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METRICS OF POSITIVE SCALAR CURVATURE ON SPHERICAL SPACE FORMS

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ABSTRACT. We use the eta invariant to show every non-simply connected spherical space form of dimension $m \geq 5$ has a countable family of non bordant metrics of positive scalar curvature.

0. **Introduction.** Let M be a compact Riemannian manifold of dimension m. One of the most elementary local invariants of M is the scalar curvature. Let $\mathcal{R}^+(M)$ be the space of metrics on M with positive scalar curvature. There are several ways to distinguish metrics. Following Gromov and Stolz, we say two metrics $g_i \in \mathcal{R}^+(M)$ are *concordant* if there exists $h \in \mathcal{R}^+(M \times I)$ which is product near the boundary such that $h|_{M\times\{i\}} = g_i$ for $i = 0, 1$. If g_0 and g_1 are in the same arc component of $\mathcal{R}^+(M)$, then g_0 and g_1 are concordant; it is not known if the converse holds. We shall define bordism of metrics presently; concordant metrics are necessarily bordant but the converse need not hold.

The group of diffeomorphisms Diff(M) acts naturally on the space of metrics of positive scalar curvature $\mathcal{R}^+(M)$; we denote the moduli space by

$$\mathcal{M}(M) := \mathcal{R}^+(M)/\operatorname{Diff}(M). \tag{0.1}$$

We say that M is a spherical space form if M is a compact manifold which admits a metric of constant sectional curvature +1; these manifolds have been classified by Wolf [19] and form a natural family to study.

THEOREM 0.1. *Let M be a spherical space form of dimension $m \geq 5$ which is not simply connected. There exists a countable family of metrics g_i on M of positive scalar curvature so that g_i is not concordant to g_j, so that g_i is not bordant to g_j, and so that g_i and g_j belong to different arc components of $\mathcal{M}(M)$ for $i \neq j$.*

It is possible to use the index theorem to construct metrics which are bordant but not concordant in a very general context, see for example Kreck and Stolz [12]. In this paper, we will use the eta invariant. At present, this is the only invariant which is known that distinguishes bordism classes of metrics. It uses the fundamental group in an essential fashion and has no counterpart in the simply connected context.

In [2], we used the eta invariant to detect non bordant metrics for spin manifolds with finite fundamental groups under certain conditions. Not all spherical space forms

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admit spin structures so the results of [2] do not suffice to prove Theorem 0.1. However, the spherical space forms do admit *twisted spin* structures in odd dimensions and *pin* structures in even dimensions. In this paper, we generalize the results of [2] to these structures in Theorems 0.2 and 0.3.

We learned in discussions with Stolz that he had independently considered these structures in a more general context; see [18]. We define these structures below. First recall some necessary definitions. Let G be a *finite* group henceforth and let $\sigma: M \longrightarrow \mathrm{BG}$ be a G structure on a closed manifold M of dimension m. If s is a spin structure on M, let

$$[(M, s, \sigma)] \in \mathrm{MSpin}_m(\mathrm{BG}) \tag{0.2}$$

be the bordism class. If we are given $g \in \mathcal{R}^+(M)$, then the bordism group $\mathrm{MSpin}_m^+(\mathrm{BG})$ is defined by introducing the equivalence relation $(M, g, s, \sigma) \simeq 0$ if there exists a compact Riemannian manifold N with boundary M such that σ and s extend over N and so that the metric g extends over N as a metric of positive scalar curvature which is product near the boundary. We say that metrics of positive scalar curvature g_0 and g_1 on a manifold M are *bordant* if

$$[(M, g_0, s, \sigma)] = [(M, g_1, s, \sigma)] \quad \text{in } \mathrm{MSpin}_m^+(\mathrm{BG}). \tag{0.3}$$

Now we define twisted spin structures. Let $\mathbf{Z}_2 = \{\pm 1\}$ be the multiplicative group with 2 elements. Let

$$1 \longrightarrow \mathbf{Z}_2 \longrightarrow \mathcal{G} \xrightarrow{\mu} G \longrightarrow 1. \tag{0.4}$$

be a central extension. This gives an action of $\mathbf{Z}_2$ on $\mathcal{G}$ by group multiplication. Define the twisted spinor group

$$\mathcal{J} = \mathcal{J}(\mathcal{G}, \mu, G) := \mathrm{Spin} \times_{\mathbf{Z}_2} \mathcal{G} \tag{0.5}$$

by identifying $(\theta, \lambda) = (-\theta, -\lambda)$ for $\theta \in \mathrm{Spin}$ and $\lambda \in \mathcal{G}$. The group Spin is a double cover of SO; $\mathcal{J}$ is a double cover of $\mathrm{SO} \times G$. A $\mathcal{J}$ structure on a manifold M is a lift of the transition functions of the tangent bundle of M from the special orthogonal group to $\mathcal{J}$. The bordism groups $\mathrm{M}\mathcal{J}_*$ and $\mathrm{M}\mathcal{J}_*^+$ are defined in the obvious way. In Theorem 1.1 we will show that M always admits a suitable twisted spin structure if M is orientable and if the universal cover of M admits a spin structure.

Let s be a $\mathcal{J}$ structure on M. The map μ defines an extension

$$1 \longrightarrow \mathrm{Spin} \longrightarrow \mathcal{J} \xrightarrow{\mu} G \longrightarrow 1. \tag{0.6}$$

Then s and μ induce a G structure $\check{\mu}: \pi_1(M) \longrightarrow G$; $\mathrm{M}\mathcal{J}_*$ is an equivariant bordism theory.

THEOREM 0.2. *Let G be a non-trivial finite group and let $\mathcal{J} = \mathcal{J}(\mathcal{G}, \mu, G)$. Let M be a connected manifold of odd dimension $m \geq 5$ which admits a $\mathcal{J}$ structure s so $\check{\mu}(s): \pi_1(M) \longrightarrow G$ is an isomorphism. Assume M admits at least one metric of positive scalar curvature.*

(1) *If $m \equiv 3$ (4), assume $\mathcal{G}$ contains an element $\lambda \neq \pm 1$ so that λ is not conjugate to either $-\lambda$ or to $-\lambda^{-1}$.*

(2) *If $m \equiv 1$ (4), assume $\mathcal{G}$ contains an element $\lambda \neq \pm 1$ so that λ is not conjugate to $-\lambda$ or to λ^{-1}.*

Then there exists a countable family of metrics g_i on M of positive scalar curvature so that g_i is not $\mathcal{J}$ bordant to g_j and so that g_i and g_j belong to different arc components of the moduli space $\mathcal{M}(M)$ for $i \neq j$.

REMARK. Suppose that M admits a spin structure s. Let $G = \pi_1(M)$ and let $\mathcal{G} = \mathbf{Z}_2 \oplus G$ be the trivial extension. Use the natural G structure on M to define a new structure $\mathcal{J}\tilde{s}$ with $\check{\mu}(\tilde{s})$ the identity map. The hypothesis of (1) hold trivially in this setting; the hypothesis of (2) hold if and only if G contains an element λ which is not conjugate to λ^{-1}. This shows that the results of [2] are a special case of Theorem 0.2.

Let M be a spherical space form of dimension m. If m is odd, then M admits a twisted spin structure with $G = \pi_1(M)$ and Theorem 0.1 follows from Theorem 0.2 in this case. However, if m is even and if $\pi_1(M) \neq 0$, then

$$M = \mathrm{RP}^m = S^m/\mathbf{Z}_2 \tag{0.7}$$

is not orientable and we must consider different structures. Recall that the orthogonal group is not connected and has two different simply connected universal covering groups ${}^{\pm}\mathrm{Pin}$. A manifold M admits a ${}^{+}$pin structure if and only if $w_2(M) = 0$; a manifold M admits a ${}^{-}$pin structure if and only if $(w_1^2 + w_2)(M) = 0$. Thus RP^{4k} admits a ${}^{+}$pin structure and RP^{4k+2} admits a ${}^{-}$pin structure; we refer to Giambalvo [6] for details. We define the bordism groups $\mathrm{M}({}^{\pm}\mathrm{Pin})_*$ and $\mathrm{M}({}^{\pm}\mathrm{Pin})_*^+$ in the obvious way. Let $\check{\mu}\colon \pi_1(M) \longrightarrow \mathbf{Z}_2$ be defined by the orientation line bundle. The remaining cases of Theorem 0.1 will follow from the following result.

THEOREM 0.3. *Let M be a connected manifold of even dimension $m = 2\ell \geq 6$ which admits at least one metric of positive scalar curvature. Let $\epsilon = (-1)^\ell$ and assume M admits a ${}^{\epsilon}$pin structure such that the orientation $\check{\mu}\colon \pi_1(M) \longrightarrow \mathbf{Z}_2$ is an isomorphism. Then there exists a countable family of metrics g_i on M of positive scalar curvature so that g_i is not ${}^{\epsilon}$Pin bordant to g_j and so that g_i and g_j belong to different arc components of $\mathcal{M}(M)$ for $i \neq j$.*

REMARK. We will show in Theorems 4.1 and 4.2 that the structures which satisfy the assumptions of Theorems 0.2 and 0.3 are exactly those where the eta invariant is non-zero.

Here is a brief outline to the paper. In Section 1, we will construct twisted spinor structures and show it is possible to push metrics of positive scalar curvature across suitable bordisms. In Section 2 we will use the Lichnerowicz formula [13] to show there are no harmonic twisted spinors or pinors if the metric has positive scalar curvature. We will use the Atiyah-Patodi-Singer index theorem [1] to lift the eta invariant from $\mathbf{R}/\mathbf{Z}$ to $\mathbf{R}$ and to define real valued bordism invariants. In Section 3, we will prove Theorems 0.1, 0.2 and 0.3 by pushing metrics across bordisms and by using the eta invariant to distinguish the resulting metrics. In Section 4, we will establish vanishing theorems for the eta invariant.

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1. **Generalized spin structures.** The following theorem shows that twisted spin structures arise naturally. Recall that if s is a $\mathcal{J}(\mathcal{G},\mu,G)$ structure, then $\check{\mu}\colon \pi_1(M) \longrightarrow G$ gives M a G structure.

THEOREM 1.1. *Let M be a connected oriented manifold with finite fundamental group whose universal cover admits a* spin *structure. Then there exists a canonical $\mathcal{J}\big(\mathcal{G},\mu,\pi_1(M)\big)$ structure s on M so that $\check{\mu}$ is the identity map. The extension $\big(\mathcal{G},\mu,\pi_1(M)\big)$ is split if and only if M admits a* spin *structure.*

PROOF. We prove the theorem by constructing s. Let $\tilde{M}$ be the universal cover of M and let $G = \pi_1(M)$ be the deck group. Lift the Riemannian metric on M to define a G invariant metric on $\tilde{M}$. Since M is orientable, the deck group action of G on $\tilde{M}$ is by orientation preserving isometries. Let PSO($\tilde{M}$) be the principal SO bundle of oriented orthonormal frames for the tangent bundle of $\tilde{M}$. Let PSPIN($\tilde{M}$) be the principal Spin bundle over $\tilde{M}$ defined by the spin structure $\tilde{s}$ on $\tilde{M}$. Let π be the associated double cover

$$\pi\colon \mathrm{PSPIN}(\tilde{M}) \longrightarrow \mathrm{PSO}(\tilde{M}). \tag{1.1}$$

If P is a principal bundle with structure group H, let $\mathcal{H}(P)$ be the group of diffeomorphisms of the total space P which commute with the action of H. Since $\tilde{M}$ is simply connected, the spin structure is unique and the map π of (1.1) induces a central $\mathbf{Z}_2$ extension

$$\mathbf{Z}_2 \longrightarrow \mathcal{H}\big(\mathrm{PSPIN}(\tilde{M})\big) \xrightarrow{\pi} \mathcal{H}\big(\mathrm{PSO}(\tilde{M})\big). \tag{1.2}$$

Let $\mathcal{G} = \pi^{-1}(G)$; the restriction of π to $\mathcal{G}$ defines a central extension

$$1 \longrightarrow \mathbf{Z}_2 \longrightarrow \mathcal{G} \xrightarrow{\mu} G \longrightarrow 1 \tag{1.3}$$

which is independent of the metric on M which is chosen; M is spin if and only if the extension (1.3) is split. Let $\mathcal{J} = \mathcal{J}(\mathcal{G},\mu,G)$.

We use the natural inclusion Spin $\longrightarrow \mathcal{J}$ to induce a $\mathcal{J}$ structure on $\tilde{M}$ from $\tilde{s}$; the associated principal bundle is defined by

$$P\mathcal{J}(\tilde{M}) := \mathrm{PSPIN}(\tilde{M}) \times_{\mathbf{Z}_2} \mathcal{J}. \tag{1.4}$$

The diagonal lift of G to $\mathcal{H}\big(\mathrm{PSPIN}(\tilde{M})\big) \times \mathcal{G}$ is not well defined since there is a $\mathbf{Z}_2$ ambiguity. It is, however, well defined in the quotient group $\mathcal{H}\big(P\mathcal{J}(\tilde{M})\big)$. If $\xi \in G$, let $\mu^{-1}(\xi) = \{\pm\lambda\} \subseteq \mathcal{G}$. We define s with the necessary properties by

$$s(\xi) := (\lambda,\lambda) = (-\lambda,-\lambda) \in \mathcal{H}\big(P\mathcal{J}(\tilde{M})\big). \tag{1.5}$$

■

We will use the following lemma to push metrics through a bordism.

LEMMA 1.2. *Let $\mathcal{J} = {}^{\pm}\mathrm{Pin}$ or let $\mathcal{J} = \mathcal{J}(\mathcal{G}, \mu, G)$ for $|G| < \infty$. Let M_i be closed manifolds of dimension $m \geq 5$ with $\mathcal{J}$ structures s_i so that*

$$[(M_1, s_1)] = [(M_2, s_2)] \quad in\ \mathrm{M}\mathcal{J}_m.$$

Assume that M_1 is connected and that $\check{\mu}_1 : \pi_1(M_1) \longrightarrow G$ is an isomorphism. Assume that M_2 admits a metric g_2 of positive scalar curvature. Then there exists a $\mathcal{J}$ bordism N and metrics of positive scalar curvature g_N on N and g_1 on M_1 so that (N, M_1) is 2 connected and so that

$$\partial(N, g_N, s_N) = (M_1, g_1, s_1) - (M_2, g_2, s_2).$$

PROOF. We generalize the argument of Miyazaki [14] and Rosenberg [15] to twisted spin structures to prove the Lemma. Let N be a bordism so

(1.6) $$\partial(N, s_N) = (M_1, s_1) - (M_2, s_2).$$

By taking connected sum, we may assume N is connected. Since M_1 is connected, (N, M_1) is 0-connected.

Let $\tilde{N} \longrightarrow N$ be the principal G bundle over N defined by $\check{\mu}$. Then

(1.7) $$\begin{array}{ccccccccc} 1 & \longrightarrow & \pi_1(\tilde{N}) & \longrightarrow & \pi_1(N) & \xrightarrow{\check{\mu}_N} & G & \longrightarrow & 1 \\ & & & & \uparrow{\scriptstyle (i_1)_*} & \circ & \uparrow{\scriptstyle =} & & \\ & & & & \pi_1(M_1) & \xrightarrow{\check{\mu}_1} & G & \longrightarrow & 1 \end{array}$$

Since $\tilde{N}$ is compact, $\pi_1(\tilde{N})$ is finitely generated. We choose embedded circles

(1.8) $$\alpha_i : S^1 \longrightarrow N$$

as generators for $\pi_1(\tilde{N})$ in $\pi_1(N)$. Since $\check{\mu}(\alpha_i) = 1$, the G structure defined by the $\mathcal{J}$ structure on the tubular neighborhood U_i of α_i in M is trivial. This is a critical point. If $\mathcal{J} = {}^{\pm}\mathrm{Pin}$, the $\check{\mu}$ structure is trivial means that the normal bundle is orientable. If $\mathcal{J} = \mathcal{J}(\mathcal{G}, \mu, G)$, the $\check{\mu}$ structure reflects the twisting by G. Thus in either case, the $\mathcal{J}$ structure on U_i is in fact a spin structure. Thus the normal bundle is trivial; we choose the trivialization

(1.9) $$U_i = S^1 \times D^{m-1}$$

to preserve the spin and hence the $\mathcal{J}$ structure. We can now do surgery on the α_i to kill $\pi_1(\tilde{N})$ while preserving the $\mathcal{J}$ structure. This shows that we may choose the $\mathcal{J}$ bordism so (N, M_1) is 1-connected.

Since $\pi_1(N) = \pi_1(M)$ is finite, $\pi_2(N)$ is finitely generated. We can choose embedded spheres $\beta_i : S^2 \longrightarrow N$ as generators for $\pi_2(N)$. Since S^2 is simply connected, the $\mathcal{J}$ structure on the normal neighborhood U_i of β_i is in fact a spin structure. Therefore the normal bundle has trivial second Stieffel-Whitney class. This implies the normal bundle is trivial so $U_i = S^2 \times D^{m-2}$. We do surgery on U_i to kill $\pi_2(N)$ and choose the $\mathcal{J}$ bordism so (N, M_1) is 2-connected.

We can choose a handle body decomposition so that N is obtained from M_1 by attaching handles of dimension at least 3; dually, N is obtained from M_2 by attaching handles of codimension at least 3. Results of Gajer [5], Gromov and Lawson [10], and Schoen and Yau [17] then permit us to push the metric of positive scalar curvature from M_2 through the bordism N to define the desired metric on M_1. ∎

REMARK. The referee has pointed out to us that this result may also be derived from the Bordism Theorem 3.3 of Rosenberg and Stolz [16]; we have presented a self-contained argument here for the convenience of the reader and to illustrate the geometry involved.

We shall need the following technical lemma later.

LEMMA 1.3. *(a) If m is odd,* $|\mathrm{MSpin}_m(B\mathbf{Z}_n)| < \infty$.

(b) Let $m = 2\ell$ and let $\epsilon = (-1)^\ell$. If m is even and if s is a ${}^\epsilon\mathrm{pin}$ *structure on* RP^m, *then* $[(\mathrm{RP}^m, s)]$ *is an element of finite order in* $\mathrm{M}({}^\epsilon\mathrm{Pin})_m$.

PROOF. Assertion (a) follows as the reduced bordism groups are rationally trivial for any space whose rational reduced homology groups vanish; assertion (b) follows from Giambalvo [6, Theorem 3.4] who computed the orders of RP^{2k} in the appropriate pin bordism groups. ∎

2. **Analytic bordism invariants.** In this section, we use the eta invariant to construct bordism invariants. We first review some facts concerning operators of Dirac type. Let M be a compact Riemannian manifold without boundary of dimension $m \geq 2$, let V be a smooth vector bundle over M, and let Q be an operator of Dirac type on the space of smooth sections to V. For $\mathrm{Re}(z) >> 0$, let

$$\eta(z, Q) := \mathrm{Tr}_{L^2}\big(Q(Q^2)^{-(z+1)/2}\big) \tag{2.1}$$

be the eta function defined by Atiyah et.al. [1]. This has a meromorphic extension to $\mathbf{C}$. The origin is a regular value and we define the following measure of the spectral asymmetry of Q:

$$\eta(Q) := \frac{1}{2}\{\eta(z, Q) + \dim\ker(Q)\}|_{z=0}. \tag{2.2}$$

Let X be a compact oriented manifold with smooth (possibly empty) boundary Y. Let $\mathcal{J} = \mathcal{J}(\mathcal{G}, \mu, G)$ and let s be a $\mathcal{J}$ structure on X. Let α be a unitary representation of $\mathcal{G}$ so that $\alpha(-\lambda) = -\alpha(\lambda)$ for all $\lambda \in \mathcal{G}$; such representations always exist. Let Δ be the fundamental spin representation; $\Delta(-\theta) = -\Delta(\theta)$ for $\theta \in \mathrm{Spin}$. Since

$$\Delta(\theta) \otimes \alpha(\lambda) = \Delta(-\theta) \otimes \alpha(-\lambda) \quad \forall \theta \in \mathrm{Spin},\ \lambda \in \mathcal{G}, \tag{2.3}$$

$\Delta \otimes \alpha$ extends to a unitary representation of $\mathcal{J}$. Let $W(X, s, \Delta, \alpha)$ be the unitary bundle over X defined by s; this is a bundle of twisted spinors. Let $\mathcal{C}(X)$ be the Clifford module bundle of the tangent bundle; Clifford multiplication gives $W(X, s, \Delta, \alpha)$ a natural $\mathcal{C}(X)$ module structure γ. Since the principal bundle $P\mathcal{J}(X)$ is a finite cover of $P\,\mathrm{SO}(X)$, the Levi-Civita connection lifts to define a connection ∇ on $P\mathcal{J}$ and on $W(X, s, \Delta, \alpha)$. We define the Dirac operator on $C^\infty\big(W(X, s, \Delta, \alpha)\big)$ by the formula

$$D(X, g_X, s, \alpha) := \gamma \circ \nabla. \tag{2.4}$$

Similarly, if X admits a $^\pm$pin structure s, let Δ be the pinor representation, let W be the bundle of pinors, let ∇ be the $^\pm$pin connection on W, and let γ give W a $\mathcal{C}(X)$ module structure; see Section 4 for further details. We define the Dirac operator on $C^\infty(W)$ by the formula

$$D(X, g_X, s) := \gamma \circ \nabla. \tag{2.5}$$

If the boundary of X is empty, then $D = D(X, g_X, s, \alpha)$ or $D = D(X, g_X, s)$ is a self-adjoint first order elliptic partial differential operator. If the boundary Y of X is non-empty, we impose Atiyah-Patodi-Singer boundary conditions. Let $\vec{\nu}$ be the inward pointing unit normal. Near Y, we decompose

$$D = \gamma_{\vec{\nu}}(\nabla_{\vec{\nu}} + A) \tag{2.6}$$

where A is a self-adjoint tangential operator of Dirac type. Let Π be the spectral projection on the non-negative eigenspaces of A; $\phi \in C^\infty\big(W(\cdot)\big)$ satisfies our boundary conditions if $\Pi(\phi|_Y) = 0$.

LEMMA 2.1. *Let s be a $\mathcal{J}$ structure on X for $\mathcal{J} = \mathcal{J}(\mathcal{G}, \mu, G)$ or $\mathcal{J} =$* $^\pm$pin. *Let g_X be a metric of positive scalar curvature on X; if $\partial X \neq \emptyset$ we assume g_X is product near the boundary. We impose Atiyah-Patodi-Singer boundary conditions on the operator D of Dirac type defined above. Then* $\ker(D) = \{0\}$.

PROOF. Let R_X be the scalar curvature of g_X. Locally, the twisting defined by α if $\mathcal{J} = \mathcal{J}(\mathcal{G}, \mu, G)$ or by the orientation if $\mathcal{J} = {}^\pm$pin does not play a role so we can use the Lichnerowicz formula [13] to see that if $\phi \in C^\infty\big(W(\cdot)\big)$, then

$$D^2\phi = -\operatorname{Tr}(\nabla^2\phi) + \frac{1}{4}R_X\phi. \tag{2.7}$$

Let $D\phi = 0$, let dx be the Riemannian measure on X, and let dy be the Riemannian measure on the boundary Y. We integrate by parts to compute that

$$\begin{aligned} 0 &= \int_X (D^2\phi, \phi)\,dx \\ &= \int_X \{(\nabla\phi, \nabla\phi) + \frac{1}{4}R_X(\phi, \phi)\}\,dx + \int_Y (\nabla_{\vec{\nu}}\phi, \phi)\,dy. \end{aligned} \tag{2.8}$$

In the usual argument that there are no harmonic spinors, the boundary integral does not appear. Here we must control the sign of this term, the boundary conditions are critical for this. Since $D\phi = 0$, $(\phi_{;\vec{\nu}})|_Y = -A(\phi|_Y)$. Since $\Pi(\phi|_Y) = 0$,

$$\int_Y (A\phi, \phi)\,dy \le 0 \quad \text{so} \quad \int_Y (\phi_{;\vec{\nu}}, \phi)\,dy \ge 0. \tag{2.9}$$

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Since $R_X > 0$, all the terms appearing in (2.8) are non-negative so $\phi = 0$. ∎

Let $\mathrm{Cl}(\mathcal{G})$ denote the ring of complex class functions on $\mathcal{G}$. The map $\alpha \longrightarrow \mathrm{Tr}(\alpha)$ associates to each representation the corresponding character; the characters of the irreducible representations of $\mathcal{J}$ form a basis for $\mathrm{Cl}(\mathcal{G})$; we shall identify a representation with its character henceforth. Let

$$\text{(2.10)} \qquad \mathrm{Cl}_0(\mathcal{G}) := \{\phi \in \mathrm{Cl}(\mathcal{G}) : \phi(1) = 0\}.$$

There are two different actions of $\mathbf{Z}_2 = \{\pm 1\}$ on $\mathcal{G}$ defined by $g \longrightarrow \pm g$ and $g \longrightarrow g^{\pm 1}$ which induce corresponding $\mathbf{Z}_2$ module structures on $\mathrm{Cl}(\mathcal{G})$ and $\mathrm{Cl}_0(\mathcal{G})$. We use the first action to decompose

$$\text{(2.11)} \qquad \mathrm{Cl}(\mathcal{G}) = \mathrm{Cl}^+(\mathcal{G}) \oplus \mathrm{Cl}^-(\mathcal{G})$$

where we define

$$\text{(2.12)} \qquad \begin{aligned} \mathrm{Cl}^{\pm}(\mathcal{G}) &:= \{\phi \in \mathrm{Cl}(\mathcal{G}) : \phi(-\lambda) = \pm\phi(\lambda)\ \forall \lambda \in \mathcal{G}\}, \\ \mathrm{Cl}_0^{\pm}(\mathcal{G}) &:= \mathrm{Cl}_0(\mathcal{G}) \cap \mathrm{Cl}^{\pm}(\mathcal{G}). \end{aligned}$$

Let s be a $\mathcal{J} = \mathcal{J}(\mathcal{G}, \mu, G)$ structure on M and let g be a metric of positive scalar curvature on M. If α satisfies $\alpha(-\lambda) = -\alpha(\lambda)$, we may extend $\Delta \otimes \alpha$ to a representation of $\mathcal{J}$ and define the associated eta invariant

$$\text{(2.13)} \qquad \eta(M, g, s, \alpha) := \eta\big(D(M, g, s, \alpha)\big) \in \mathbf{R}.$$

Since the eta invariant is real valued and additive with respect to direct sums, we may extend the eta invariant to a linear map

$$\text{(2.14)} \qquad \eta(M, g, s, \cdot) \colon \mathrm{Cl}^-(\mathcal{G}) \longrightarrow \mathbf{C}.$$

The class functions in $\mathrm{Cl}^+(\mathcal{G})$ play no role since the corresponding representations do not extend to $\mathcal{J}$.

LEMMA 2.2. *Let m be odd and let $\mathcal{J} = \mathcal{J}(\mathcal{G}, \mu, G)$. The eta invariant extends to a map in bordism*

$$\eta \colon \mathrm{M}\mathcal{J}_m^+ \otimes_{\mathbf{Z}} \mathrm{Cl}_0^-(\mathcal{G}) \longrightarrow \mathbf{C}.$$

REMARK. In Section 4, we will decompose $\mathrm{Cl}_0^-(\mathcal{G}) = \mathcal{A}(\mathcal{G}) \oplus \mathcal{B}(\mathcal{G})$ into the ± 1 eigenspaces with respect to the $\mathbf{Z}_2$ module structure induced by the action $g \longrightarrow g^{\pm 1}$. We will show in Lemma 4.4 that η is trivial on $\mathcal{B}$ if $m \equiv 3\ (4)$ and trivial on $\mathcal{A}$ if $m \equiv 1\ (4)$.

PROOF. We use the index theorem for manifolds with boundary of Atiyah, Patodi, and Singer [1]. Let $[(M, g, s)] = 0$ in $\mathrm{M}\mathcal{J}_m^+$. This means there is a manifold N whose boundary is M so that the structure s extends over N and so the metric g extends over N as a metric of positive scalar curvature g_N which is product near the boundary M.

We must show that if α_i are representations of $\mathcal{G}$ so that $\dim(\alpha_1) = \dim(\alpha_2)$ and so that $\alpha_i(-\lambda) = -\alpha_i(\lambda)$, then

$$\eta\big(D(M,g,s,\alpha_1)\big) - \eta\big(D(M,g,s,\alpha_2)\big) = 0. \tag{2.15}$$

We decompose the bundle W over N into the half spin bundles

$$W(N,s,\Delta,\alpha_i) = W(N,s,\Delta^+,\alpha_i) \oplus W(N,s,\Delta^-,\alpha_i) \tag{2.16}$$

and decompose the corresponding Dirac operators over N in the form

$$\begin{aligned} D(N,g_N,s,\alpha_i) &= D^+(N,g_N,s,\alpha_i) + D^-(N,g_N,s,\alpha_i) \quad \text{for} \\ D^{\pm}\colon C^\infty\big(W(N,s,\Delta^{\pm},\alpha_i)\big) &\to C^\infty\big(W(N,s,\Delta^{\mp},\alpha_i)\big). \end{aligned} \tag{2.17}$$

Near the boundary M, we decompose the corresponding tangential operators

$$\begin{aligned} A(N,g_N,s,\alpha_i) &= A_i^+ + A_i^- \quad \text{for} \\ A^{\pm}\colon C^\infty\big(W(N,s,\Delta^{\pm},\alpha_i)\big)\big|_M &\to C^\infty\big(W(N,s,\Delta^{\pm},\alpha_i)\big)\big|_M. \end{aligned} \tag{2.18}$$

We identify $W(N,s,\Delta^{\pm},\alpha_i)|_M = W(M,s,\Delta,\alpha_i)$; under this identification

$$A_i^{\pm} = \epsilon D(M,g_M,s,\alpha_i) \tag{2.19}$$

where $\epsilon = \epsilon(m) = \pm 1$ is universally defined and reflects certain normalizing sign conventions which are not relevant to our argument. The boundary conditions decouple and define elliptic boundary conditions for $D^{\pm}$. We use Lemma 2.1 to see that $\ker(A) = \{0\}$ so

$$(D^{\pm})^* = D^{\mp}. \tag{2.20}$$

Let $\hat{A}(g_N)$ be the differential form on N defined by the metric g_N whose representative in de Rham cohomology gives the $\hat{A}$-genus. The index theorem of Atiyah, Patodi, and Singer [1] then yields

$$\operatorname{index}\{D^+(N,g_N,s,\alpha_i)\} = \dim(\alpha_i)\int_N \hat{A}(g_N) - \epsilon\eta(M,g,s,\alpha_i). \tag{2.21}$$

Since $\dim(\alpha_1) - \dim(\alpha_2) = 0$,

(2.22)

$$-\epsilon\{\eta(M,g,s,\alpha_1) - \eta(M,g,s,\alpha_2)\} = \operatorname{index}\{D^+(N,g_N,s,\alpha_1)\} - \operatorname{index}\{D^+(N,g_N,s,\alpha_2)\}.$$

By Lemma 2.1, there are no harmonic spinors and the index vanishes since g_N is a metric of positive scalar curvature. ∎

If s is a $^{\pm}$pin structure on an even dimensional manifold M, let

$$\eta(M,g,s) := \eta\big(D(M,g,s)\big) \in \mathbf{R}. \tag{2.23}$$

LEMMA 2.3. *Let m be even. The eta invariant extends to a map in bordism*

$$\eta\colon \mathrm{M}(^{\pm}\mathrm{Pin})_m^+ \to \mathbf{R}.$$

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PROOF. We use the $^{\pm}$pin complex; this elliptic complex is defined for odd dimensional $^{\pm}$pin manifolds with even dimensional boundary. Since the dimension of N is odd, there is no interior integrand in the Atiyah Patodi Singer index formula;

$$\operatorname{index}\{D^{+}(N, g_N, s)\} = \epsilon\eta(M, g, s). \tag{2.24}$$

The theorem follows since there are no harmonic pinors by Lemma 2.1. ∎

REMARK. The crucial difference between $\mathcal{J}(\mathcal{G}, \mu, G)$ and $^{\pm}$pin lies in (a) the parity of the dimensions and (b) there is no need to twist with a representation of virtual dimension 0.

3. **Constructing metrics of positive scalar curvature.** Let M admit a $\mathcal{J}$ structure s and a metric of positive scalar curvature g. If $\mathcal{J} = \mathcal{J}(\mathcal{G}, \mu, G)$ and if m is odd, we say the eta invariant of (M, g, s) is non-zero if there exists $\phi \in \mathrm{Cl}_0^-(\mathcal{G})$ so that $\eta(M, g, s, \phi) \neq 0$. If $\mathcal{J} = {}^{\pm}$pin and if m is even, we say the eta invariant of (M, g, s) is non-zero if $\eta(M, g, s) \neq 0$.

THEOREM 3.1. *Let $\mathcal{J} = \mathcal{J}(\mathcal{G}, \mu, G)$ for G finite and let m be odd or let $\mathcal{J} = {}^{\pm}$Pin and let m be even. Assume there exists $(\check{M}, \check{g}, \check{s})$ so that $[(\check{M}, \check{s})]$ has finite order in $\mathrm{M}\mathcal{J}_m$ and so that the eta invariant of $(\check{M}, \check{g}, \check{s})$ is non zero. Let M be a connected manifold of dimension $m \geq 5$ which admits a $\mathcal{J}$ structure so that $\check{\mu}$ is an isomorphism. Assume M admits a metric of positive scalar curvature. Then there exists a countable family of metrics g_i on M of positive scalar curvature so that g_i is not $\mathcal{J}$ bordant to g_j and so that g_i and g_j belong to different arc components of $\mathcal{M}(M)$ for $i \neq j$.*

PROOF. Let $\mathcal{J} = \mathcal{J}(\mathcal{G}, \mu, G)$; the argument for $\mathcal{J} = {}^{\pm}$Pin is similar. Choose $\phi \in \mathrm{Cl}_0^-(\mathcal{G})$ so that

$$\eta(\check{M}, \check{g}, \check{s}, \phi) \neq 0. \tag{3.1}$$

Choose ν so $\nu[(\check{M}, \check{s})] = 0$ in $\mathrm{M}\mathcal{J}_m$. Let $i \in \mathbf{Z}$. Use Lemma 1.2 to push the metric of positive scalar curvature on

$$(M, g_0, s) \,\dot{\sqcup}\, \nu i(\check{M}, \check{g}, \check{s}) \tag{3.2}$$

across a bordism

$$[(M, s)] = [(M, s)] + \nu i[(\check{M}, \check{s})] \tag{3.3}$$

to define a new metric g_i of positive scalar curvature on M. Then

$$\begin{aligned} &\eta(M, g_i, s, \phi) = \eta(M, g_0, s, \phi) + \nu i\eta(\check{M}, \check{g}, \check{s}, \phi) \quad \text{so} \\ &\eta(M, g_j, s, \phi) - \eta(M, g_j, s, \phi) = \nu(i - j)\eta(\check{M}, \check{g}, \check{s}, \phi) \neq 0 \quad \text{for } i \neq j. \end{aligned} \tag{3.4}$$

This shows that g_i is not bordant to g_j for $i \neq j$.

Let $\mathrm{Diff}^{\mathcal{J}}(M)$ be the subgroup of diffeomorphisms of M which preserve the $\mathcal{J}$ structure, the orientation, and act as the identity on the fundamental group. It is immediate from the definition that

$$(3.5) \qquad \eta(M, h^*g, h^*s, \phi) = \eta(M, g, s, \phi).$$

If $h \in \mathrm{Diff}^{\mathcal{J}}(M)$, then $h^*s = s$. Consequently h^*g_i and h^*g_j give different eta invariants and hence are not concordant metrics for $i \neq j$. Consequently g_i and g_j are in different arc components of $\mathcal{R}^+(M)/\,\mathrm{Diff}^{\mathcal{J}}(M)$ for $i \neq j$.

Since M is compact and since $\mathcal{J}$ is a finite extension of SO, the set of inequivalent $\mathcal{J}$ structures on M is finite. Since the fundamental group of M is finite, $\mathrm{Diff}^{\mathcal{J}}(M)$ is a subgroup of finite index of $\mathrm{Diff}(M)$. The theorem now follows. ∎

PROOF OF THEOREM 0.3. We embed ${}^{\epsilon}$pin in the complexification ^{c}pin to define a ^{c}pin structure on $\mathrm{RP}^{2\ell}$; every ^{c}pin structure on $\mathrm{RP}^{2\ell}$ is the complexification of a ${}^{\epsilon}$pin structure. We computed the eta invariant of the ^{c}pin operator in [7, Lemma 1.3.8 and Theorem 3.2.12]; this operator is the ${}^{\epsilon}$pin operator for $\mathrm{RP}^{2\ell}$. We showed that that if the ${}^{\epsilon}$pin structure s on $\mathrm{RP}^{2\ell}$ is chosen suitably, then

$$(3.6) \qquad \eta(\mathrm{RP}^{2\ell}, s) = 2^{-\ell-1} \neq 0.$$

By Lemma 1.3, $(\mathrm{RP}^{2\ell}, s)$ has finite order in $\mathrm{M}({}^{\epsilon}\mathrm{Pin})$. Since $\mathrm{RP}^{2\ell}$ admits a metric of constant positive scalar curvature, Theorem 0.3 follows from Theorem 3.1. ∎

REMARK. There are two inequivalent ${}^{\epsilon}$pin or ^{c}pin structures on $\mathrm{RP}^{2\ell}$ which are related by twisting with the orientation line bundle. This twisting replaces the pinor operator D by $-D$ and changes the sign of the eta invariant but not the fact that it is non-zero.

PROOF OF THEOREM 0.2. Let $H = \langle\lambda\rangle$ be the cyclic subgroup of $\mathcal{G}$ generated by λ. The natural map Φ: Spin $\times H \longrightarrow \mathcal{J}$ induces a natural map of bordism groups

$$(3.7) \qquad \Phi\colon \mathrm{MSpin}_m(BH) \longrightarrow \mathrm{M}\mathcal{J}_m.$$

By Lemma 1.3, $\mathrm{MSpin}_m(BH)$ is finite. Thus if $[(M, s, \sigma)] \in \mathrm{MSpin}_m(BH)$, then $\Phi\big([(M, s, \sigma)]\big)$ has finite order in $\mathrm{M}\mathcal{J}_m$. If $\alpha \in \mathrm{Cl}_0^-(\mathcal{G})$, let $r(\alpha) \in \mathrm{Cl}_0^-(H)$ be the restriction. Then

$$(3.8) \qquad \eta\big(M, s, \sigma, r(\alpha)\big) = \eta\big(\Phi(M, s, \sigma), \alpha\big).$$

We complete the proof by finding M which admits a metric of positive scalar curvature and which has non-vanishing eta invariant.

Suppose first $m = 4k - 1$. Choose $\pm 1 \neq \lambda \in \mathcal{G}$ so λ is not conjugate to $-\lambda$ or $-\lambda^{-1}$. This implies λ^{-1} is not conjugate to $-\lambda$ or to $-\lambda^{-1}$. Thus we can find $\alpha \in \mathrm{Cl}_0^-(\mathcal{G})$ which takes values in $\{0, \pm 1\}$ such that

$$(3.9) \qquad \alpha(\lambda) = \alpha(\lambda^{-1}) = 1.$$

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Let $I_{2k} \in U(2k)$ be the identity matrix, let n be the order of λ, and let

$$\tau(\lambda) := e^{2\pi i/n} \cdot I_{2k}\colon H \longrightarrow U(2k) \tag{3.10}$$

define a fixed point free representation of $H = \langle\lambda\rangle$. Let

$$M := S^{4k-1}/\tau(H) \tag{3.11}$$

be the resulting spherical space form. The manifold M admits a unique spin structure if n is odd and admits two inequivalent spin structures if n is even, see for example [3, Section 4]. We used a suitably chosen spin structure and the natural H structure σ together with work of Donnelly [4] in [2, Lemma 2.2] to show that

$$\eta(M,s,\sigma,\alpha) = n^{-1} \sum_{1\leq j<n} \alpha(\lambda^j)|1 - e^{2\pi ij/n}|^{-2k}. \tag{3.12}$$

We also showed, [2, Lemma 4.1], that if $\ell \geq 3$, then

$$\sum_{2\leq j\leq n-2} |1 - e^{2\pi ij/n}|^{-\ell} < 2|1 - e^{2\pi i/n}|^{-\ell}. \tag{3.13}$$

Since $\alpha(\lambda) = \alpha(-\lambda) = 1$ and since $\alpha(\lambda^j) \in \{0, \pm 1\}$, the values in (3.12) when $j = 1$ and $j = n - 1$ swamp the remaining values so η is non-zero. This completes the proof if $m = 4k - 1$.

Suppose next $m = 4k+1$. Choose $\pm 1 \neq \lambda \in \mathcal{G}$ so λ is not conjugate to λ^{-1} or to $-\lambda$; this implies λ^{-1} is not conjugate to λ or to $-\lambda^{-1}$. Thus we can find $\alpha \in \mathrm{Cl}_0^-(\mathcal{J})$ taking values in $\{0, \pm 1\}$ so that

$$\alpha(\lambda) = 1 \quad \text{and} \quad \alpha(\lambda^{-1}) = -1. \tag{3.14}$$

Let L be the tautological complex line bundle over CP^1 and let 1^ν be the trivial complex bundle of dimension ν over CP^1. Let

$$\tilde{M} := S(L \oplus L \oplus 1^{2(k-1)}) \tag{3.15}$$

be the sphere bundle of fiber dimension $4k - 1$ over CP^1. We use the complex structure on $L \oplus L \oplus 1^{2(k-1)}$ to define a natural S^1 action on $\tilde{M}$. By shrinking the size of the fiber spheres, we can give $\tilde{M}$ a metric of positive scalar curvature which is S^1 invariant. Since the underlying real vector bundle of $L \oplus L$ is trivial, $\tilde{M}$ is diffeomorphic to $\mathrm{CP}^1 \times S^{4k-1}$ and admits a unique spin structure s.

Let n be the order of λ and let $\tau(\lambda) = e^{2\pi i/n}$ define a fixed point free action of $H = \langle\lambda\rangle$ on $\tilde{M}$. Let

$$M := \tilde{M}/\tau(H). \tag{3.16}$$

We give M the natural spin structure s and H structure σ, see [3, Section 4]. We complete the proof of Theorem 0.2 by showing the eta invariant of M is non-zero. Let

$$\beta(j,n,k) := (1 + e^{2\pi ij/n})(1 - e^{2\pi ij/n})^{-1}|1 - e^{2\pi ij/n}|^{-2k}.$$

In [2, Lemma 4.2] we showed that if $m = 5$ and $k = 2$, then

$$\eta(M, s, \sigma, \alpha) = -n^{-1} \sum_{1 \le j \le n-1} \alpha(\lambda^j)\beta(j, n, k). \tag{3.17}$$

A similar calculation shows that this identity holds for general k. Since $\alpha(\lambda) = -\alpha(-\lambda) = 1$, since $\alpha(\lambda^j) \in \{0, \pm 1\}$, and since $\beta(j, n, k) = -\beta(n-j, n, k)$ we may use (3.13) to see η is non-zero. ∎

PROOF OF THEOREM 0.1. Let M be a spherical space form of dimension $m \geq 5$ with non trivial fundamental group. If $m \equiv 0$ (4), then $M = \mathrm{RP}^m$ admits a $^+$pin structure; if $m \equiv 2$ (4), then $M = \mathrm{RP}^m$ admits a $^-$pin structure. Therefore Theorem 0.1 follows from Theorem 0.3 if m is even. If $m \equiv 3$ (4), then M is spin and the conditions of Theorem 0.2 are satisfied. If $m \equiv 1$ (4) and if $|G|$ is odd, then M is spin and no non-trivial element of G is conjugate to its inverse so the conditions of Theorem 0.2 are satisfied. If $m \equiv 1$ (4) and if $|G|$ is even, then the 2-Sylow subgroup G_2 of $\pi_1(M)$ is cyclic and non-trivial; see Wolf [19] for details. We use Theorem 1.1 to give M a canonical $\mathcal{J} = \mathcal{J}(\mathcal{G}, \mu, G)$ structure. Since M is not spin, the extension is not split so the 2-Sylow subgroup $\mathcal{G}_2 = \mu^{-1}(G_2)$ of $\mathcal{G}$ is cyclic. It now follows that no two different elements of $\mathcal{G}_2$ are conjugate in $\mathcal{G}$ and the conditions of Theorem 0.2 are satisfied. Thus Theorem 0.1 follows from Theorem 0.2 if m is odd. ∎

4. **Vanishing theorems for the eta invariant.**

The hypothesis of Theorems 0.2 and 0.3 are exactly those where the eta invariant is non-zero. More precisely

THEOREM 4.1. *Let M be a closed manifold of dimension m which admits a metric of positive scalar curvature and a $\mathcal{J} = \mathcal{J}(\mathcal{G}, \mu, G)$ structure s for G finite.*

(1) If $m \equiv 3$ (4), assume every element $\lambda \neq \pm 1$ of $\mathcal{G}$ is conjugate to either $-\lambda$ or to $-\lambda^{-1}$.

(2) If $m \equiv 1$ (4), assume every element $\lambda \neq \pm 1$ of $\mathcal{G}$ is conjugate to either $-\lambda$ or to λ^{-1}.

Then $\eta(M, g, s, \alpha) = 0$ for all $\alpha \in \mathrm{Cl}_0^-(\mathcal{G})$.

THEOREM 4.2. *Let M be a closed manifold of dimension m which admits a metric of positive scalar curvature and a $^\epsilon$pin structure s.*

(1) If $m \equiv 0$ (4), assume s is a $^-$pin structure.

(2) If $m \equiv 2$ (4), assume s is a $^+$pin structure.

Then $\eta(M, g, s) = 0$.

We shall need several technical lemmas to prove Theorems 4.1 and 4.2. We shall say $D \simeq \pm\bar{D}$ if there is a natural unitary equivalence between these two operators.

LEMMA 4.3. *Let D be the (s)pinor operator defined by a $\mathcal{J}$ structure s on M.*

(a) If $(m, \mathcal{J}) \in \{(4k-1, \mathrm{Spin}), (4k, {}^+\mathrm{Pin}), (4k+2, {}^-\mathrm{Pin})\}$, then $\bar{D} \simeq D$.

(b) If $(m, \mathcal{J}) \in \{(4k+1, \mathrm{Spin}), (4k+2, {}^{+}\mathrm{Pin}), (4k, {}^{-}\mathrm{Pin})\}$, *then* $\bar{D} \simeq -D$.

PROOF. Let $m = 2\ell - 1$. There are two inequivalent irreducible left $\mathcal{C}(\mathbf{R}^m)$ modules $(\Delta_m^{\pm}, \gamma_m^{\pm})$. If ω is the orientation class, then

(4.1) $$\gamma_m^{\pm}(\omega) = \pm\epsilon(\ell) \cdot I$$

where $\epsilon(\ell) = \pm 1$ if ℓ is even and $\epsilon(\ell) = \pm i$ if ℓ is odd; the precise sign convention is irrelevant here. Thus

(4.2) $$(\Delta_m^{+}, -\gamma_m^{+}) \simeq (\Delta_m^{-}, \gamma_m^{-}).$$

Furthermore, if we take the complex conjugate,

(4.3) $$(\bar{\Delta}_m^{\pm}, \bar{\gamma}_m^{\pm}) = \begin{cases} (\Delta_m^{+}, \gamma_m^{+}) & \text{if } \ell \equiv 0\ (2), \\ (\Delta_m^{-}, \gamma_m^{-}) = (\Delta_m^{+}, -\gamma_m^{+}) & \text{if } \ell \equiv 1\ (2). \end{cases}$$

Since the spin operator is defined by

(4.4) $$D = \gamma_m^{+} \circ \nabla^{+},$$

we conclude $\bar{D} \simeq D$ if $m = 4k - 1$ and $\bar{D} \simeq -D$ if $m = 4k + 1$. This completes the proof of the lemma if m is odd.

Let $m = 2\ell$ and let $\epsilon = \pm$ be a choice of sign; the Clifford algebra $\mathcal{C}^{\epsilon}(\mathbf{R}^{\nu})$ is defined by the identity

(4.5) $$v * w + w * v = \epsilon(v, w) \cdot I.$$

In our previous discussion, we have taken $\epsilon = -$ so $\mathcal{C} = \mathcal{C}^{-}$; this distinction was inessential previously since we were working with spinors but becomes crucial now that we are working with pinors. Let ${}^{\epsilon}\mathrm{Pin}$ be the subgroup of $\mathcal{C}^{\epsilon}(\mathbf{R}^m)$ generated by the unit sphere of $\mathbf{R}^m$ under Clifford multiplication. Let χ be the orientation representation; this is defined by

(4.6) $$\chi(\theta) := \begin{cases} 1 & \text{if } \theta \in \mathrm{Spin}, \\ -1 & \text{if } \theta \in {}^{\epsilon}\mathrm{Pin} - \mathrm{Spin}. \end{cases}$$

The canonical representation ρ of ${}^{\epsilon}\mathrm{Pin}$ on $\mathbf{R}^m$ is then defined by

(4.7) $$\rho(\theta)v := \chi(\theta)\theta * v * \theta^{-1}.$$

Let $\sigma(\theta)\xi = \theta * \xi$ define the canonical representation of ${}^{\epsilon}\mathrm{Pin}$ on $\mathcal{C}^{\epsilon}$.

Let $\{e_i\}$ be an orthonormal basis for $\mathbf{R}^m$ and let

(4.8) $$\omega := i^{\ell+1} e_1 * \cdots * e_m \in \mathcal{C}^{\epsilon}(\mathbf{R}^m).$$

Then $\omega^2 = -1$ and $\theta * \omega = \chi(\theta)\omega * \theta$. Let

(4.9) $$\gamma(v)\xi := v * \omega * \xi$$

define a map $\gamma\colon \mathbf{R}^m \longrightarrow \mathrm{End}(\mathcal{C})$. We compute that

$$\begin{aligned}\gamma(v)\gamma(w)+\gamma(w)\gamma(v) &= v*\omega*w*\omega+w*\omega*v*\omega\\ &= -\omega^2\{v*w+w*v\}=\epsilon(v,w)\cdot I.\end{aligned} \tag{4.10}$$

This shows γ defines a $\mathcal{C}^\epsilon$ module structure on $\mathcal{C}$. We show that γ intertwines $\rho\otimes\sigma$ and σ by computing that

$$\begin{aligned}\gamma\big(\rho(\theta)v\big)\big(\sigma(\theta)\xi\big) &= \chi(\theta)\theta*v*\theta^{-1}*\omega*\theta*\xi\\ &= \chi^2(\theta)\theta*v*\omega*\xi=\sigma(\theta)\big(\gamma(v)\xi\big).\end{aligned} \tag{4.11}$$

Let $\sigma(M)$ be the bundle defined by the ${}^\epsilon$pin structure and let

$$\gamma\colon T^*M\otimes\sigma(M)\longrightarrow\sigma(M) \tag{4.12}$$

be the Clifford module structure. If Δ is the fundamental pinor representation, $2^\ell\Delta\simeq\sigma(M)$.

Let $m=2\ell$ and let $\delta=i^{\ell+1}$ be the normalizing constant of (4.8). If $\epsilon=-$, let $D=\gamma\circ\nabla$; modulo a possible sign convention and a factor of 2^ℓ, this is the pinor operator described in Section 2. Since $\sigma(M)$ is a real vector bundle over M, $\bar\nabla=\nabla$. If $m=4k$, then δ is imaginary so $\bar\gamma=-\gamma$ and $\bar D=-D$. If $m=4k+2$, then δ is real so $\bar\gamma=\gamma$ and $\bar D=-D$. If $\epsilon=+$, let $D=i\gamma\circ\nabla$; the factor of i ensures that D is self-adjoint. If $m=4k+2$, $\bar\gamma=\gamma$ and $\bar D=-D$. If $m=4k$, then $\bar\gamma=-\gamma$ and $\bar D=D$. ∎

PROOF OF THEOREM 4.2. Let $m=2\ell$ be even. Since $\eta\in\mathbf{R}$,

$$\eta(D)=\bar\eta(D)=\eta(\bar D)=\epsilon\eta(D)\quad\text{for }\epsilon=\pm1. \tag{4.13}$$

The desired vanishing now follows from Lemma 4.3. ∎

We shall need one additional technical fact before proving Theorem 4.1. Decompose $\mathrm{Cl}_0^-(\mathcal{G})=\mathcal{A}(\mathcal{G})\oplus\mathcal{B}(\mathcal{G})$ where

$$\begin{aligned}\mathcal{A}(\mathcal{G}) &:= \{\alpha\in\mathrm{Cl}_0^-(\mathcal{G}):\alpha(\lambda^{-1})=\alpha(\lambda)\ \forall\lambda\in\mathcal{G}\}\\ \mathcal{B}(\mathcal{G}) &:= \{\alpha\in\mathrm{Cl}_0^-(\mathcal{G}):\alpha(\lambda^{-1})=-\alpha(\lambda)\ \forall\lambda\in\mathcal{G}\}.\end{aligned} \tag{4.14}$$

LEMMA 4.4. *(a) If $m\equiv3$ (4) and if $\alpha\in\mathcal{B}(\mathcal{G})$, then $\eta(M,g,s,\alpha)=0$.*
(b) If $m\equiv1$ (4) and if $\alpha\in\mathcal{A}(\mathcal{G})$, then $\eta(M,g,s,\alpha)=0$.

PROOF. Let $\check\mu\colon\pi_1(M)\longrightarrow G$ be the associated G structure. Let M_1 be the principal G bundle defined by $\check\mu$; M_1 inherits a natural spin structure. Let D_1 be the spin operator on M_1 and let D_α be the twisted spin operator on M. By Lemma 4.3, $\bar D_1\simeq\epsilon D_1$ where $\epsilon=\epsilon(m)=\pm1$. Let $E_1(t)$ be the eigenspaces of D_1. If $\theta\in\mathcal{H}\big(\mathrm{PSpin}(M_1)\big)$, let $e_1(t,\theta)$ be the induced unitary morphism of $E_1(t)$. We define the equivariant eta function by

$$\eta(\theta,D_1):=\Big\{\sum_t\mathrm{sign}(t)|t|^{-z}\,\mathrm{Tr}\big(e_1(t,\theta)\big)\Big\}_{z=0}. \tag{4.15}$$

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If $\xi \in G$, let $\mu^{-1}(\xi) = (\theta, \lambda)$. Note that

$$\eta(-\theta, D_1) = -\eta(\theta, D_1) \quad \text{and} \quad \alpha(-\lambda) = -\alpha(\lambda). \tag{4.16}$$

Thus $\eta\big(\theta(\xi), \lambda(\xi)\big)$ is independent of the lift chosen. We compute that

$$\eta(D_\alpha) = |G|^{-1} \sum_{\xi \in \mathcal{G}} \eta\big(\theta(\xi), D_1\big)\alpha\big(\lambda(\xi)\big). \tag{4.17}$$

Since the complex conjugate of $\mathrm{Tr}\big(e_1(t,\theta)\big)$ is $\mathrm{Tr}\big(e_1(t,\theta^{-1})\big)$,

$$\eta(\theta, \bar{D}_1) = \eta(\theta^{-1}, D_1). \tag{4.18}$$

Since $\bar{D}_1 = \epsilon D_1$,

$$\begin{aligned}\eta(D) &= \frac{1}{2}\big(\eta(D) + \epsilon\eta(\bar{D})\big)\\ &= \frac{1}{2}|G|^{-1} \sum_{\xi\in G}\Big\{\eta\big(\theta(\xi), D_1\big)\alpha\big(\lambda(\xi)\big) + \epsilon\eta\big(\theta^{-1}(\xi), D_1\big)\alpha\big(\lambda(\xi)\big)\Big\}\\ &= \frac{1}{2}|G|^{-1} \sum_{\xi\in G}\eta\big(\theta(\xi), D_1\big)\Big\{\alpha\big(\lambda(\xi)\big) + \epsilon\alpha\big(\lambda(\xi)^{-1}\big)\Big\}.\end{aligned} \tag{4.19}$$

Thus the eta invariant is trivial if $\alpha(\lambda) + \epsilon\alpha(\lambda^{-1}) = 0$. ∎

The converse of Lemma 4.4 holds. As we shall not need this result, we omit the proof; it follows from the same arguments used to prove [2, Lemma 4.2].

LEMMA 4.5. *(a) If $m \equiv 3\ (4)$ and if $\eta(M,g,s,\alpha) = 0$ for all (M,g,s), then $\alpha \in \mathcal{B}(\mathcal{G})$.*
(b) If $m \equiv 1\ (4)$ and if $\eta(M,g,s,\alpha) = 0$ for all (M,g,s), then $\alpha \in \mathcal{A}(\mathcal{G})$.

REMARK. Let $\mathcal{A}(\mathcal{G})^*$ and $\mathcal{B}(\mathcal{G})^*$ be the dual vector spaces. Lemma 4.5 shows that the eta invariant can be interpreted as defining a surjective map

$$\eta\colon M\mathcal{J}^+_{4k-1} \otimes_{\mathbf{Z}} \mathbf{C} \longrightarrow \mathcal{A}(\mathcal{G})^* \longrightarrow 0, \quad \text{and}$$
$$\eta\colon M\mathcal{J}^+_{4k+1} \otimes_{\mathbf{Z}} \mathbf{C} \longrightarrow \mathcal{B}(\mathcal{G})^* \longrightarrow 0.$$

PROOF OF THEOREM 4.1. Let $m = 2\ell - 1$. The group $\mathcal{G}$ satisfies the assumptions of Theorem 4.1(1) if and only if $\mathcal{A}(\mathcal{G}) = 0$ or equivalently $\mathrm{Cl}_0^-(\mathcal{G}) = \mathcal{B}(\mathcal{G})$. Similarly, M satisfies the assumptions of Theorem 4.1(2) if and only if $\mathcal{B}(\mathcal{G}) = 0$ or equivalently $\mathrm{Cl}_0^-(\mathcal{G}) = \mathcal{A}(\mathcal{G})$. Theorem 4.1 now follows from Lemma 4.4. ∎

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