisfy $\|\cdot\|\ge\|\cdot\|_\infty$ (see [6] for more information and other examples).

Proposition 2.5. *The following assertions hold.*

(1) *For every complete metric space X, the spaces $C_b(X)$, $C_b^u(X)$ satisfy the axioms (A_1)–(A_4^G) and the space $\mathrm{Lip}_b^\alpha(X)$ $(0<\alpha\le 1)$ satisfies the axioms (A_1)–(A_4^F).*

(2) *If X is a Banach space having a bump function (that is, a function with nonempty and bounded support) in $A=C_b^k(X)$ with $k\in\mathbb{N}^*$ (respectively, in $A=C_b^{1,\alpha}(X)$ with $0<\alpha\le 1$ or $A=C_b^{1,u}(X)$), then A satisfies the axioms (A_1)–(A_4^F).*

Proof. The axiom (A_1) follows from the definitions of the spaces and their norms (see [6]). The axiom (A_2) is clear. The axiom (A_3) is easy and can be found in [16, Remark 2.5] (see also [16, Proposition 1.4]). The axiom (A_4^β) follows from [6, Proposition 2.5] and [6, Proposition 2.6]. □

Note that the existence of a bump function in $C_b(X)$, $C_b^u(X)$ or in $\mathrm{Lip}_b^\alpha(X)$ with $0<\alpha\le 1$ is always true by using the metric d on X. This is not always the case when X is a Banach space for the spaces of smooth functions $A=C_b^k(X)$ $(k\in\mathbb{N}^*$, $C_b^{1,\alpha}(X)$ $(0<\alpha\le 1)$ or $C_b^{1,u}(X)$. In the last examples the existence of a bump function is connected to the geometry of the Banach space X. For more information on the existence of a bump function in $C_b^k(X)$ $(k\in\mathbb{N}^*)$, $C_b^{1,\alpha}(X)$ $(0<\alpha\le 1)$ or $C_b^{1,u}(X)$, we refer to the book of Deville *et al.* [15].

2.3. Some useful lemmas. We need the following lemmas in the proof of Theorem 1.2. We begin by Lemma 2.8 that is a consequence of the variational principle of Deville *et al.* [14] (see also Deville and Revalski [16]).

Definition 2.6. Let (X,d) be a metric space and $f: X\longrightarrow\mathbb{R}\cup\{+\infty\}$ be a function with nonempty domain. We say that f has a strong minimum at $x\in X$ if $\inf_X f=f(x)$ and $d(x_n,x)\to 0$ whenever $f(x_n)\to f(x)$.

Theorem 2.7 (Deville *et al.* [14]; Deville and Revalski [16]). *Let (X,d) be a complete metric space and $A\subset C_b(X)$ be a space satisfying the axioms (A_1) and (A_3). Let f be a lower semicontinuous and bounded from below function with nonempty domain. Then*

$$\sigma(f):=\{\varphi\in A/f-\varphi \text{ does not attain a strong minimum on } X\}$$

is σ-porous; in particular, it is of the first Baire category and so $A\setminus\sigma(f)$ is a dense subset of A.

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LEMMA 2.8. *Under the hypothesis of Theorem 2.7, we have that for every lower semicontinuous and bounded from below function with nonempty domain* $f : X \to \mathbb{R} \cup \{+\infty\}$, *the set*

$$D(f) := \{x \in X / \exists \varphi_x \in A : f - \varphi_x \text{ has a strong minimum at } x\}$$

is dense in $\mathrm{dom}(f)$.

PROOF. Let $x \in \mathrm{dom}(f)$ and $n \in \mathbb{N}^*$. Thanks to the axiom (A_3), there exist a positive constant M_n and a function $h_n : X \to [0, 1]$ such that $h_n \in A$, $\|h_n\| \leq M_n$, $h_n(x) = 1$ and $\mathrm{diam}(\mathrm{supp}(h_n)) < 1/n$. Let us set $\lambda_x^n := f(x) - \inf_X(f) + 3/n$ and, applying Theorem 2.7 to the function $f - \lambda_x^n h_n$, there exist $x_n \in X$ and $\varphi \in A$ such that $\|\varphi\| < 1/n$ and $f - \lambda_x^n h_n - \varphi$ has a strong minimum at x_n. Suppose that $d(x, x_n) \geq 1/n$. Since $\mathrm{diam}(\mathrm{supp}(h_n)) < 1/n$ and $x \in \mathrm{supp}(h_n)$, $h_n(x_n) = 0$. Thus,

$$\begin{aligned}
\inf_X(f) - \varphi(x_n) &\leq f(x_n) - \varphi(x_n) \\
&= f(x_n) - \lambda_x^n h_n(x_n) - \varphi(x_n) \\
&< f(x) - \lambda_x^n h_n(x) - \varphi(x) \\
&= f(x) - \lambda_x^n - \varphi(x).
\end{aligned}$$

We deduce that $\lambda_x^n < f(x) - \inf_X(f) + 2/n$, which is a contradiction with the choice of λ_x^n. So, $d(x, x_n) < 1/n$ and $x_n \in D(f)$. It follows that $D(f)$ is dense in $\mathrm{dom}(f)$. □

LEMMA 2.9. *Let Z be a Banach space and $h, k : Z \to \mathbb{R}$ be two continuous and convex functions. Suppose that the function $z \to l(z) := \max(h(z), k(z))$ is Fréchet (respectively, Gâteaux) differentiable at some point $z_0 \in Z$. Then either h or k (maybe both h and k) is Fréchet (respectively, Gâteaux) differentiable at z_0 and $l'(z_0) = h'(z_0)$ or $l'(z_0) = k'(z_0)$.*

PROOF. We give the proof for the Fréchet differentiability; the Gâteaux differentiability is similar. Suppose without loss of generality that $l(z_0) = h(z_0)$; let us prove that h is Fréchet differentiable at z_0 and that $l'(z_0) = h'(z_0)$. For each $z \neq 0$,

$$0 \leq \frac{h(z_0 + z) + h(z_0 - z) - 2h(z_0)}{\|z\|} \leq \frac{l(z_0 + z) + l(z_0 - z) - 2l(z_0)}{\|z\|}.$$

Since l is convex and Fréchet differentiable at z_0, the right-hand side in the above inequalities tends to 0 when z tends to 0. This implies that h is Fréchet differentiable at z_0 by the convexity of h. Now, if we denote $f = h - l$, then $f(z_0) = 0$, $f \leq 0$ and $f'(z_0)$ exists. Thus, for all $z \in Z$,

$$f'(z_0)(z) = \lim_{t \longrightarrow 0^+} \frac{1}{t}(f(z_0 + tz) - f(z_0)) \leq 0.$$

This implies that $\|f'(z_0)\| = 0$. Thus, $h'(z_0) = l'(z_0)$. □

LEMMA 2.10. *Let (X, d) be a complete metric space and $A \subset C_b(X)$ satisfy the axioms (A_1), (A_2) and (A_4^β). Let $(\lambda_n)_n \subset \mathbb{R}$ be such that $|\lambda_n| = 1$ for all $n \in \mathbb{N}$ and let $(x_n)_n \subset X$. Suppose that $\lambda_n \delta_{x_n}$ converges for the topology τ_β (the norm topology or the weak-star topology) to some point $Q \in A^*$. Then $(\lambda_n)_n$ converges in $\mathbb{R}$ to some real number λ such that $|\lambda| = 1$, the sequence $(x_n)_n$ converges to some point x in (X, d) and we have $Q = \lambda\delta_x$.*

PROOF. Since $\lambda_n \delta_{x_n}$ converges for the topology τ_β to some point $Q \in A^*$, $\lambda_n \delta_{x_n}(\varphi) \to Q(\varphi)$ for all $\varphi \in A$. Since A contains the constants, we have $\lambda_n \to Q(1) := \lambda$ with $|\lambda| = 1$. Now, since $(\lambda_n)_n$ converges to λ and $\lambda_n \delta_{x_n}$ converges for the topology τ_β to Q, by dividing by λ_n we obtain that δ_{x_n} converges for the topology τ_β to $Q/\lambda \in A^*$. The property P^β implies that $(x_n)_n$ converges to some point $x \in X$ and in consequence that δ_{x_n} converges for the topology τ_β to δ_x. By the uniqueness of the limit, we have that $Q = \lambda\delta_x$. □

3. The proof of Theorem 1.2

The proof of Theorem 1.2 is divided into the four steps below. The part (2) ⇒ (1) is easy. We prove the part (1) ⇒ (2). Let us begin by recalling the hypotheses of Theorem 1.2 and fixing some notation which will appear in the proof. Let X and Y be two complete metric spaces and $A \subset C_b(X)$ and $B \subset C_b(Y)$ be two Banach spaces satisfying the axioms (A_1)–(A_4^β) with the same β, where $\beta = F$ or $\beta = G$. Let $T : A \to B$ be an isomorphism and $T^* : B^* \to A^*$ the adjoint of T. Recall that T^* is norm to norm continuous as well as weak-star to weak-star continuous. By I_X, we denote the identity map on A and by I_Y we denote the identity map on B. Recall that by the β-differentiability, we mean the Fréchet differentiability if $\beta = F$ and the Gâteaux differentiability if $\beta = G$. By the τ_β topology, we mean the norm topology if $\beta = F$ and the weak-star topology if $\beta = G$. There exists a connection between the β-differentiability and the property P^β that we are going to use in this section. The purpose is to be able to identify the Dirac masses in the dual space A^*. The β-differentiability is a good tool for it. It indeed allows us to see, thanks to [6, Theorem 2.8], that the Dirac masses correspond to the β-differentiability of $f^\times$ defined below, at some well-chosen points of the space A whenever this space has the property P^β. The existence of these 'good' points of A will be guaranteed by Lemma 2.8.

Let $f : X \to \mathbb{R} \cup \{+\infty\}$ be a lower semicontinuous and bounded from below function with nonempty domain. We recall from [6] that the conjugacy $f^\times$ of f and the second conjugacy $f^{\times\times}$ are defined as follows:

$$\begin{aligned} f^\times : A &\to \mathbb{R} \\ \varphi &\to \sup_{x \in X}\{\varphi(x) - f(x)\}, \\ f^{\times\times} : X &\to \mathbb{R} \cup \{+\infty\} \\ x &\mapsto f^{\times\times}(x) := \sup_{\varphi \in A}\{\varphi(x) - f^\times(\varphi)\}. \end{aligned}$$

The function $f^\times$ is clearly convex and Lipschitz continuous (see for instance [6, Proposition 2.1]) but the function $f^{\times\times}$ is not convex in general even if X is a vector space. However, under the axiom (A_3), we have from [6, Theorem 2.2] that $f^{\times\times} = f$ for each bounded from below and lower semicontinuous function on X. This fact will be used in Step (4) of the proof of Theorem 1.2. We also need the following elementary lemma.

Lemma 3.1. *We have*

$$\sup_{x\in X}\{|\varphi(x)| - f(x)\} = \max(f^\times(\varphi), f^\times(-\varphi))$$

for all $\varphi \in A$.

Proof. Since $|t| = \max(t, -t)$ on $\mathbb{R}$, by inverting the supremum and the maximum,

$$\begin{aligned}\sup_{x\in X}\{|\varphi(x)| - f(x)\} &= \max\Big(\sup_{x\in X}\{\varphi(x) - f(x)\}, \sup_{x\in X}\{-\varphi(x) - f(x)\}\Big)\\ &= \max(f^\times(\varphi), f^\times(-\varphi)).\end{aligned}$$

□

3.1. Step 1. The map T has the canonical form. Let $f : X \to \mathbb{R} \cup \{+\infty\}$ and $g : Y \to \mathbb{R} \cup \{+\infty\}$ be lower semicontinuous and bounded from below functions with nonempty domains. Suppose that for all $\varphi \in A$,

$$\sup_{y\in Y}\{|T\varphi(y)| - g(y)\} = \sup_{x\in X}\{|\varphi(x)| - f(x)\}.$$

Lemma 3.2. *There exist a map* $\pi : \overline{\mathrm{dom}(g)} \to \overline{\mathrm{dom}(f)}$ *and a map* $\varepsilon : \overline{\mathrm{dom}(g)} \to \{\pm 1\}$ *such that for all* $y \in \overline{\mathrm{dom}(g)}$, *we have* $T^*\delta_y = \varepsilon(y)\delta_{\pi(y)}$ *or, equivalently,* $T\varphi(y) = \varepsilon(y)\varphi \circ \pi(y)$ *for all* $y \in \overline{\mathrm{dom}(g)}$ *and all* $\varphi \in A$.

Proof. By Lemma 2.8, the set $D(g)$ is dense in $\overline{\mathrm{dom}(g)}$. Let $y \in D(g)$ and $\tilde{\psi}_y \in B$ be such that $g - \tilde{\psi}_y$ has a strong minimum at y. Let $c \in \mathbb{R}$ be such that $c > (1/2)(g^\times(-\tilde{\psi}_y) - g^\times(\tilde{\psi}_y))$ and put $\psi_y = c + \tilde{\psi}_y$. The function $g - \psi_y$ has also a strong minimum at y and satisfies by the choice of c the inequality $g^\times(\psi_y) > g^\times(-\psi_y)$. Since $g^\times$ and so also $g^\times \circ (-I_Y)$ are continuous (even Lipschitz functions, see for instance [6, Proposition 2.1]), there exists an open neighbourhood $O(\psi_y) \subset B$ of ψ_y such that $g^\times(\psi) > g^\times(-\psi)$ for all $\psi \in O(\psi_y)$. Thus, we have $\max(g^\times(\psi), g^\times(-\psi)) = g^\times(\psi)$ on the open set $O(\psi_y)$ of B. Since $g - \psi_y$ has a strong minimum at y, [6, Theorem 2.8] guarantees the β-differentiability of $g^\times$ at ψ_y with the derivative $(g^\times)'(\psi_y) = \delta_y$. Since the functions $\psi \to \max(g^\times(\psi), g^\times(-\psi))$ and $\psi \to g^\times(\psi)$ coincide on the open set $O(\psi_y)$, we conclude that $\psi \to \max(g^\times(\psi), g^\times(-\psi))$ is also β-differentiable at ψ_y with the same derivative $(g^\times)'(\psi_y) = \delta_y$. On the other hand, there exists $\varphi_y \in A$ such that $\psi_y = T\varphi_y$ by the surjectivity of T. The composition of a β-differentiable function with a linear and continuous map is again β-differentiable. Thus, we have the β-differentiability of the composite map $\varphi \to \max(g^\times(T\varphi), g^\times(-T\varphi))$ at φ_y on

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A and the chain rule formula gives $\delta_y \circ T$ as the derivative of the function $\varphi \to \max(g^\times(T\varphi), g^\times(-T\varphi))$ at φ_y. But by hypothesis and by Lemma 3.1, we have that $\max(g^\times(T\varphi), g^\times(-T\varphi)) = \max(f^\times(\varphi), f^\times(-\varphi))$ for all $\varphi \in A$. We deduce that the function $\varphi \to \max(f^\times(\varphi), f^\times(-\varphi))$ is also β-differentiable at φ_y on A with the same derivative $\delta_y \circ T$. Lemma 2.9 implies that either the function $\varphi \to f^\times(\varphi)$ or the function $\varphi \to f^\times(-\varphi) = (f^\times \circ (-I_X))(\varphi_y)$ is β-differentiable at φ_y with the derivative given by the derivative of $\varphi \to \max(f^\times(\varphi), f^\times(-\varphi))$ at φ_y that is here $\delta_y \circ T$. If it is the function $\varphi \to f^\times(\varphi)$, [6, Theorem 2.8] asserts that there exists $\pi(y) \in D(f)$ such that $(f^\times)'(\varphi_y) = \delta_{\pi(y)}$. If it is the function $\varphi \to f^\times(-\varphi)$, then by composition with $-I_X$, we have that $f^\times$ is β-differentiable at $-\varphi_y$ and so [6, Theorem 2.8] guarantees the existence of a point $\pi(y) \in X$ such that $(f^\times)'(-\varphi_y) = \delta_{\pi(y)}$. Using the chain rule formula, we obtain that $(f^\times \circ (-I_X))'(\varphi_y) = (f^\times)'(-\varphi_y) \circ (-I_X) = \delta_{\pi(y)} \circ (-I_X) = -\delta_{\pi(y)}$. By identifying the derivatives of the two equal functions $\varphi \to \max(g^\times(T\varphi), g^\times(-T\varphi)) = \max(f^\times(\varphi), f^\times(-\varphi))$, we obtain that $\delta_y \circ T = \delta_{\pi(y)}$ or $\delta_y \circ T = -\delta_{\pi(y)}$. Let us put $\varepsilon(y) = \pm 1$. Then we have proved that

$$\forall y \in D(g) \exists \pi(y) \in D(f) / T^*\delta_y := \delta_y \circ T = \varepsilon(y)\delta_{\pi(y)}. \tag{3.1}$$

Now let y be any point of $\overline{\operatorname{dom}(g)}$; there exists by Lemma 2.8 a sequence $(y_n)_n \subset D(g)$ such that $y_n \to y$. The property P^β (axiom A_4^β) implies that $\delta_{y_n} \xrightarrow{\tau_\beta} \delta_y$. Since T^* is τ_β to τ_β continuous (here τ_β is the norm or the weak-star topology), $T^*\delta_{y_n} \xrightarrow{\tau_\beta} T^*\delta_y$. Since $(y_n)_n \subset D(g)$, from the formula (3.1) there exists $\pi(y_n) \in D(f)$ such that $T^*\delta_{y_n} = \varepsilon(y_n)\delta_{\pi(y_n)}$. So, we have $\varepsilon(y_n)\delta_{\pi(y_n)} \xrightarrow{\tau_\beta} T^*\delta_y$. Lemma 2.10 implies the existence of a real number $\varepsilon(y) = \pm 1$ and some point $\pi(y) \in X$ such that $\varepsilon(y_n) \to \varepsilon(y)$ in $\mathbb{R}$ and $\pi(y_n) \to \pi(y)$ in X. Thus, $\pi(y) \in \overline{D(f)} = \overline{\operatorname{dom}(f)}$. Lemma 2.10 implies also that $T^*\delta_y = \varepsilon(y)\delta_{\pi(y)}$. Thus, we have proved that there exist a map $\pi : \overline{\operatorname{dom}(g)} \to \overline{\operatorname{dom}(f)}$ and a map $\varepsilon : \overline{\operatorname{dom}(g)} \to \{-1, 1\}$ such that for all $y \in \overline{\operatorname{dom}(g)}$, we have $T^*\delta_y = \varepsilon(y)\delta_{\pi(y)}$. □

3.2. Step 2. The map π is bijective.

Proof. Lemma 3.2 applied to T^{-1} implies also the existence of a map $\pi' : \overline{\operatorname{dom}(f)} \to \overline{\operatorname{dom}(g)}$ and a map $\varepsilon' : \overline{\operatorname{dom}(f)} \to \{-1, 1\}$ such that for all $x \in \overline{\operatorname{dom}(f)}$, we have $(T^{-1})^*\delta_x = \varepsilon'(x)\delta_{\pi'(x)}$. We obtain then

$$\begin{aligned}\delta_x &= T^*(\varepsilon'(x)\delta_{\pi'(x)})\\ &= \varepsilon'(x)T^*(\delta_{\pi'(x)})\\ &= \varepsilon'(x)\varepsilon(\pi(x))\delta_{\pi(\pi'(x))}.\end{aligned}$$

By applying the above identity to the constant function 1, we obtain that $\varepsilon'(x)\varepsilon(\pi(x)) = 1$. On the other hand, since the space A separates the points of X (axiom (A_3)), we obtain $\pi(\pi'(x)) = x$. This reasoning applies for all $x \in \overline{\operatorname{dom}(f)}$. By inverting the roles of T and T^{-1}, we have also $\pi'(\pi(y)) = y$ for all $y \in \overline{\operatorname{dom}(g)}$. Thus, π is bijective. □

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3.3. Step 3. The maps ε, π and π^{-1} are continuous.

Proof. Let $y_n \in \overline{\mathrm{dom}(g)}$ be such that $y_n \to y \in \overline{\mathrm{dom}(g)}$. Let us prove that $\varepsilon(y_n) \to \varepsilon(y)$ in $\mathbb{R}$ and $\pi(y_n) \to \pi(y)$ in $\overline{\mathrm{dom}(f)}$. Indeed, by the property P^β (axiom A_4^β), we have that $\delta_{y_n} \xrightarrow{\tau_\beta} \delta_y$. Since T is τ_β to τ_β continuous, $T^*\delta_{y_n} \xrightarrow{\tau_\beta} T^*\delta_y$ for the τ_β topology in A^*. In other words, $\varepsilon(y_n)\delta_{\pi(y_n)} \xrightarrow{\tau_\beta} \varepsilon(y)\delta_{\pi(y)}$, which implies by Lemma 2.10 that $\varepsilon(y_n) \to \varepsilon(y)$ in $\mathbb{R}$ and $\delta_{\pi(y_n)} \xrightarrow{\tau_\beta} \delta_{\pi(y)}$ in A^*. Again by the property P^β, we have $\pi(y_n) \to \pi(y)$ in $\overline{\mathrm{dom}(f)}$. Thus, ε and π are continuous. The same argument applied to T^{-1} shows that π^{-1} is also continuous. □

3.4. Step 4. The formula $g = f \circ \pi$ on $\overline{\mathrm{dom}(g)}$.

Proof. This formula follows from the previous steps together with [6, Theorem 2.2]. Indeed, by hypothesis,

$$\sup_{y\in Y}\{|T\varphi(y)| - g(y)\} = \sup_{x\in X}\{|\varphi(x)| - f(x)\}$$

for all $\varphi \in A$. Since g (respectively, f) is equal to $+\infty$ on $Y \setminus \overline{\mathrm{dom}(g)}$ (respectively, on $X \setminus \overline{\mathrm{dom}(f)}$),

$$\sup_{y\in\overline{\mathrm{dom}(g)}}\{|T\varphi(y)| - g(y)\} = \sup_{x\in\overline{\mathrm{dom}(f)}}\{|\varphi(x)| - f(x)\}$$

for all $\varphi \in A$. Since $\overline{\mathrm{dom}(g)}$ and $\overline{\mathrm{dom}(f)}$ are homeomorphic by Steps (2) and (3), by replacing $T\varphi(y)$ in the above formula with its expression $\varepsilon(y)\varphi(\pi(y))$ for $y \in \overline{\mathrm{dom}(g)}$ (see Step (1)),

$$\sup_{y\in\overline{\mathrm{dom}(g)}}\{|\varphi(\pi(y))| - g(y)\} = \sup_{x\in\overline{\mathrm{dom}(f)}}\{|\varphi(x)| - f(x)\}$$

for all $\varphi \in A$. Since π is bijective, we obtain by the change of variable $x = \pi(y)$,

$$\sup_{x\in\overline{\mathrm{dom}(f)}}\{|\varphi(x)| - g(\pi^{-1}(x))\} = \sup_{x\in\overline{\mathrm{dom}(f)}}\{|\varphi(x)| - f(x)\}$$

for all $\varphi \in A$. The above formula is also true for the functions $\varphi - \inf_X(\varphi) \geq 0$ for all $\varphi \in A$ since A contains the constants. Replacing φ by $\varphi - \inf_X(\varphi) \geq 0$,

$$\sup_{x\in\overline{\mathrm{dom}(f)}}\{\varphi(x) - \inf_X(\varphi) - g(\pi^{-1}(x))\} = \sup_{x\in\overline{\mathrm{dom}(f)}}\{\varphi(x) - \inf_X(\varphi) - f(x)\}$$

for all $\varphi \in A$. So,

$$\sup_{x\in\overline{\mathrm{dom}(f)}}\{\varphi(x) - g(\pi^{-1}(x))\} = \sup_{x\in\overline{\mathrm{dom}(f)}}\{\varphi(x) - f(x)\}$$

for all $\varphi \in A$. Let us denote by $i_{\overline{\mathrm{dom}(f)}}$ the lower semicontinuous indicator function, which is equal to 0 on $\overline{\mathrm{dom}(f)}$ and equal to $+\infty$ otherwise. The above formula can be written as follows:

$$\sup_{x\in X}\{\varphi(x) - (g(\pi^{-1}(x)) + i_{\overline{\mathrm{dom}(f)}})\} = \sup_{x\in X}\{\varphi(x) - (f(x) + i_{\overline{\mathrm{dom}(f)}})\}$$

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for all $\varphi \in A$. In other words, by using the notation of the conjugacy,

$$(g \circ \pi^{-1} + i_{\overline{\mathrm{dom}(f)}})^{\times}(\varphi) = (f + i_{\overline{\mathrm{dom}(f)}})^{\times}(\varphi)$$

for all $\varphi \in A$. By passing to the second conjugacy,

$$(g \circ \pi^{-1} + i_{\overline{\mathrm{dom}(f)}})^{\times\times}(x) = (f + i_{\overline{\mathrm{dom}(f)}})^{\times\times}(x)$$

for all $x \in X$. Since the functions $f + i_{\overline{\mathrm{dom}(f)}}$ and $g \circ \pi^{-1} + i_{\overline{\mathrm{dom}(f)}}$ are bounded from below and lower semicontinuous on X, by [6, Theorem 2.2], each of these functions coincides with its second conjugacy. Thus,

$$g \circ \pi^{-1} + i_{\overline{\mathrm{dom}(f)}} = f + i_{\overline{\mathrm{dom}(f)}},$$

which is equivalent to $g \circ \pi^{-1} = f$ on $\overline{\mathrm{dom}(f)}$ as well as $g = f \circ \pi$ on $\overline{\mathrm{dom}(g)}$. □

3.5. Remarks. By imitating the proof of Theorem 1.2, we obtain easily the following version where the map ϵ coincides with the constant function 1 on Y.

THEOREM 3.3. *Let X and Y be two complete metric spaces and $A \subset C_b(X)$ and $B \subset C_b(Y)$ be two Banach spaces satisfying the axioms A_1–A_4^{β} (with the same property β). Let $T : A \to B$ be an isomorphism and let $f : X \to \mathbb{R} \cup \{+\infty\}$ and $g : Y \to \mathbb{R} \cup \{+\infty\}$ be lower semicontinuous and bounded from below functions with nonempty domains. Then we have* (1) ⇔ (2).

(1) $g^{\times} \circ T = f^{\times}$, *that is, for all $\varphi \in A$,*

$$\sup_{y \in Y}\{T\varphi(y) - g(y)\} = \sup_{x \in X}\{\varphi(x) - f(x)\}.$$

(2) *There exists a homeomorphism $\pi : \overline{\mathrm{dom}(g)} \to \overline{\mathrm{dom}(f)}$ such that, for all $y \in \overline{\mathrm{dom}(g)}$ and all $\varphi \in A$,*

$$T\varphi(y) = \varphi \circ \pi(y)$$

and

$$g(y) = f \circ \pi(y).$$

REMARK 3.4. (i) In Theorems 1.2 and 3.3, we can have more information about the homeomorphism π. The more the spaces A and B are regular, the more the homeomorphism π is also. This is due to the fact that the condition $\varphi \circ \pi \in B$ for all $\varphi \in A$ implies a certain regularity on π. See for instance Theorem 44 in the paper of Garrido and Jaramillo [19] for Lipschitz continuous functions; see also the papers of Gutiérrez and Llavona [21] and Jaramillo *et al.* [22] for weakly C^k functions and the paper of Bachir and Lancien [8] for weakly C^k functions on spaces with the Schur property.

(ii) Under the hypothesis of Theorems 1.2 or 3.3, the isomorphism T is not necessarily an isometry for the norm $\|\cdot\|_{\infty}$, but if we assume in addition that $\overline{\mathrm{dom}(f)} = X$ and $\overline{\mathrm{dom}(g)} = Y$, then the condition (1) of the theorems implies that T is an isometry for the norm $\|\cdot\|_{\infty}$; this follows from the formula (4.1) in the theorems.

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4. Applications

This section is concerned with some simple applications of our main result. For various results around the Banach–Stone theorem, we can consult for example the works of Amir [1], Araujo [2, 3], Araujo and Font [4], Behrends [10, 11], Cambern [12], Cengiz [13], Garrido and Jaramillo [18], Garrido *et al.* [17], Jarosz [23], Jarosz and Pathak [24], Vieira [27] and Weaver [28].

4.1. Partial isometries. Let X and Y be two complete metric spaces and $A \subset C_b(X)$ and $B \subset C_b(Y)$ be two Banach spaces. Let $T : A \to B$ be an isomorphism. We say that T is partially isometric (for the norm $\|\cdot\|_\infty$) if there exist a nonempty closed subset E of X and a nonempty closed subset F of Y such that $\sup_{y\in F}|T\varphi(y)| = \sup_{x\in E}|\varphi(x)|$ for all $\varphi \in A$. There are examples where an isomorphism is not an isometry but is partially isometric. Indeed, let $K \subset X$ and $L \subset Y$ be two nonhomeomorphic compact spaces such that $C(K)$ and $C(L)$ are isomorphic and let $T_1 : C(K) \to C(L)$ be an isomorphism (T_1 cannot be isometric by the classical Banach–Stone theorem). Note that Milutin proved in [25] that if K and L are both uncountable compact metric spaces, then $C(K)$ and $C(L)$ are always linearly isomorphic. Let $E \subset X$ and $F \subset Y$ be two homeomorphic closed spaces such that $E \cap K = \emptyset$ and $F \cap L = \emptyset$ and let $\pi : F \to E$ be a homeomorphism. Let us consider the map $T : C_b(K \cup E) \to C_b(L \cup F)$ defined by $T(\varphi)(y) = T_1(\varphi_{|K})(y)$ if $y \in L$ and $T(\varphi)(y) = \varphi \circ \pi(y)$ if $y \in F$ for all $\varphi \in C_b(K \cup E)$. Here $\varphi_{|K}$ denotes the restriction of φ to K. The map T is an isomorphism (not isometric) satisfying $\sup_{y\in F}|T\varphi(y)| = \sup_{x\in E}|\varphi(x)|$ for all $\varphi \in C_b(K \cup E)$. As an immediate consequence of Theorem 1.2, we obtain the following generalization of the Banach–Stone theorem in the complete metric framework.

Corollary 4.1. *Let X and Y be two complete metric spaces. Let $A \subset C_b(X)$ and $B \subset C_b(Y)$ be two Banach spaces satisfying the axioms (A_1)–(A_4^β) (with the same β). Let E be a nonempty closed subset of X and F be a nonempty closed subset of Y. Let $T : A \to B$ be an isomorphism. Then*

$$\sup_{y\in F}|T\varphi(y)| = \sup_{x\in E}|\varphi(x)|$$

for all $\varphi \in A$ if and only if there exist a homeomorphism $\pi : F \to E$ and a continuous map $\varepsilon : F \to \{\pm 1\}$ such that for all $y \in F$ and all $\varphi \in A$, we have $T\varphi(y) = \varepsilon(y)\varphi \circ \pi(y)$.

Proof. It is enough to apply Theorem 1.2 with the indicator functions $f = i_E$ and $g = i_F$, where i_E (respectively, i_F) is equal to 0 on E (respectively, on F) and $+\infty$ otherwise. □

Let us mention here that the first result about the vector-valued Banach–Stone theorem is due to Behrends [10, 11] (see also [23]). For the noncompact vector-valued Banach–Stone theorem, see [2, 3, 5]. We know from [10] that the existence of an isometric isomorphism between $C(K_1, C(L)) \cong C(K_1 \times L)$ and $C(K_2, C(L)) \cong C(K_2 \times L)$ does not imply in general that K_1 and K_2 are homeomorphic. This is due to the fact that in general: ($K_1 \times L$ and $K_2 \times L$ are homeomorphic) $\nRightarrow$ (K_1 and

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K_2 are homeomorphic). We give below a condition under which the existence of a particular isomorphism between $C(K_1 \times L)$ and $C(K_2 \times L)$ implies that K_1 and K_2 are homeomorphic.

COROLLARY 4.2. *Let X, Y, M_1 and M_2 be complete metric spaces. Let $T : C_b(X \times M_1) \to C_b(Y \times M_2)$ be an isomorphism. Let $m_1 \in M_1$ and $m_2 \in M_2$. Then*

$$\sup_{y \in Y} |T\varphi(y, m_2)| = \sup_{x \in X} |\varphi(x, m_1)|$$

for all $\varphi \in C_b(X \times M_1)$ if and only if there exist a homeomorphism $\pi : Y \to X$ and a continuous map $\varepsilon : Y \to \{\pm 1\}$ such that for all $y \in Y$ and all $\varphi \in C_b(X \times M)$, we have $T\varphi(y, m_2) = \varepsilon(y)\varphi(\pi(y), m_1)$.

PROOF. We apply Corollary 4.1 with $E = X \times \{m_1\}$ and $F = Y \times \{m_2\}$. □

4.2. Groups and isomorphism. Let X be a complete metric space and E be a nonempty closed subspace of X. By $(IS(C_b(X)), \circ)$, we denote the group (for the law $\circ$ of composition of maps) of all isomorphisms from $C_b(X)$ onto itself. We define now the following sets:

$$\mathrm{Isom}_E(C_b(X)) = \{T \in IS(C_b(X)) : \sup_{y \in E} |T\varphi(y)| = \sup_{x \in E} |\varphi(x)|, \forall \varphi \in C_b(X)\},$$
$$\mathrm{Isom}(C_b(E)) = \{S : C_b(E) \to C_b(E) : S \text{ isomorphism isometric}\}.$$

Let us define the restriction map $R_E : C_b(X) \to C_b(E)$ by $R_E : \varphi \mapsto \varphi_{|E}$, where $\varphi_{|E}$ denotes the restriction of $\varphi \in C_b(X)$ to E. By N_E, we denote the subset of $IS(C_b(X))$ defined by

$$\begin{aligned} N_E &:= \{T \in IS(C_b(X)) : R_E \circ T = R_E\} \\ &= \{T \in IS(C_b(X)) : (T\varphi)_{|E} = \varphi_{|E}, \varphi \in C_b(X)\}. \end{aligned}$$

Clearly, $\mathrm{Isom}_E(C_b(X))$ and $\mathrm{Isom}(C_b(E))$ are groups and N_E is a subgroup of $\mathrm{Isom}_E(C_b(X))$. The purpose of the following result is to give a relation between these three groups.

THEOREM 4.3. *Let X be a complete metric space and E be a nonempty closed subspace of X. Then N_E is a normal subgroup of $\mathrm{Isom}_E(C_b(X))$ and the group quotient $\mathrm{Isom}_E(C_b(X))/N_E$ is isomorphic to a subgroup of $\mathrm{Isom}(C_b(E))$. If moreover we suppose that $X \setminus E$ is also closed (in particular, X is nonconnected here), then*

$$\mathrm{Isom}_E(C_b(X))/N_E \cong \mathrm{Isom}(C_b(E)).$$

PROOF. The proof will be complete if we construct a group homomorphism

$$\Lambda : \mathrm{Isom}_E(C_b(X)) \to \mathrm{Isom}(C_b(E))$$

such that $\operatorname{Ker} \Lambda = N_E$, since the kernel of a group homomorphism is always a normal subgroup and $\mathrm{Isom}_E(C_b(X))/\operatorname{Ker} \Lambda \cong \mathrm{Im}\Lambda$. If moreover Λ is surjective, then we obtain that $\mathrm{Isom}_E(C_b(X))/\operatorname{Ker} \Lambda \cong \mathrm{Isom}(C_b(E))$.

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(i) *The construction of the map* Λ. For each $T \in \mathrm{Isom}_E(C_b(X))$, there exist by Corollary 4.1 a homeomorphism $\pi_T : E \to E$ and a continuous function $\epsilon_T : E \to \{\pm 1\}$ such that $T\varphi(e) = \epsilon_T(e)\varphi \circ \pi_T(e)$ for all $\varphi \in C_b(X)$ and all $e \in E$. Let us denote by $\widehat{T} : C_b(E) \to C_b(E)$ the map defined by $\widehat{T}\psi(e) = \epsilon_T(e)\psi \circ \pi_T(e)$ for all $\psi \in C_b(E)$ and all $e \in E$. Clearly, $\widehat{T} \in \mathrm{Isom}(C_b(E))$, since π_T is a homeomorphism from E onto itself. We define Λ as follows: $\Lambda T = \widehat{T}$. This map is well defined. Indeed, let $T, S \in \mathrm{Isom}_E(C_b(X))$. If $T = S$, then $\epsilon_T(e)\varphi \circ \pi_T(e) = \epsilon_S(e)\varphi \circ \pi_S(e)$ for all $\varphi \in C_b(X)$ and all $e \in E$. Since $1 \in C_b(X)$, we obtain that $\epsilon_T = \epsilon_S$ and, since $C_b(X)$ separates the points of X and also the points of E, $\pi_T = \pi_S$. Thus, $\widehat{T} = \widehat{S}$.

(ii) *The map* Λ *is a group homomorphism.* This fact is obvious.

(iii) *We have that* $\mathrm{Ker}\,\Lambda = N_E$. Indeed, let I be the identity map on $C_b(X)$ and i the identity map on E. Recall that for each $T \in \mathrm{Isom}_E(C_b(X))$, there exist a homeomorphism $\pi_T : E \to E$ and a continuous function $\epsilon_T : E \to \{\pm 1\}$ such that $T\varphi(e) = \epsilon_T(e)\varphi \circ \pi_T(e)$ for all $\varphi \in C_b(X)$ and all $e \in E$. So,

$$\begin{aligned} \mathrm{Ker}\,\Lambda &:= \{T \in \mathrm{Isom}_E(C_b(X)) : \Lambda T = I\} \\ &= \{T \in \mathrm{Isom}_E(C_b(X)) : \epsilon_T \equiv 1; \pi_T = i\} \\ &= \{T \in \mathrm{Isom}_E(C_b(X)) : R_E \circ T = R_E\} \\ &:= N_E. \end{aligned} \tag{4.1}$$

(iv) If moreover we assume that $X \setminus E$ is also closed, then the map Λ is surjective. Indeed, let $L \in \mathrm{Isom}(C_b(E))$. By Corollary 1.3, there exist a homeomorphism $\pi : E \to E$ and a continuous function $\epsilon : E \to \{\pm 1\}$ such that $L\psi(e) = \epsilon(e)\psi \circ \pi(e)$ for all $\psi \in C_b(E)$ and all $e \in E$. Let us define $T : C_b(X) \to C_b(X)$ as follows: for all $\varphi \in C_b(X)$, $T\varphi(x) = \epsilon(x)\varphi \circ \pi(x)$ if $x \in E$ and $T\varphi(x) = \varphi(x)$ if $x \in X \setminus E$. Since E and $X \setminus E$ are closed subsets of X, T is a well-defined isomorphism and satisfies $\sup_{y \in E} |T\varphi(y)| = \sup_{x \in E} |\varphi(x)|, \forall \varphi \in C_b(X)$. So, $T \in \mathrm{Isom}_E(C_b(X))$ and we have $L = \Lambda T$. □

Let $f : X \to \mathbb{R}$ be a lower semicontinuous and bounded below function with $\mathrm{dom}(f) = X$. Let us define the sets $\mathcal{H}_f(X)$ and $\mathrm{Isom}_f(C_b(X))$ as follows.

$$\mathcal{H}_f(X) = \{\pi : X \to X : \text{ homeomorphism such that } f \circ \pi = f\},$$
$$\mathrm{Isom}_f(C_b(X)) := \{T : C_b(X) \to C_b(X), \text{ isomorphism}/ f^{\times} \circ T = f^{\times}\}.$$

Clearly, $(\mathcal{H}_f(X), \circ)$ and $(\mathrm{Isom}_f(C_b(X)), \circ)$ are groups for the composition law.

Proposition 4.4. *Let X be a complete metric space. Then the following assertions hold.*

(1) *The groups* $(\mathcal{H}_f(X), \circ)$ *and* $(\mathrm{Isom}_f(C_b(X)), \circ)$ *are isomorphic.*
(2) *The group* $\mathrm{Isom}_0(C_b(X))$ *(with* $f \equiv 0$*) coincides with the set of all isometric isomorphisms* $T : C_b(X) \to C_b(X)$ *such that* $T1 = 1$.

Proof. For the part (1), we prove that the following map is an isomorphism of groups:

$$\begin{aligned} \chi : \mathcal{H}_f(X) &\to \mathrm{Isom}_f(C_b(X)) \\ \pi &\mapsto [\hat{\pi} : \varphi \mapsto \varphi \circ \pi]. \end{aligned}$$

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It is clear that χ is well defined and is a group morphism. The injectivity of χ follows from the fact that $C_b(X)$ separates the points of X and its surjectivity follows from Theorem 3.3.

For the part (2), let T be an isometric isomorphism such that $T1 = 1$. We obtain from Corollary 1.3 a homeomorphism $\pi : X \to X$ and a continuous function $\epsilon : X \to \{\pm 1\}$ such that $T\varphi(x) = \epsilon(x)\varphi \circ \pi(x)$ for all $x \in X$ and all $\varphi \in C_b(X)$. Since $T1 = 1$, $\epsilon \equiv 1$ and so we have $T\varphi = \varphi \circ \pi$ for all $\varphi \in C_b(X)$. This implies that $\sup_{x\in X} T\varphi(x) = \sup_{x\in X} \varphi(x)$ for all $\varphi \in C_b(X)$ or, equivalently, $0^\times \circ T = 0^\times$. So, we have $T \in \mathrm{Isom}_0(C_b(X))$. Conversely, if $T \in \mathrm{Isom}_0(C_b(X))$, from Theorem 3.3, there exists a homeomorphism $\pi : X \to X$ such that $T\varphi = \varphi \circ \pi$; in particular, T is isometric and satisfies $T1 = 1$. □

4.3. Isometries of the space of lower semicontinuous and bounded functions. We are interested in this section in the isometries between spaces of lower semicontinuous and bounded functions defined on complete metric spaces. Let X be a complete metric space. We denote by $\mathrm{SCI}_b(X)$ the set of all lower semicontinuous and bounded functions $f : X \to \mathbb{R}$ ($\overline{\mathrm{dom}(f)} = X$). We define the metric ρ on $\mathrm{SCI}_b(X)$ as follows:

$$\rho(f_1, f_2) = \|f_1 - f_2\|_\infty; \quad \forall (f_1, f_2) \in \mathrm{SCI}_b(X) \times \mathrm{SCI}_b(X).$$

Note that the space $(\mathrm{SCI}_b(X), +)$ is a monoid having 0 as identity element and that the maximal group of $(\mathrm{SCI}_b(X), +)$ is exactly the group $(C_b(X), +)$. We prove below that the Banach–Stone theorem is also true for the metric monoid structure of $(\mathrm{SCI}_b(X), +, \rho)$. For other examples of Banach–Stone-type theorems for monoid structures, we refer to the paper [7].

Theorem 4.5. *Let X and Y be two complete metric spaces and let $\Phi : (\mathrm{SCI}_b(X), +, \rho) \to (\mathrm{SCI}_b(Y), +, \rho)$ be a map. Then (1) $\Leftrightarrow$ (2) $\Leftrightarrow$ (3).*

(1) *The map $\Phi : (\mathrm{SCI}_b(X), +, \rho) \to (\mathrm{SCI}_b(Y), +, \rho)$ is an isometric isomorphism of monoids such that $\Phi 1 \geq 0$.*
(2) *There exists a homeomorphism $\pi : Y \to X$ such that $\Phi f = f \circ \pi$ for all $f \in \mathrm{SCI}_b(X)$.*
(3) *The map $\Phi : (\mathrm{SCI}_b(X), +, \rho) \to (\mathrm{SCI}_b(Y), +, \rho)$ is an isometric isomorphism of monoids such that $\Phi 1 = 1$.*

Proof. The part (2) $\Rightarrow$ (3) $\Rightarrow$ (1) is trivial. Let us prove the part (1) $\Rightarrow$ (2). Since a monoid isomorphism sends the maximal group to the maximal group, we have that the restriction $T := \Phi_{|C_b(X)}$ of Φ to $C_b(X)$ is a group isomorphism from $(C_b(X), +)$ onto $(C_b(Y), +)$. Since Φ is isometric for ρ and $\rho(f, 0) = \|f\|_\infty$ for all $f \in \mathrm{SCI}_b(X)$, we obtain that $\|T\varphi\|_\infty = \|\varphi\|_\infty$ for all $\varphi \in C_b(X)$. Thus, T is an isometric group isomorphism between $(C_b(X), +, \|\cdot\|_\infty)$ and $(C_b(Y), +, \|\cdot\|_\infty)$ and so an isometric isomorphism of Banach spaces. It follows from Corollary 1.3 that there exist a homeomorphism $\pi : Y \to X$ and a continuous function $\epsilon : Y \to \{\pm 1\}$ such that $T\varphi(y) = \epsilon(y)\varphi \circ \pi(y)$ for all $y \in Y$ and $\varphi \in C_b(X)$. Since $T1 = \Phi 1 \geq 0$, $\epsilon \equiv 1$ and so $T\varphi = \varphi \circ \pi$ for all $\varphi \in C_b(X)$.

On the other hand, we know that $\rho(T\varphi, \Phi f) = \rho(\varphi, f)$ for all $(\varphi, f) \in C_b(X) \times \mathrm{SCI}_b(X)$, since Φ is isometric. In other words, for all $(\varphi, f) \in C_b(X) \times (\mathrm{SCI}_b(X))$,

$$\sup_{y\in Y}\{|T\varphi(y) - \Phi f(y)|\} = \sup_{x\in X}\{|\varphi(x) - f(x)|\}.$$

Replacing T by its expression,

$$\sup_{y\in Y}\{|\varphi \circ \pi(y) - \Phi f(y)|\} = \sup_{x\in X}\{|\varphi(x) - f(x)|\}.$$

By changing the variable $\pi(y)$ by x in the left-hand member of the above equality,

$$\sup_{x\in\in X}\{|\varphi(x) - (\Phi f) \circ \pi^{-1}(x)|\} = \sup_{x\in X}\{|\varphi(x) - f(x)|\}.$$

For each $(\varphi, f) \in C_b(X) \times (\mathrm{SCI}_b(X))$, there exists a real number $c(\varphi, f) \in \mathbb{R}$ such that $\varphi + c(\varphi, f) \geq \max((\Phi f) \circ \pi^{-1}, f)$ (it is enough to choose a very big positive number). Since $\varphi + c(\varphi, f) \in C_b(X)$, replacing φ by $\varphi + c(\varphi, f)$ in the above equality,

$$\sup_{x\in\in X}\{\varphi(x) + c(\varphi, f) - (\Phi f) \circ \pi^{-1}(x)\} = \sup_{x\in X}\{\varphi(x) + c(\varphi, f) - f(x)\},$$

which is equivalent to

$$\sup_{x\in\in X}\{\varphi(x) - (\Phi f) \circ \pi^{-1}(x)\} = \sup_{x\in X}\{\varphi(x) - f(x)\}.$$

In other words, with the notation of the conjugacy,

$$((\Phi f) \circ \pi^{-1})^{\times}(\varphi) = f^{\times}(\varphi), \quad \forall \varphi \in C_b(X).$$

By using the second conjugacy and [6, Theorem 2.2], we get $(\Phi f) \circ \pi^{-1} = f$. Thus, $\Phi f = f \circ \pi$ for all $\varphi \in C_b(X)$. □

The following corollary concerns the extension of isometries.

Corollary 4.6. *Let X and Y be two complete metric spaces and $A \subset C_b(X)$ and $B \subset C_b(Y)$ be two Banach spaces satisfying the axioms (A_1)–(A_4^{β}) with the same β. For each isometric isomorphism (for the norm $\|\cdot\|_\infty$) T from A onto B such that $T1 = 1$, there exists a unique map $\tilde{T}$ extending T to an isometric isomorphism of monoids from $(\mathrm{SCI}_b(X), +, \rho)$ onto $(\mathrm{SCI}_b(Y), +, \rho)$.*

Proof. Since T is an isometric isomorphism, by Corollary 1.3, there exist $\pi : Y \to X$ a homeomorphism and $\epsilon : Y \to \{\pm 1\}$ continuous such that $T\varphi(y) = \epsilon(y)\varphi \circ \pi(y)$ for all $y \in Y$ and $\varphi \in A$. Since $T1 = 1$, $\epsilon \equiv 1$ and so $T\varphi = \varphi \circ \pi$ for all $\varphi \in A$. Now it is clear that the map $\tilde{T} : f \mapsto f \circ \pi$ for all $f \in \mathrm{SCI}_b(X)$ is an isometric isomorphism of monoids between $(\mathrm{SCI}_b(X), +, \rho)$ and $(\mathrm{SCI}_b(Y), +, \rho)$, which is an extension of T. For the uniqueness of $\tilde{T}$, let Φ be an isometric isomorphism of monoids from $(\mathrm{SCI}_b(X), +, \rho)$ onto $(\mathrm{SCI}_b(Y), +, \rho)$ such that $\Phi_{|A} = T$. By Theorem 4.5, there exists a homeomorphism $\pi_\Phi : Y \to X$ such that $\Phi f = f \circ \pi_\Phi$ for all $f \in \mathrm{SCI}_b(X)$. Since $\Phi_{|A} = T$ and A separates the points of X, $\pi_\Phi = \pi$ and so $\Phi = \tilde{T}$. □

4.4. Operators between products of function spaces. This section concentrates on the representations of some class of operators between products of function spaces.

Let X, Y be complete metric spaces and Z, W be Banach spaces. We denote by $C_b(X, Z)$ the space of all bounded Z-valued continuous functions. When $Z = \mathbb{R}$, we us the notation $C_b(X)$ for $C_b(X, \mathbb{R})$. We define the norm $\|\cdot\|_{\infty,1}$ on $C_b(X) \times C_b(X, Z)$ by

$$\|(\varphi, \psi)\|_{\infty,1} := \sup_{x \in X}\{|\varphi(x)| + \|\psi(x)\|_Z\}$$

for all $(\varphi, \psi) \in C_b(X) \times C_b(X, Z)$. Let $A \subset C_b(X)$ and $B \subset C_b(Y)$ be Banach subspaces and let $A' \subset C_b(X, Z)$ and $B' \subset C_b(Y, W)$ be any sets. Let H be an operator

$$\begin{aligned} H : A \times A' &\to B \times B' \\ (\varphi, \psi) &\mapsto (H_1(\varphi, \psi), H_2(\varphi, \psi)). \end{aligned}$$

Definition 4.7. We say that the operator H satisfies the property (P) if:

(a) for all $(\varphi, \psi) \in A \times A'$, we have $H_2(\varphi, \psi) = H_2(0, \psi)$ (H_2 depends only on the second variable);

(b) for all $\psi \in A'$, the map $\varphi \mapsto H_1(\varphi, \psi)$ is an isomorphism from A onto B.

Example 4.8. (1) Suppose that X and Y are homeomorphic complete metric spaces and let $\mathcal{H}_{0,0}(Y, X)$ be the set of all homeomorphisms from Y onto X. Let $\lambda : C_b(X) \to \mathcal{H}_{0,0}(Y, X)$ be any map. Let $A = A' = C_b(X)$ and $B = B' = C_b(Y)$. Let us define H by

$$H(\varphi, \psi) = (\varphi \circ \lambda(\psi), \psi \circ \lambda(\psi)).$$

Then H is not linear but satisfies the property (P). Note that in this case, H satisfies also $\|H(\varphi, \psi)\|_{\infty,1} = \|(\varphi, \psi)\|_{\infty,1}$ for all $(\varphi, \psi) \in A \times A'$. For example, we can set $\lambda : \psi \mapsto e^{\|\psi\|_\infty}\pi$ for a fixed homeomorphism π from a Banach space Y onto a Banach space X.

(2) Let $T : A \to B$ be an isomorphism and $S : A' \to B'$ be any map. Then the operator $H := (T, S)$ defined by $H(\varphi, \psi) = (T\varphi, S\psi)$ for all $(\varphi, \psi) \in A \times A'$ satisfies the property (P).

We are interested now in the operators (not necessarily linear) satisfying the property (P) and preserving the norm $\|\cdot\|_{\infty,1}$. The following theorem gives a canonical representation of such maps. Let us note that a nonlinear map H preserving norms, that is, such that $\|H(\varphi, \psi)\|_{\infty,1} = \|(\varphi, \psi)\|_{\infty,1}$ for all $(\varphi, \psi) \in A \times A'$, is not an isometry in general.

Theorem 4.9. *Let X, Y be complete metric spaces and Z, W be Banach spaces. Let $A \subset C_b(X)$ and $B \subset C_b(Y)$ be Banach spaces satisfying the axioms (A_1)–(A_4^β) (with the same β) and let $A' \subset C_b(X, Z)$ and $B' \subset C_b(Y, W)$ be any subsets. Let $H : A \times A' \to B \times B'$ be a map satisfying the property (P). Then (1) $\Leftrightarrow$ (2).*

(1) *For all $(\varphi, \psi) \in A \times A'$, we have $\|H(\varphi, \psi)\|_{\infty,1} = \|(\varphi, \psi)\|_{\infty,1}$.*

(2) *For all $\psi \in A'$, there exist a homeomorphism $\pi_\psi : Y \to X$ and a continuous function $\varepsilon_\psi : Y \to \{\pm 1\}$ such that for all $(\varphi, \psi) \in A \times A'$ and all $y \in Y$,*

$$H_1(\varphi, \psi)(y) = \varepsilon_\psi(y)\varphi \circ \pi_\psi(y),$$

and

$$\|H_2(\varphi, \psi)(y)\|_W := \|H_2(0, \psi)(y)\|_W = \|\psi \circ \pi_\psi(y)\|_Z.$$

Proof. Suppose that H satisfies (1). So,

$$\sup_{y\in Y}\{|H_1(\varphi, \psi)(y)| + \|H_2(\varphi, \psi)(y)\|_W\} = \sup_{x\in X}\{|\varphi(x)| + \|\psi(x)\|_Z\}$$

for all $(\varphi, \psi) \in A \times A'$. By the property (P),

$$\sup_{y\in Y}\{|H_1(\varphi, \psi)(y)| + \|H_2(0, \psi)(y)\|_W\} = \sup_{x\in X}\{|\varphi(x)| + \|\psi(x)\|_Z\}$$

for all $(\varphi, \psi) \in A \times A'$. For each fixed $\psi \in A'$, since $H_1(\cdot, \psi)$ is an isomorphism from A onto B by the property (P), then by applying Theorem 1.2 to the spaces A and B which satisfy the axioms (A_1)–(A_4^β) with the isomorphism $T_\psi := H_1(\cdot, \psi)$ and the continuous and bounded function $f_\psi(\cdot) = -\|\psi(\cdot)\|_Z$ on X and $g_\psi(\cdot) = -\|H_2(0, \psi)(\cdot)\|_W$ on Y, we obtain a homeomorphism $\pi_\psi : Y \to X$ and a continuous map $\epsilon_\psi : Y \to \{\pm 1\}$ such that for all $y \in Y$ and all $\varphi \in A$,

$$H_1(\phi, \psi)(y) = \epsilon_\psi(y)\varphi \circ \pi_\psi(y)$$

and

$$\|H_2(0, \psi)(y)\|_W = \|\psi(\pi_\psi(y))\|_Z.$$

So, we proved that (1) ⇒ (2). The converse is clear. □

We explore now the case where the operator H is linear, isometric for the norm $\|\cdot\|_{\infty,1}$ and satisfies the property (P).

Corollary 4.10. *Let X, Y be complete metric spaces and Z, W be Banach spaces. Let $A \subset C_b(X)$ and $B \subset C_b(Y)$ be Banach spaces satisfying the axioms (A_1)–(A_4^β) (with the same β) and let $A' \subset C_b(X, Z)$ be a linear subspace containing the constants and $B' \subset C_b(Y, W)$ be any linear subspace. Let $H : A \times A' \to B \times B'$ be a map satisfying the property (P). Then (1) ⇔ (2).*

(1) *The map H is a linear isometry for the norm $\|\cdot\|_{\infty,1}$.*

(2) *There exist a homeomorphism $\pi : Y \to X$, a continuous function $\varepsilon : Y \to \{\pm 1\}$ and a linear and isometric map $U_y : Z \to W$ for each $y \in Y$ such that for each $z \in Z$ the map $y \mapsto U_y(z)$ is continuous from Y into W and, for all $(\varphi, \psi) \in A \times A'$ and all $y \in Y$,*

$$H_1(\varphi, \psi)(y) = H_1(\varphi, 0)(y) = \varepsilon(y)\varphi \circ \pi(y)$$

and

$$H_2(\varphi, \psi)(y) := H_2(0, \psi)(y) = U_y(\psi \circ \pi(y)).$$

If moreover we assume that H is surjective and B' contains the constants, then U_y is also surjective.

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Proof. From Theorem 4.9, for all $\psi \in A'$ there exist a homeomorphism $\pi_\psi : Y \to X$ and a continuous function $\varepsilon_\psi : Y \to \{\pm 1\}$ such that for all $\varphi \in A$ and all $y \in Y$,

$$H_1(\varphi,\psi)(y) = \varepsilon_\psi(y)\varphi\circ\pi_\psi(y) \quad \text{and} \quad \|H_2(0,\psi)(y)\|_W = \|\psi\circ\pi_\psi(y)\|_Z.$$

Using the linearity of H_1, we have that $H_1(\varphi,0) = H_1(\varphi,\psi) - H_1(0,\psi)$ for all $(\varphi,\psi) \in A\times A'$. Thus, we obtain that $\epsilon_0(\cdot)\varphi\circ\pi_0 = \epsilon_\psi(\cdot)\varphi\circ\pi_\psi$ for all $(\varphi,\psi)\in A\times A'$. By replacing φ by 1, since A contains the constants, we obtain $\epsilon_0 = \epsilon_\psi$ for all $\psi\in A'$ and so $\epsilon := \epsilon_\psi$ does not depend on ψ. It follows that $\varphi\circ\pi_0 = \varphi\circ\pi_\psi$ for all $(\varphi,\psi)\in A\times A'$. For each fixed $\psi\in A'$, since the space A separates the points of X, we obtain that $\pi_0 = \pi_\psi$. So, $\pi := \pi_\psi$ does not depend on ψ. Finally,

$$\begin{aligned} H_1(\varphi,\psi)(y) &= \varepsilon(y)\varphi\circ\pi(y), \quad \forall y\in Y,\ \forall\varphi\in A,\\ \|H_2(0,\psi)(y)\|_W &= \|\psi\circ\pi(y)\|_Z, \quad \forall y\in Y,\ \forall\psi\in A'. \end{aligned} \tag{4.2}$$

First, let us observe that for each $y\in Y$, we have $Z = \{\psi\circ\pi(y) : \psi\in A'\}$, since A' contains the constant functions ψ_z for each $z\in Z$, where ψ_z is defined by $\psi_z(x) := z$ for all $x\in X$. Now, for each $y\in Y$, we define the map U_y as follows:

$$\begin{aligned} U_y : Z = \{\psi\circ\pi(y) : \psi\in A'\} &\to W\\ \psi\circ\pi(y) &\mapsto H_2(0,\psi)(y). \end{aligned}$$

This map is well defined. Indeed, let $\psi_1,\psi_2\in A'$. Suppose that $\psi_1\circ\pi(y) = \psi_2\circ\pi(y)$ or, equivalently, that $(\psi_1-\psi_2)\circ\pi(y) = 0$. By the formula (4.2),

$$\|H_2(0,\psi_1-\psi_2)(y)\|_W = \|(\psi_1-\psi_2)\circ\pi(y)\|_Z = 0.$$

Thus, by linearity, we have $H_2(0,\psi_1)(y) = H_2(0,\psi_2)(y)$. So, U_y is well defined and we can write, for all $y\in Y$ and $\psi\in A'$,

$$H_2(0,\psi)(y) = U_y(\psi\circ\pi(y)). \tag{4.3}$$

The fact that U_y is linear and isometric follows from the linearity of H_2 and the formula (4.2). For each $z\in Z$, the map $y\mapsto U_y(z)$ is continuous from Y into W, since $U_y(z) = H_2(0,\psi_z)(y)$ and $H_2(0,\psi_z)$ is an element of $C_b(Y,W)$. If furthermore H is surjective, then H_2 will also be surjective since it depends only on the second variable by the property (P). If moreover B' contains the constant maps, we deduce from (4.3) that U_y is also surjective. □

We consider now another example of norm which reveals the Volterra operators in a natural way in connection with isometries. We denote by $C_0^1(\mathbb{R})$ the space of all continuously differentiable functions ψ such that ψ and ψ' are uniformly bounded and $\psi(0) = 0$. We define the norm $\|\cdot\|'_{\infty,1}$ on $C_b(\mathbb{R})\times C_0^1(\mathbb{R})$ by

$$\|(\varphi,\psi)\|'_{\infty,1} := \sup_{x\in\mathbb{R}}\{|\varphi(x)| + |\psi'(x)|\}$$

for all $(\varphi,\psi)\in C_b(\mathbb{R})\times C_0^1(\mathbb{R})$ and we consider the norm $\|\cdot\|_{\infty,1}$ on $C_b(\mathbb{R})\times C_b(\mathbb{R})$.

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COROLLARY 4.11. *Let $H := (T, L) : C_b(\mathbb{R}) \times C_b(\mathbb{R}) \to C_b(\mathbb{R}) \times C_0^1(\mathbb{R})$ be a map such that $T : (C_b(\mathbb{R}), \|\cdot\|_\infty) \to (C_b(\mathbb{R}), \|\cdot\|_\infty)$ is an isomorphism and $L : C_b(\mathbb{R}) \to C_0^1(\mathbb{R})$ is a linear map. Then we have* (1) $\Leftrightarrow$ (2).

(1) *The linear map H is an isometry, that is, $\|H(\varphi,\psi)\|'_{\infty,1} = \|(\varphi,\psi)\|_{\infty,1}$ for all $(\varphi,\psi) \in C_b(\mathbb{R}) \times C_b(\mathbb{R})$.*

(2) *There exist a homeomorphism $\pi : \mathbb{R} \to \mathbb{R}$ and two constants $\epsilon \equiv \pm 1$ and $\lambda \equiv \pm 1$ such that for all $\varphi \in C_b(\mathbb{R})$ and all $x \in \mathbb{R}$,*

$$T\varphi(x) = \varepsilon\varphi \circ \pi(x) \quad \textit{and} \quad L\varphi(x) = \lambda \int_0^x \varphi \circ \pi(t)\,dt$$
$$= \epsilon\lambda \int_0^x T\varphi(t)\,dt.$$

PROOF. Let us denote by $D : C_0^1(\mathbb{R}) \to C_b(\mathbb{R})$ the linear map defined by $D\psi = \psi'$ for all $\psi \in C_0^1(\mathbb{R})$. So, the map $D \circ L : C_b(\mathbb{R}) \to C_b(\mathbb{R})$ is linear. Let $\tilde{H} := (T, D \circ L)$; then $\|H(\varphi,\psi)\|'_{\infty,1} = \|(\varphi,\psi)\|_{\infty,1}$ if and only if $\|\tilde{H}(\varphi,\psi)\|_{\infty,1} = \|(\varphi,\psi)\|_{\infty,1}$ for all $(\varphi,\psi) \in C_b(\mathbb{R}) \times C_b(\mathbb{R})$. By applying Corollary 4.10 to the map $\tilde{H}$, we obtain the existence of a homeomorphism $\pi : Y \to X$ and a continuous function $\varepsilon, \lambda : \mathbb{R} \to \{\pm 1\}$ such that $T\varphi(x) = \varepsilon(x)\varphi \circ \pi(x)$ and $(L\varphi)'(x) = (D \circ L)\varphi(x) = \lambda(x)\varphi \circ \pi(x)$ for all $x \in \mathbb{R}$ and $\varphi \in C_b(\mathbb{R})$. Since $\mathbb{R}$ is a connected space, $\epsilon \equiv \pm 1$ and $\lambda \equiv \pm 1$. It follows that $T\varphi(x) = \varepsilon\varphi \circ \pi(x)$ and $L\varphi(x) = \lambda \int_0^x \varphi \circ \pi(t)dt$ for all $x \in \mathbb{R}$ and all $\varphi \in C_b(\mathbb{R})$. □

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