Bull. Aust. Math. Soc. **97** (2018), 15–25
doi:10.1017/S0004972717000788

MORE ON A CERTAIN ARITHMETICAL DETERMINANT

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(Received 26 June 2017; accepted 8 July 2017; first published online 17 October 2017)

Abstract

Let $n \geq 1$ be an integer and f be an arithmetical function. Let $S = \{x_1, \ldots, x_n\}$ be a set of n distinct positive integers with the property that $d \in S$ if $x \in S$ and $d|x$. Then $\min(S) = 1$. Let $(f(S)) = (f(\gcd(x_i, x_j)))$ and $(f[S]) = (f(\mathrm{lcm}(x_i, x_j)))$ denote the $n \times n$ matrices whose (i, j)-entries are f evaluated at the greatest common divisor of x_i and x_j and the least common multiple of x_i and x_j, respectively. In 1875, Smith ['On the value of a certain arithmetical determinant', *Proc. Lond. Math. Soc.* **7** (1875–76), 208–212] showed that $\det(f(S)) = \prod_{l=1}^{n}(f * \mu)(x_l)$, where $f * \mu$ is the Dirichlet convolution of f and the Möbius function μ. Bourque and Ligh ['Matrices associated with classes of multiplicative functions', *Linear Algebra Appl.* **216** (1995), 267–275] computed the determinant $\det(f[S])$ if f is multiplicative and, Hong, Hu and Lin ['On a certain arithmetical determinant', *Acta Math. Hungar.* **150** (2016), 372–382] gave formulae for the determinants $\det(f(S \setminus \{1\}))$ and $\det(f[S \setminus \{1\}])$. In this paper, we evaluate the determinant $\det(f(S \setminus \{x_t\}))$ for any integer t with $1 \leq t \leq n$ and also the determinant $\det(f[S \setminus \{x_t\}])$ if f is multiplicative.

2010 *Mathematics subject classification*: primary 11C20; secondary 11A05, 15B36.

Keywords and phrases: factor-closed set, squarefree integer, arithmetical function, Dirichlet convolution, Smith's determinant.

1. Introduction

Let n be a positive integer. In 1875, Smith [15] published his famous result stating that the determinant of the $n \times n$ matrix $(\gcd(i, j))_{1 \leq i,j \leq n}$, having the greatest common divisor $\gcd(i, j)$ of i and j as the (i, j)-entry for all integers i and j between 1 and n, is equal to $\prod_{k=1}^{n} \varphi(k)$, where φ is Euler's totient function. Throughout, let f be an arithmetical function and $S = \{x_1, \ldots, x_n\}$ be a set of n distinct positive integers. Let $(f(\gcd(x_i, x_j)))_{1 \leq i,j \leq n}$ and $(f(\mathrm{lcm}(x_i, x_j)))_{1 \leq i,j \leq n}$ denote the $n \times n$ matrices whose (i, j)-entries are f evaluated at the greatest common divisor $\gcd(x_i, x_j)$ and the least common multiple $\mathrm{lcm}(x_i, x_j)$ of x_i and x_j, respectively. Smith [15] also showed that

$$\det(\mathrm{lcm}(x_i, x_j))_{1 \leq i,j \leq n} = \prod_{i=1}^{n} \varphi(x_i)\pi(x_i)$$

The research was partially supported partially by National Science Foundation of China Grant Nos. 11371260, 11771304 and 11671218.

and

$$\det(f(\gcd(x_i, x_j)))_{1\le i,j\le n} = \prod_{i=1}^{n}(f * \mu)(x_i)$$

if S is *factor closed* (that is, $d \in S$ if $x \in S$ and $d|x$), where $f * \mu$ is the Dirichlet convolution of f and the Möbius function μ and π is the multiplicative function defined for any prime power p^r by $\pi(p^r) := -p$. One hundred and twenty years later, Bourque and Ligh [3] showed that if S is factor closed and f is a multiplicative function such that $f(x) \neq 0$ for all $x \in S$, then

$$\det(f(\mathrm{lcm}(x_i, x_j)))_{1\le i,j\le n} = \prod_{i=1}^{n} f(x_i)^2 (f^{-1} * \mu)(x_i),$$

where f^{-1} is defined for any positive integer x by $f^{-1}(x) := 1/f(x)$ if $f(x) \neq 0$, and 0 otherwise.

Since Smith's paper, this area has been studied intensely. Apostol [2] points out that Smith's determinant has connections with Ramanujan's sum and its generalisations (see also [10, 14, 17]). Haukkanen, Wang and Sillanpää [6] review papers relating to Smith's determinant and present a common structure in the language of posets (further developed in [1, 13]). Weber [16] investigates gcd quadratic forms $\sum x_i x_j F((i, j))$ and their connections with the Riemann zeta function. The asymptotic behaviour of the eigenvalues of gcd and lcm matrices and their generalisations has also been investigated (see [11] and references therein). Multidimensional determinants have been considered (see [5] for a review and [10, 17] for recent developments). Related determinants involving other multiplicative functions or multiple gcd-closed sets are considered in [4, 7, 8].

Let us recall that a positive integer is called *squarefree* if it is divisible by no other perfect square than 1. In 2016, Hong *et al.* [9] showed that if S is factor closed,

$$\det(f(S\setminus\{1\})) = \sum_{\substack{l=1 \\ x_l \text{ squarefree}}}^{n} \prod_{\substack{k=1 \\ k\neq l}}^{n} (f * \mu)(x_k),$$

and if f is multiplicative and $f(x) \neq 0$ for all $x \in S$,

$$\det(f[S\setminus\{1\}]) = \left(\prod_{l=1}^{n} f(x_l)^2\right) \sum_{\substack{l=1 \\ x_l \text{ squarefree}}}^{n} \prod_{\substack{k=1 \\ k\neq l}}^{n} (f^{-1} * \mu)(x_k).$$

In this paper, we address the problem of calculating the determinants of the following $(n-1)\times(n-1)$ matrices:

$$(f(S\setminus\{x_t\})) = (f(\gcd(x_i, x_j)))_{\substack{1\le i,j\le n \\ i\neq t,\, j\neq t}}$$

and

$$(f[S\setminus\{x_t\}]) = (f(\mathrm{lcm}(x_i, x_j)))_{\substack{1\le i,j\le n \\ i\neq t,\, j\neq t}},$$

where S is factor closed and x_t is any given element of S. Our main results can be stated as follows.

https://doi.org/10.1017/S0004972717000788 Published online by Cambridge University Press

THEOREM 1.1. *Let $n \geq 2$ be an integer and let t be an integer with $1 \leq t \leq n$. Let f be an arithmetical function and $S = \{x_1, \ldots, x_n\}$ be factor closed. Then*

$$\det(f(\gcd(x_i, x_j)))_{\substack{1 \leq i, j \leq n \\ i \neq t, j \neq t}} = \sum_{\substack{l = 1 \\ x_t | x_l, \ x_l / x_t \text{ squarefree}}}^{n} \prod_{\substack{k = 1 \\ k \neq l}}^{n} (f * \mu)(x_k).$$

Moreover, if f is multiplicative, then

$$\det(f(\mathrm{lcm}(x_i, x_j)))_{\substack{1 \leq i, j \leq n \\ i \neq t, j \neq t}} = \Big(\prod_{\substack{l = 1 \\ l \neq t}}^{n} f(x_l)^2\Big) \sum_{\substack{l = 1 \\ x_t | x_l, \ x_l / x_t \text{ squarefree}}}^{n} \prod_{\substack{k = 1 \\ k \neq l}}^{n} (f^{-1} * \mu)(x_k).$$

Theorem 1.1 extends the results of Smith, Bourque and Ligh, and Hong, Hu and Lin. If $x_t = \max(S)$, then Theorem 1.1 reduces to the theorems of Smith [15] and Bourque and Ligh [3]. If $x_t = \min(S)$, then Theorem 1.1 gives [9, Theorem 2]. The problem of removing elements from the set S (and inserting elements into S) was also considered in [12] in the more general setting of posets using partitioned matrices.

For any positive integer x, we let $\omega(x)$ and $\mathrm{rad}(x)$ stand for the number and the product of all distinct prime divisors of x, respectively. Taking $f = I$ in Theorem 1.1, where $I(x) := x$ for any positive integer x, gives the following result.

THEOREM 1.2. *Let $n \geq 2$ be an integer and let t be an integer with $1 \leq t \leq n$. Let $S = \{x_1, \ldots, x_n\}$ be factor closed. Then*

$$\det((\gcd(x_i, x_j)))_{\substack{1 \leq i, j \leq n \\ i \neq t, j \neq t}} = \Big(\prod_{l=1}^{n} \varphi(x_l)\Big) \sum_{\substack{k = 1 \\ x_t | x_k, \ x_k / x_t \text{ squarefree}}}^{n} \frac{1}{\varphi(x_k)}$$

and

$$\det((\mathrm{lcm}(x_i, x_j)))_{\substack{1 \leq i, j \leq n \\ i \neq t, j \neq t}} = \frac{(-1)^{\sum_{l=1}^{n} \omega(x_l)}}{x_t^2} \Big(\prod_{l=1}^{n} \mathrm{rad}(x_l) \varphi(x_l)\Big) \sum_{\substack{k = 1 \\ x_t | x_k, \ x_k / x_t \text{ squarefree}}}^{n} \frac{(-1)^{\omega(x_k)} x_k^2}{\mathrm{rad}(x_k) \varphi(x_k)}.$$

For any real number x, $\lfloor x \rfloor$ stands for the largest integer that is less than or equal to x. Taking $S = \{1, 2, \ldots, n\}$ in Theorem 1.2 gives the following result.

THEOREM 1.3. *Let $n \geq 2$ be an integer and let t be an integer with $1 \leq t \leq n$. Then*

$$\det((\gcd(i, j)))_{\substack{1 \leq i, j \leq n \\ i \neq t, j \neq t}} = \Big(\prod_{l=1}^{n} \varphi(l)\Big) \sum_{\substack{k = 1 \\ k \text{ squarefree}}}^{\lfloor n/t \rfloor} \frac{1}{\varphi(tk)}$$

and

$$\det((\mathrm{lcm}(i, j)))_{\substack{1 \leq i, j \leq n \\ i \neq t, j \neq t}} = (-1)^{\sum_{l=1}^{n} \omega(l)} \Big(\prod_{l=1}^{n} \mathrm{rad}(l) \varphi(l)\Big) \sum_{\substack{k = 1 \\ k \text{ squarefree}}}^{\lfloor n/t \rfloor} \frac{(-1)^{\omega(tk)} k^2}{\mathrm{rad}(tk) \varphi(tk)}.$$

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Taking $t = 1$ in Theorem 1.3, gives [9, Theorem 1].

The proof of Theorem 1.1 is similar to the proofs of the results in Smith [15] and Hong *et al.* [9], but more complicated.

We organise this paper as follows. In Section 2, we present several lemmas which are needed in the proof of Theorem 1.1. In Section 3, we first give the proof of Theorem 1.1 and then apply Theorem 1.1 to show Theorems 1.2 and 1.3.

2. Preliminary lemmas

In this section, we present several lemmas that are needed in the next section.

Lemma 2.1. *Let $n \geq 2$ be an integer and let $S = \{x_1, \dots, x_n\}$ be a factor-closed set such that $x_1 < \cdots < x_n$.*

(i) *The smallest element of S is $x_1 = 1$.*
(ii) *Let t be an integer with $1 \leq t \leq n$. The set $S \setminus \{x_t\}$ is factor closed if and only if x_t is not a proper divisor of any element of S.*
(iii) *The set $S \setminus \{1\}$ is not factor closed.*
(iv) *The set $S \setminus \{x_n\}$ is factor closed.*

Proof. Parts (i) and (ii) are easy deductions from the definition of a factor-closed set and parts (iii) and (iv) follow from part (ii). □

Lemma 2.2 [9]. *Let $m \geq 2$ be an integer and f be an arithmetical function. Define the arithmetical function F_m for any positive integer n by*

$$F_m(n) := \sum_{d|n} \mu\Big(\frac{n}{d}\Big) f(\gcd(m, d)).$$

Then

$$F_m(n) = \begin{cases} (f * \mu)(n) & \text{if } n \mid m, \\ 0 & \text{otherwise}. \end{cases}$$

Lemma 2.3. *Let m and n be positive integers such that $m|n$ and $m < n$ and let f be an arithmetical function. Then*

$$\sum_{\substack{m|d|n \\ d \geq 2}} f\Big(\frac{n}{d}\Big) = (f * \mathbf{1})\Big(\frac{n}{m}\Big) - f\Big(\frac{n}{m}\Big)\delta(m),$$

where the arithmetical functions $\mathbf{1}$ and δ are defined by $\mathbf{1}(x) = 1$ for any positive integer x and $\delta(x) = 1$ if $x = 1$ and 0 otherwise.

Proof. For any integer d with $m|d|n$, we can write $d = mk$ with an integer $k \geq 1$. So

$$\sum_{\substack{m|d|n \\ d \geq 2}} f\Big(\frac{n}{d}\Big) = \sum_{\substack{mk|n \\ mk \geq 2}} f\Big(\frac{n}{mk}\Big) = \sum_{\substack{k|\frac{n}{m} \\ mk \geq 2}} f\Big(\frac{n}{mk}\Big). \tag{2.1}$$

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If $m = 1$, the right-hand side of (2.1) is

$$\sum_{k|n} f\Big(\frac{n}{k}\Big)\mathbf{1}(k) - f(n) = (f * \mathbf{1})(n) - f(n) = (f * \mathbf{1})\Big(\frac{n}{m}\Big) - f\Big(\frac{n}{m}\Big)\delta(m).$$

If $m > 1$, then $mk \geq 2$ for any positive integer k and so the right-hand side of (2.1) is

$$\sum_{k|\frac{n}{m}} f\Big(\frac{n}{mk}\Big)\mathbf{1}(k) = (f * \mathbf{1})\Big(\frac{n}{m}\Big) = (f * \mathbf{1})\Big(\frac{n}{m}\Big) - f\Big(\frac{n}{m}\Big)\delta(m)$$

as expected. This ends the proof of Lemma 2.3. □

By the well-known result that $\sum_{d|n} \mu(d) = 0$ for any integer n with $n > 1$, taking $f = \mu$ in Lemma 2.3 and noting that $\mathbf{1} * \mu = \delta$ gives the following corollary.

Corollary 2.4 [9, Lemma 5] . *Let m and n be positive integers with m dividing n and $m < n$. Then*

$$\sum_{\substack{m|d|n \\ d \geq 2}} \mu\Big(\frac{n}{d}\Big) = \begin{cases} (-1)^{1+\omega(n)} & \text{if } m = 1 \text{ and } n \text{ is squarefree,} \\ 0 & \text{otherwise.} \end{cases}$$

Lemma 2.5. *Let $n \geq 2$ be an integer and $a_1, \ldots, a_n$ and $b_1, \ldots, b_n$ be $2n$ elements in a commutative ring. Let $M = \operatorname{diag}(a_1^2, b_2, \ldots, b_n) + M_1 + M_2$, where M_1 and M_2 are the $n \times n$ matrices defined by:*

(i) *the first row of M_1 is $(0, -a_2b_1, \ldots, -a_nb_1)$ and all other elements are zero;*
(ii) *the first column of M_2 is $(0, a_2, \ldots, a_n)^T$ and all other elements are zero.*

Then

$$\det(M) = \sum_{i=1}^{n} a_i^2 \prod_{\substack{k=1 \\ k \neq i}}^{n} b_k.$$

Proof. Write $M := (m_{ij})_{n\times n}$. Let A_i be the minor of m_{i1}. Then $A_1 = b_2 \cdots b_n$ and, for $2 \leq i \leq n$, all the elements of the $(i-1)$th column of A_i are zero except that the first element equals $-a_ib_1$. Consequently

$$A_i = (-1)^{1+i-1}(-a_ib_1)b_2 \cdots b_{i-1}b_{i+1} \cdots b_n = (-1)^{i+1}a_i \prod_{\substack{j=1 \\ j \neq i}}^{n} b_j. \tag{2.2}$$

By the Laplace expansion theorem and (2.2),

$$\begin{aligned} \det(A) &= \sum_{i=1}^{n} (-1)^{i+1} m_{i1} A_i = a_1^2 A_1 + \sum_{i=2}^{n} (-1)^{i+1} a_i A_i \\ &= a_1^2 \prod_{j=2}^{n} b_j + \sum_{i=2}^{n} (-1)^{i+1} a_i \cdot (-1)^{i+1} a_i \prod_{\substack{j=1 \\ j\neq i}}^{n} b_j = \sum_{i=1}^{n} a_i^2 \prod_{\substack{j=1 \\ j \neq i}}^{n} b_j \end{aligned}$$

as required. This concludes the proof of Lemma 2.5. □

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3. Proofs of Theorems 1.1 to 1.3

For a positive integer x, we introduce the auxiliary arithmetical function u_x defined for any positive integer y by

$$u_x(y) := \begin{cases} \mu\Big(\dfrac{y}{x}\Big) & \text{if } x|y, \\ 0 & \text{otherwise.} \end{cases}$$

Proof of Theorem 1.1. For any set $S = \{x_1, \ldots, x_n\}$, we define $S_\sigma := \{x_{\sigma(1)}, \ldots, x_{\sigma(n)}\}$, where σ is a permutation on $\{1, \ldots, n\}$. Then $(f(S)) = P^T(f(S))P$ for any arithmetical function f, where P is the $n \times n$ permutation matrix whose ith row is equal to $(0, \ldots, 0, 1, 0, \ldots, 0)$ with a 1 in the $\sigma(i)$th place, for $1 \le i \le n$. It follows that $\det(f(S)) = \det(f(S_\sigma))$ and $\det(f[S]) = \det(f[S_\sigma])$. So we can rearrange the elements of S in any case of necessity. Without loss of generality, we assume that $x_1 < \cdots < x_n$ in what follows. Then, by Lemma 2.1, $x_1 = 1$.

Define the $n \times n$ matrix $A = (a_{ij})$ as follows: $a_{tt} := 1$, $a_{it} := 0$ if $i \neq t$, and $a_{ij} := f(\gcd(x_i, x_j))$ for all integers i and j with $1 \le i, j \le n$ and $j \neq t$.

Let R_1 and T_1 be the empty set. For each integer r with $2 \le r \le n$, define two subsets R_r and T_r of S by

$$R_r := \{x_d : x_d | x_r, 1 \le d < r\}, \quad T_r := R_r \backslash \{x_t\}.$$

Then, R_r is nonempty and $R_r \cup \{x_r\}$ is factor closed, but T_r may be empty for any integer r with $2 \le r \le n$.

For each integer r with $2 \le r \le n$ and each integer d with $x_d \in R_r$, multiply the entries of the dth row of A by $\mu(x_r/x_d)$ and then add them to the corresponding entries of the rth row of A. We obtain a new $n \times n$ matrix, denoted by $B := (b_{ij})$.

Lemma 3.1. *For all integers i and j with $1 \le i, j \le n$,*

$$b_{ij} = \begin{cases} u_{x_t}(x_i) & \text{if } j = t, \\ (f * \mu)(x_i) & \text{if } j \neq t \text{ and } x_i | x_j, \\ 0 & \text{otherwise.} \end{cases}$$

Proof. For any integers i and j with $1 \le i, j \le n$, since $R_i \cup \{x_i\}$ is factor closed,

$$b_{ij} = a_{ij} + \sum_{x_d \in R_i} \mu\Big(\frac{x_i}{x_d}\Big) a_{dj} = \sum_{x_d | x_i} \mu\Big(\frac{x_i}{x_d}\Big) a_{dj}. \tag{3.1}$$

Since $a_{tt} = 1$ and $a_{it} = 0$ if $i \neq t$, it follows that

$$b_{it} = \sum_{x_d | x_i} \mu\Big(\frac{x_i}{x_d}\Big) a_{dt} = u_{x_t}(x_i) a_{tt} = u_{x_t}(x_i)$$

as desired.

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Now let j be an integer different from t and between 1 to n. Since S is factor closed and $a_{kj} = f(\gcd(x_k, x_j))$ for any integer k with $1 \le k \le n$, it follows from Lemma 2.2 and (3.1) that

$$b_{ij} = \sum_{x_d|x_i} \mu\Big(\frac{x_i}{x_d}\Big) f(\gcd(x_j, x_d)) = \sum_{d|x_i} \mu\Big(\frac{x_i}{d}\Big) f(\gcd(x_j, d))$$
$$= \begin{cases} (f * \mu)(x_i) & \text{if } x_i|x_j \text{ and } j \ne t, \\ 0 & \text{if } x_i \nmid x_j \text{ and } j \ne t. \end{cases}$$

Therefore Lemma 3.1 is proved. □

Next, for each integer r with $r \ne t$ and $1 \le r \le n$ and each integer d with $x_d \in T_r$ (if T_r is nonempty), multiply the entries of the dth column of B by $\mu(x_r/x_d)$ and add them to the corresponding entries of the rth column of B, to arrive at the $n \times n$ matrix $C := (c_{ij})$.

Lemma 3.2. *For all integers i and j with $1 \le i, j \le n$,*

$$c_{ij} = \begin{cases} u_{x_t}(x_i) & \text{if } j = t, \\ (f * \mu)(x_j) & \text{if } j \ne t \text{ and } i = j, \\ -\mu\Big(\dfrac{x_j}{x_t}\Big)(f * \mu)(x_i) & \text{if } j > t \text{ and } x_i|x_t|x_j, \\ 0 & \text{otherwise}. \end{cases}$$

Proof. Let i and j be integers between 1 and n. If $j = t$, then clearly $c_{it} = b_{it} = u_{x_t}(x_i)$. In the following, we suppose $j \ne t$, so that

$$c_{ij} = b_{ij} + \sum_{x_d \in T_j} \mu\Big(\frac{x_j}{x_d}\Big) b_{id} = \sum_{\substack{x_d|x_j \\ d \ne t}} \mu\Big(\frac{x_j}{x_d}\Big) b_{id}. \tag{3.2}$$

Consider the following three cases.

Case 1: $i \ge j$. For each integer d with $x_d \in T_j$, we have $d \ne t$ and $d < j \le i$, which implies that $x_i \nmid x_d$. Also, $b_{id} = 0$ since $d \ne t$. From Lemma 3.1,

$$c_{ij} = b_{ij} + \sum_{x_d \in T_j} \mu\Big(\frac{x_j}{x_d}\Big) \times 0 = b_{ij} = \begin{cases} (f * \mu)(x_i) & \text{if } i = j, \\ 0 & \text{if } j < i. \end{cases}$$

Case 2: $i < j$ and $x_i \nmid x_j$. For each integer d with $x_d \in T_j$, we must have $x_i \nmid x_d$. Otherwise, $x_i|x_d$ and, from $x_d|x_j$, we deduce that $x_i|x_j$ which contradicts the assumption $x_i \nmid x_j$. Since $x_i \nmid x_j$ and $x_i \nmid x_d$, by Lemma 3.1, $b_{ij} = 0$ and $b_{id} = 0$ for each d with $x_d \in T_j$. Hence, by (3.2), $c_{ij} = 0$.

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Case 3: $i < j$ and $x_i | x_j$. Since S is factor closed, $x_1 = 1$ and $x_j > 1$,

$$
\begin{aligned}
c_{1j} &= \sum_{\substack{x_d | x_j \\ d \neq t}} \mu\Big(\frac{x_j}{x_d}\Big)(f * \mu)(1) = (f * \mu)(1)\Big(\sum_{x_d | x_j} \mu\Big(\frac{x_j}{x_d}\Big) - u_{x_t}(x_j)\Big) \\
&= (f * \mu)(1)\Big(\sum_{d | x_j} \mu\Big(\frac{x_j}{d}\Big) - u_{x_t}(x_j)\Big) \\
&= -(f * \mu)(1) u_{x_t}(x_j) = \begin{cases} -(f * \mu)(x_1)\mu\Big(\dfrac{x_j}{x_t}\Big) & \text{if } x_t | x_j, \\ 0 & \text{otherwise.} \end{cases}
\end{aligned} \tag{3.3}
$$

Now take $i > 1$. Then $x_i > 1$. By (3.2), Lemma 3.1, Corollary 2.4 and noting that S is factor closed,

$$
\begin{aligned}
c_{ij} &= \sum_{\substack{x_i | x_d | x_j \\ d \neq t}} \mu\Big(\frac{x_j}{x_d}\Big)(f * \mu)(x_i) = (f * \mu)(x_i) \sum_{\substack{x_i | x_d | x_j \\ d \neq t}} \mu\Big(\frac{x_j}{x_d}\Big) \\
&= \begin{cases} (f * \mu)(x_i)\Big(\displaystyle\sum_{x_i | x_d | x_j} \mu\Big(\frac{x_j}{x_d}\Big) - \mu\Big(\frac{x_j}{x_t}\Big)\Big) & \text{if } x_i | x_t | x_j, \\ (f * \mu)(x_i) \displaystyle\sum_{x_i | x_d | x_j} \mu\Big(\frac{x_j}{x_d}\Big) & \text{otherwise} \end{cases} \\
&= \begin{cases} (f * \mu)(x_i)\Big(\displaystyle\sum_{x_i | d | x_j} \mu\Big(\frac{x_j}{d}\Big) - \mu\Big(\frac{x_j}{x_t}\Big)\Big) & \text{if } x_i | x_t | x_j, \\ (f * \mu)(x_i) \displaystyle\sum_{x_i | d | x_j} \mu\Big(\frac{x_j}{d}\Big) & \text{otherwise} \end{cases} \\
&= \begin{cases} -(f * \mu)(x_i)\mu\Big(\dfrac{x_j}{x_t}\Big) & \text{if } x_i | x_t | x_j, \\ 0 & \text{otherwise.} \end{cases}
\end{aligned} \tag{3.4}
$$

Taking (3.3) and (3.4) together gives the evaluation of c_{ij} in this case.

Finally, combining Cases 1 to 3 gives the desired result and proves Lemma 3.2. □

We continue the proof of Theorem 1.1. Obviously,

$$\det(f(S \setminus \{x_t\})) = \det(A) = \det(B) = \det(C).$$

By Lemma 3.2, the tth column of C is

$$(0, \ldots, 0, 1, u_{x_t}(x_{t+1}), \ldots, u_{x_t}(x_n))^T,$$

the tth row of C is

$$(0, \ldots, 0, 1, -u_{x_t}(x_{t+1})(f * \mu)(x_t), \ldots, -u_{x_t}(x_n)(f * \mu)(x_t)),$$

the diagonal elements of C are

$$(f * \mu)(x_1), \ldots, (f * \mu)(x_{t-1}), 1, (f * \mu)(x_{t+1}), \ldots, (f * \mu)(x_n),$$

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and $c_{ij} = 0$ for all integers i and j with $1 \le j \le t-1$ and $i \ne j$, or $t+1 \le i, j \le n$ and $i \ne j$. Since $u_y(x)^2 = 1$ if x/y is squarefree and $u_{x_t}(x_l) = 0$ for any integer l with $1 \le l < t$ and, for any square matrices P and Q,

$$\det\begin{pmatrix} P & * \\ O & Q \end{pmatrix} = \det(P) \cdot \det(Q),$$

by Lemmas 2.5 and 3.2,

$$\begin{aligned}
\det(C) = {} & \det(\mathrm{diag}((f * \mu)(x_1), \ldots, (f * \mu)(x_{t-1}))) \sum_{l=t}^{n} (u_{x_t}(x_l))^2 \prod_{\substack{k=t \\ k \ne l}}^{n} (f * \mu)(x_k) \\
= {} & \Big(\prod_{k=1}^{t-1} (f * \mu)(x_k) \Big) \sum_{l=t}^{n} (u_{x_t}(x_l))^2 \prod_{\substack{k=t \\ k \ne l}}^{n} (f * \mu)(x_k) \\
= {} & \sum_{l=t}^{n} (u_{x_t}(x_l))^2 \prod_{\substack{k=1 \\ k \ne l}}^{n} (f * \mu)(x_k) = \sum_{l=1}^{n} (u_{x_t}(x_l))^2 \prod_{\substack{k=1 \\ k \ne l}}^{n} (f * \mu)(x_k) \\
= {} & \sum_{\substack{l=1 \\ x_t | x_l,\, x_l/x_t \text{ squarefree}}}^{n} \prod_{\substack{k=1 \\ k \ne l}}^{n} (f * \mu)(x_k)
\end{aligned}$$

as desired. This finishes the proof of the first part of Theorem 1.1.

We are now in a position to prove the second part of Theorem 1.1. Since f is multiplicative, $f(\gcd(x_i, x_j)) f(\mathrm{lcm}(x_i, x_j)) = f(x_i) f(x_j)$. It follows that

$$(f(\mathrm{lcm}(x_i, x_j)))_{\substack{1 \le i,j \le n \\ i \ne t,\, j \ne t}} = \Lambda \cdot (f^{-1}(\gcd(x_i, x_j)))_{\substack{1 \le i,j \le n \\ i \ne t,\, j \ne t}} \cdot \Lambda,$$

where $\Lambda := \mathrm{diag}(f(x_1), \ldots, f(x_{t-1}), f(x_{t+1}), \ldots, f(x_n))$ is the $(n-1) \times (n-1)$ diagonal matrix with $f(x_1), \ldots, f(x_{t-1}), f(x_{t+1}), \ldots, f(x_n)$ as its diagonal elements. So

$$\det(f(\mathrm{lcm}(x_i, x_j)))_{\substack{1 \le i,j \le n \\ i \ne t,\, j \ne t}} = \Big(\prod_{\substack{i=1 \\ i \ne t}}^{n} f(x_i)^2 \Big) \cdot \det(f^{-1}(\gcd(x_i, x_j)))_{\substack{1 \le i,j \le n \\ i \ne t,\, j \ne t}}.$$

Thus, the first part of Theorem 1.1 applied to f^{-1} gives the expected formula. This completes the proof of Theorem 1.1. □

Proof of Theorem 1.2. We apply Theorem 1.1 with $f = I$. Note that $I * \mu = \varphi$ and $(I^{-1} * \mu)(x) = \pi(x)\varphi(x)/x^2 = (-1)^{\omega(x)} \mathrm{rad}(x) \varphi(x)/x^2$ for any positive integer x. So by Theorem 1.1,

$$\begin{aligned}
\det(\gcd(x_i, x_j))_{\substack{1 \le i,j \le n \\ i \ne t,\, j \ne t}} & = \sum_{\substack{k=1 \\ x_t | x_k,\ x_k/x_t \text{ squarefree}}}^{n} \prod_{\substack{k=1 \\ k \ne l}}^{n} \varphi(x_k) \\
& = \Big(\prod_{l=1}^{n} \varphi(x_l) \Big) \sum_{\substack{k=1 \\ x_t | x_k,\ x_k/x_t \text{ squarefree}}}^{n} \frac{1}{\varphi(x_k)}
\end{aligned}$$

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and, similarly,

$$
\begin{aligned}
&\det(\operatorname{lcm}(x_i, x_j))_{\substack{1 \le i, j \le n \\ i \ne t, j \ne t}} \\
&= \Big(\prod_{\substack{l=1 \\ l \ne t}}^{n} x_l^2\Big) \sum_{\substack{k=1 \\ x_t | x_k,\ x_k/x_t \text{ squarefree}}}^{n} \prod_{\substack{l=1 \\ l \ne k}}^{n} \frac{\pi(x_l)\varphi(x_l)}{x_l^2} \\
&= \frac{(-1)^{\sum_{l=1}^{n} \omega(x_l)}}{x_t^2} \Big(\prod_{l=1}^{n} \operatorname{rad}(x_l)\varphi(x_l)\Big) \sum_{\substack{k=1 \\ x_t | x_k,\ x_k/x_t \text{ squarefree}}}^{n} \frac{(-1)^{\omega(x_k)} x_k^2}{\operatorname{rad}(x_k)\varphi(x_k)}
\end{aligned}
$$

as required. This completes the proof of Theorem 1.2.

Proof of Theorem 1.3. Applying Theorem 1.2 with $x_i = i$ for $1 \le i \le n$,

$$
\begin{aligned}
\det((\gcd(i, j)))_{\substack{1 \le i, j \le n \\ i \ne t, j \ne t}} &= \Big(\prod_{l=1}^{n} \varphi(l)\Big) \sum_{\substack{k=1 \\ t | k,\ k/t \text{ squarefree}}}^{n} \frac{1}{\varphi(k)} \\
&= \Big(\prod_{l=1}^{n} \varphi(l)\Big) \sum_{\substack{k=1 \\ k \text{ is squarefree}}}^{\lfloor n/t \rfloor} \frac{1}{\varphi(kt)}
\end{aligned}
$$

and

$$
\begin{aligned}
&\det((\operatorname{lcm}(i, j)))_{\substack{1 \le i, j \le n \\ i \ne t, j \ne t}} \\
&= \frac{(-1)^{\sum_{l=1}^{n} \omega(l)}}{t^2} \Big(\prod_{l=1}^{n} \operatorname{rad}(l)\varphi(l)\Big) \sum_{\substack{k=1 \\ t | k,\ k/t \text{ squarefree}}}^{n} \frac{(-1)^{\omega(k)} k^2}{\operatorname{rad}(k)\varphi(k)} \\
&= \frac{(-1)^{\sum_{l=1}^{n} \omega(l)}}{t^2} \Big(\prod_{l=1}^{n} \operatorname{rad}(l)\varphi(l)\Big) \sum_{\substack{k=1 \\ k \text{ is squarefree}}}^{\lfloor n/t \rfloor} \frac{(-1)^{\omega(tk)} (tk)^2}{\operatorname{rad}(tk)\varphi(tk)} \\
&= (-1)^{\sum_{l=1}^{n} \omega(l)} \Big(\prod_{l=1}^{n} \operatorname{rad}(l)\varphi(l)\Big) \sum_{\substack{k=1 \\ k \text{ is squarefree}}}^{\lfloor n/t \rfloor} \frac{(-1)^{\omega(tk)} k^2}{\operatorname{rad}(tk)\varphi(tk)}
\end{aligned}
$$

as desired. The proof of Theorem 1.3 is complete.

Acknowledgement

The authors would like to thank the anonymous referee for the careful reading of the manuscript and helpful comments.

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