Canad. Math. Bull. Vol. 28 (1), 1985

THE FIBRE OF THE DOUBLE SUSPENSION IS AN H-SPACE

BY
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ABSTRACT. In this paper we show that the homotopy-theoretic fibre of the double suspension map $E^2: S^{2n-1} \to \Omega^2 S^{2n+1}$ is an H-space.

1. **Introduction**. Let $C(n)$ denote the homotopy-theoretic fibre of the double suspension map $E^2: S^{2n-1} \to \Omega^2 S^{2n+1}$. The goal of this paper is the proof of the following theorem.

THEOREM 1.1. *$C(n)$ is an H-space for all n.*

This lends support to Mahowald's conjecture that $C(n)$ is a loop space for all n and perhaps even double loop space. If $n = 1$, the theorem is trivial since $C(1) = \Omega_0^3 S^3$, so from now on we will assume $n > 1$. The only other integers for which S^{2n-1} is an H-space are $n = 2$ and $n = 4$, but even in these cases E^2 is not an H-map. However, if we invert the prime 2, then S^{2n-1} becomes an H-space and E^2 becomes an H-map for all n so the theorem has content only at the prime 2. Thus from now on we will assume all spaces and maps to be localized at 2.

The proof is based on Sugawara's criterion [9] that a space X is an H-space if and only if the canonical map $\alpha: X \to \Omega SX$ has a retraction up to homotopy. We construct a map $r: \Omega SC(n) \to C(n)$ and prove that r induces an isomorphism on homotopy in degree $4n - 3$ (the least non-vanishing degree). We then conclude the proof by appealing to the following theorem which is proved in Section 3.

THEOREM 3.1. *Let $f: C(n) \to C(n)$ be such that f induces an isomorphism on $H_{4n-3}(C(n); Z/2Z)$, $n > 1$. Then f is a homotopy equivalence. (Recall that we have localized at 2.)*

The idea that it is useful in homotopy theory to show that certain spaces have this property (that any self-map which induces an isomorphism on the lowest non-vanishing homotopy group must be a homotopy equivalence) is due to Fred Cohen. It was first introduced into the literature by Cohen, Moore and Neisendorfer in [6] where the property was given the name "atomic". The fact that $C(n)$ is atomic for $n > 1$ is certainly well known to all the authors of [3]. The proof given in Section 3 is a straightforward application of the ideas used by Cohen and Mahowald in [5].

Received by the editors February 10, 1984, and, in revised form, April 11, 1984.
AMS Subject Classification No.: 55P45.
The author is a Natural Sciences and Engineering Research Council of Canada University Research Fellow.

REMARK. Fred Cohen has shown that $C(1)$ is also atomic at the prime 2 but the proof is more difficult in this case.

The fact that $C(2)$ is an H-space also follows from the decomposition

$$\Omega^2 S^5\{2\} \approx \Omega^2 S^3\langle 3\rangle \times C(2)$$

recently proved by Fred Cohen, where $X\{2\}$ denotes the homotopy-theoretic fibre of the squaring map on the H-space X and $S^3\langle 3\rangle$ denotes the 3-connective cover of S^3. In fact, Fred Cohen and the author have recently shown that

$$\Omega^3 S^9\{2\} \approx C(2) \times \Omega C(4)$$

which shows that $C(2)$ is a homotopy abelian H-space. However, it is shown in [3] that a decomposition of this form will not be possible for arbitrary n.

2. **Proof of the main theorem**. In this section, we give the proof of the main theorem modulo some atomicity statements contained in Section 3. Throughout this section, $n > 1$ and all spaces and maps have been localized at 2.

Let j denote the homotopy class of the inclusion of the fibre into the total space of the fibration $C(n) \xrightarrow{j} S^{2n-1} \xrightarrow{E^2} \Omega^2 S^{2n+1}$. The double adjoint of $E^2 \circ j$ is $S^2 j : S^2 C(n) \to S^{2n+1}$ so $S^2 j \simeq 0$. Let $H : \Omega SX \to \Omega S(X \wedge X)$ denote the second Hopf-invariant map. By naturality of the Hopf-invariant map, we have a homotopy commutative diagram

$$\begin{array}{ccccc} & & \Omega SC(n) & \xrightarrow{H} & \Omega S(C(n) \wedge C(n)) \\ & & \downarrow{\scriptstyle \Omega Sj} & & \downarrow{\scriptstyle \Omega S(j \wedge j)} \\ S^{2n-1} & \xrightarrow{E} & \Omega S^{2n} & \xrightarrow{H} & \Omega S^{4n-1} \end{array}$$

in which the bottom line is James' EHP fibration sequence [7]. We can factor $j \wedge j$ as $(1_{S^{2n-1}} \wedge j) \circ (j \wedge 1_{C(n)})$. The first of these is null homotopic because $n > 1$ and $S^2 j \simeq 0$ and so $j \wedge j \simeq 0$. Thus ΩSj lifts to a map $\lambda : \Omega SC(n) \to S^{2n-1}$ such that $E \circ \lambda \simeq \Omega Sj$. Now $E^2 \circ \lambda = \Omega E \circ E \circ \lambda \simeq \Omega E \circ \Omega Sj = \Omega(E \circ Sj)$ and $E \circ Sj$ is the adjoint of the null homotopic map $E^2 \circ j$. So $E^2 \circ \lambda \simeq 0$ and thus λ lifts to a map $r : \Omega SC(n) \to C(n)$ such that $j \circ r \simeq \lambda$. Theorem 1.1 follows immediately from the following theorem and Theorem 3.1.

THEOREM 2.1. *r induces an isomorphism on π_{4n-3}.*

PROOF. Let $\alpha : C(n) \to \Omega SC(n)$ denote the canonical map. From James' EHP fibration sequence, we obtain a principal fibration $\Omega^2 S^{4n-1} \to S^{2n-1} \xrightarrow{E} \Omega S^{2n}$. Let $\phi : \Omega^2 S^{4n-1} \times S^{2n-1} \to S^{2n-1}$ denote the action of the fibre on the total space in this principal fibration. Because $E \circ \lambda \circ \alpha \simeq E \circ j$, there exists $d : C(n) \to \Omega^2 S^{4n-1}$ such that $\phi \circ (d, \lambda \circ \alpha) \simeq j$. However $\pi_{4n-3}(C(n)) \cong Z/2Z$ and $\pi_{4n-3}(\Omega^2 S^{4n-1}) \cong Z_{(2)}$ and so d induces zero on π_{4n-3}. Thus $\lambda_{\#}\alpha_{\#} = j_{\#}$ on π_{4n-3}.

CASE 1. *$n \neq 2$ or 4.*

In this case the map $j_{\#} : \pi_{4n-3}(C(n)) \to \pi_{4n-3}(S^{2n-1})$ is a monomorphism (the nonzero element going to $[\iota_{2n-1}, \iota_{2n-1}]$) and so from the fact that $j_\# r_\# \alpha_\# = \lambda_\# \alpha_\# = j_\#$ we can conclude that $r_\# \alpha_\# = I$ and so r induces an isomorphism on π_{4n-3}.

CASE 2. *$n = 2$ or 4.*

In these Hopf-invariant 1 cases, $j_\# = 0$ on π_{4n-3} so we must argue differently. Since $n = 2$ or 4, $E: S^{2n-1} \to \Omega S^{2n}$ has a retraction. Thus the fact that $E \circ j \circ r \circ \alpha \simeq E \circ \lambda \circ \alpha \simeq E \circ j$ implies that $j \circ r \circ \alpha \simeq j$.

CASE 2a. *$n = 4$.*

From Toda's computations [10], the portion of the long exact homotopy sequence

$$\pi_{18}(S^7) \to \pi_{18}(\Omega^2 S^9) \to \pi_{17}(C(4)) \to \pi_{17}(S^4) \to \pi_{17}(\Omega^2 S^9)$$

is

$$Z_8 \oplus Z_2 \xrightarrow{\cong} Z_8 \oplus Z_2 \to \pi_{17}(C(4)) \to Z_8 \oplus Z_2 \xrightarrow{\begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}} Z_8 \oplus Z_2.$$

Thus $\pi_{17}(C(4)) \cong Z_2$ and $j_\#: \pi_{17}(C(4)) \to \pi_{17}(S^7)$ is a monomorphism. So $j \circ r \circ \alpha = j$ implies that $r_\# \circ \alpha_\#$ is the identity $\pi_{17}(C(4))$. But then $r_\#$ is an isomorphism on π_{13} since if $r_\#$ is zero on π_{13} we get a contradiction by letting $h = r \circ \alpha$ in Corollary 3.4.

CASE 2b. *$n = 2$.*

From Toda's computations, we have that

$$\pi_8(S^3) \to \pi_8(\Omega^2 S^5) \to \pi_7(C(2)) \to \pi_7(S^3) \to \pi_7(\Omega^2 S^5)$$

is

$$Z_2 \xrightarrow{0} Z_2 \to \pi_7(C(2)) \to Z_2 \xrightarrow{0} Z_2.$$

Thus $\pi_7(C(2))$ has 4 elements and $j_\#: \pi_7(C(2)) \to \pi_7(S^3)$ is onto. The inclusion $i: S^5 \cup_2 e^6 \to C(2)$ induces an isomorphism in degree 7 for connectivity reasons so we see that $\pi_7(C(2)) \cong \pi_7(S^5 \cup_2 e^6) \cong Z_4$. Since $j \circ r \circ \alpha = j$, we get $r_\# \circ \alpha_\# = uI$ where $u = 1$ or 3 and I denotes the identity. But then $r_\# \alpha_\# - I$ cannot be an isomorphism so we must have $r_\#$ nonzero on π_5 since $r_\# = 0$ on π_5 would again contradict Corollary 3.4 with $h = r \circ \alpha$. □

3. **Atomicity of $C(n)$.** Throughout this section, $n > 1$, all spaces and maps are assumed to have been localized at 2, and all homology is understood to be with $Z/2Z$ coefficients.

THEOREM 3.1. *Let $f: C(n) \to C(n)$ be such that f induces an isomorphism on $H_{4n-3}(C(n))$. Then f is a homotopy equivalence.*

PROOF. From the defining fibration of $C(n)$, we get an induced map $\theta: \Omega^3 S^{2n+1} \to C(n)$ which induces a surjection or homology. Let x denote the nonzero element of $H_{2n-2}(\Omega^3 S^{2n+1})$. Then

https://doi.org/10.4153/CMB-1985-015-6 Published online by Cambridge University Press

$$H_*(\Omega^3 S^{2n+1}) \cong Z/2Z\Big[\big\{\underbrace{Q_1 \dots Q_1}_{j}\ \underbrace{Q_2 \dots Q_2}_{k} x\big\}_{\substack{j\geq 0\\ k\geq 0}}\Big]$$

as a Hopf algebra, with the generators primitive, where Q_i denotes a Dyer-Lashof operation written in lower notation. Let $a(j, k) = \theta_*(\underbrace{Q_1 \dots Q_1}_{j}\ \underbrace{Q_2 \dots Q_2}_{k} x)$. Then $H_*(C(n)) \cong Z/2Z[\{a_{jk}\}_{(j,k)\neq(0,0)}]$ as a coalgebra with the generators primitive. From the Nishida relations the actions of Sq^1_* and Sq^2_* are given by

$$Sq^1_* a(j,k) = \begin{Bmatrix} a(j-1,k)^2 & j \geq 2 \\ 0 & j = 1 \\ a(1,k-1) & j = 0,\ k \geq 1 \end{Bmatrix},$$

$$Sq^2_* a(j,k) = \begin{Bmatrix} 0 & j \geq 2 \\ a(2,k-1) & j = 1,\ k \geq 1 \\ 0 & j = 1,\ k = 0 \\ a(0,k-1)^2 & j = 0,\ k \geq 2 \\ 0 & j = 0,\ k = 1 \end{Bmatrix}.$$

If f fails to induce an isomorphism on homotopy, let N be the least dimension in which it fails to do so. Let F be the homotopy-fibre of the canonical map from $C(n)$ to the infinite mapping-telescope $\varinjlim_f C(n)$. Proceeding as in [6] or [8], we see that F is $(N - 1)$-connected and that there is a map $S^N \cup_{2^r} e^{N+1} \to F$ for some r such that the composite

$$H_*(S^n \cup_{2^r} e^{N+1}) \to H_*(F) \to H_*(C(n))$$

is a monomorphism in degrees $N, N + 1$. Thus in $H_*(C(n))$, we have a pair of non-zero homology classes y and z in consecutive dimensions N, $N + 1$ such that

(1) y and z are primitive,

(2) $\beta^{(r)} z = y$ for some r,

(3) $y \in \ker Sq^{2^t}_*$ for all t, and $z \in \ker Sq^{2^t}_*$ for all $t \geq 1$.

By inspection of the action of Sq^1_* and Sq^2_* the only possibilities are $y = a(1, 0) = \theta_*(Q_1 x)$, $z = a(0, 1) = \theta_*(Q_2 x)$ or else $y = a(1, 0)^2 = \theta_*(Q_0 Q_1 x)$, $z = a(2, 0) = \theta_*(Q_1 Q_1 x)$. By hypothesis f induces an isomorphism in degree $4n - 3$ so that we cannot have $y = \theta_*(Q_1 x)$. Therefore to prove the theorem it suffices to prove the following Lemma which shows that degree $8n - 6$ is also not the least nonvanishing degree.

LEMMA 3.2. *Under the hypothesis of Theorem 3.1*, $f_*(\theta_*(Q_0 Q_1 x)) = \theta_*(Q_0 Q_1 x)$.

PROOF. For ease of notation, we will suppress θ_* and $\Omega\theta_*$ throughout this proof. Consider $\Omega f: \Omega C(n) \to \Omega C(n)$ and let $w \in H_{2n-3}(\Omega^4 S^{2n-1})$ denote the nonzero element. Since $f_*(Q_1 x) = Q_1 x$, commutativity with Sq^1_* forces $f_*(Q_2 x) = Q_2 x$. Thus we must have $\Omega f_*(Q_3 w) = Q_3 w$. Since Ωf is an H-map, this forces $(\Omega f_*)(Q_0 Q_3 w)$

$= Q_0Q_3w$ and commutativity with Sq^1_* now forces $\Omega f_*(Q_1Q_3w) = Q_1Q_3w$. Applying Sq^2_* now gives $(\Omega f)_*(Q_1Q_2w) = Q_1Q_2w$ and so applying the homology suspension gives $f_*(Q_0Q_1x) = Q_0Q_1x$ as desired. □

This concludes the proof that $C(n)$ is atomic. In Section 2, we also made use of the following extenison of this result to handle the cases $n = 2$ and $n = 4$.

THEOREM 3.3. *Let $g : \Omega C(n) \to \Omega C(n)$ be such that g induces an isomorphism on $H_{4n-4}(C(n))$. Then g is a homotopy equivalence.*

PROOF. The proof is similar to that of Theorem 3.1 but is slightly more tedious. It will be left as an exercise. □

COROLLARY 3.4. *Let $h : C(n) \to C(n)$ be such that h induces zero on $H_{4n-3}C(n))$. Then $h_\# - I$ is an isomorphism on π_q for all q.*

PROOF. Apply Theorem 3.3 with $g = (\Omega h)(1_{\Omega C(n)})^{-1}$ in $[\Omega C(n), \Omega C(n)]$. □

REMARK 1. To obtain as much of Corollary 3.4 as is needed in Section 2, one need only demonstrate that g, satisfying the hypotheses of Theorem 3.3, induces an isomorphism on homology through $H_{18}(\ )$ when $n = 4$ and through $H_8(\ )$ when $n = 2$. The $n = 4$ case is a complete triviality since after $H_{13}(\Omega C(4))$ the next lowest nonvanishing homology group is $H_{24}(\Omega C(4))$ and the $n = 2$ case is not that much harder.

REMARK 2. Theorem 3.1 actually follows from Theorem 3.3 by letting $g = \Omega f$.

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