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Václav Šimerka: quadratic forms and factorization

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Abstract

In this article we show that the Czech mathematician Václav Šimerka discovered the factorization of $\frac{1}{9}(10^{17}-1)$ using a method based on the class group of binary quadratic forms more than 120 years before Shanks and Schnorr developed similar algorithms. Šimerka also gave the first examples of what later became known as Carmichael numbers.

Introduction

According to Dickson [**6**, I. p. 172], the number

$$N = 1111111111111111 = \frac{10^{17}-1}{9} \tag{1}$$

was first factored by Le Lasseur in 1886, and the result was published by Lucas in the same year. Actually the factorization of N already appeared as a side result in a forgotten memoir [**22**] published in 1858 by Václav Šimerka, in which he presented his ideas on composition of positive definite forms, computation of class numbers, and the prime factorization of large integers such as N.

In the following we denote the *binary quadratic form* $Q(x,y) = Ax^2 + Bxy + Cy^2$ by (A, B, C); the *discriminant* of Q is $\operatorname{disc} Q = B^2 - 4AC$. The form (A, B, C) is *positive definite* if and only if $A > 0$ and $\Delta < 0$, and *primitive* if $\gcd(A, B, C) = 1$. We say that a form Q *represents* an integer m if there exist integers x, y with $Q(x, y) = m$; it represents m *primitively* if x and y can be chosen coprime. Two forms Q, Q' are called *equivalent* if there exist integers a, b, c, d with $ad - bc = 1$ and $Q'(x, y) = Q(ax + by, cx + dy)$; it is easily seen that the discriminant is an invariant of this action. If Q primitively represents m, then Q is equivalent to a form (m, B, C); in particular, forms representing 1 are equivalent to the principal form $Q_0 = (1, h, m)$, with $h \in \{0, 1\}$ and m defined by $\Delta = 4m + h$. Gauss has shown in his Disquisitiones Arithmeticae that the equivalence classes of primitive forms with discriminant Δ form a finite group with respect to *composition*; this group with unit element $[Q_0]$ is called the *class group* of forms with discriminant Δ. The order of the class group is called the *class number* and is denoted by h_Δ or simply by h. For a modern presentation of Gauss composition see Flath [**9**] and Bhargava [**3**].

Now consider the binary quadratic form

$$Q = (2, 1, 1388888888888889)$$

with discriminant $\Delta = -N$, where N is the number in (1). If we knew that the class number $h = 107019310$ was (a multiple of) the order of $[Q]$ in $\mathrm{Cl}(-N)$, then a simple calculation would reveal that

$$Q^{h/2} \sim (2071723, 2071723, 1341323520),$$

from which we could read off the factorization

$$N = 2071723 \cdot 5363222357.$$

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(The explanation for this fact is Gauss's genus theory, which predicts that the ambiguous forms (forms of order dividing 2 in the class group) are closely related to the factorizations of the discriminant Δ.) This idea for factoring integers was later rediscovered by Daniel Shanks in the 1970s; subsequent work on this idea led Shanks to introduce the notion of infrastructure, which was given a 'modern' interpretation by Lenstra (see for example Schoof [19]), and has played a major role in algorithmic number theory since then.

In [22], Šimerka explains Gauss's theory of composition using the language from Legendre's *Théorie des Nombres*. The rest of his article [22] is dedicated to the calculation of the order of (the equivalence class of) a quadratic form in the class group, and an application to factoring integers.

In this article we will review Šimerka's work and explain some of his calculations so that the readers may convince themselves that [22] contains profound ideas and important results.

1. *A short biography*

Václav Šimerka (in his German publications, Šimerka used the germanized name Wenzel instead of Václav) was born on 20 December 1819, in Hochwesseln (Vysokém Veselí). He studied philosophy and theology in Königgrätz, was ordained in 1845 and worked as a chaplain in Žlunice near Jičín. He started studying mathematics and physics in 1852 and became a teacher at the gymnasium of Budweis. He did not get a permanent appointment there, and in 1862 became priest in Jenšovice, near Vusoké Mýto. Today, Šimerka is remembered for his textbook on algebra (1863); its appendix contained an introduction to calculus and is the first Czech textbook on calculus. Šimerka died in Praskačka near Königgrätz (Praskačce u Hradce Králové) on 26 December 1887.

Šimerka's contributions to the theory of factoring have not been noticed at all, and his name does not occur in any history of number theory except Dickson's: see [6, II, p. 196] for a reference to Šimerka's article [25], which deals with the diophantine problem of rational triangles. In [6, III, p. 67], Dickson even refers to [22] in connection with the composition of binary quadratic forms.

In [23], Šimerka gave a detailed presentation of a large part of Legendre's work on sums of three squares. In [26], Šimerka proved that $7 \cdot 2^{14} + 1 \mid F_{12}$ and $5 \cdot 2^{25} + 1 \mid F_{23}$ (these factors had just been obtained by Pervouchin), where F_n denotes the nth Fermat number. In [27], Šimerka listed the Carmichael numbers [28]

$$n = 561, 1105, 1729, 2465, 2821, 6601, 8911$$

long before Korselt [14] gave criteria hinting at their existence and Carmichael [4] gave what was believed to be the first example. All of Šimerka's examples are products of three prime factors, and there are no others below 10 000.

I do not know where Šimerka acquired his knowledge of number theory. Šimerka was familiar with Legendre's '*Essais de Théorie des Nombres*' and Gauss's '*Disquisitiones Arithmeticae*'[†], as well as with publications by Scheffler [17] on diophantine analysis[‡], by Dirichlet [7] and Lipschitz [15] on the class number of forms with nonsquare discriminants, and by Arndt [1] on a simple proof of the principal genus theorem. Since the articles by Lipschitz and Arndt

[†]Actually it seems to me that Šimerka did not work through the whole *Disquisitiones*; his articles suggest that he learned composition from Gauss and then worked through Legendre's *Théorie des Nombres* with Gauss's notion of composition. In fact Šimerka regularly claimed to have discovered the 'periodicity' of forms, which is essentially a trivial consequence of the group structure provided by composition.

[‡]This is an interesting book, which contains not only the basic arithmetic of the integers up to quadratic reciprocity, but also discusses topics such as continued fractions in Gaussian integers, which are explained using geometric diagrams, and the quadratic reciprocity law in $\mathbb{Z}[i]$.

appeared in 1857 and 1859, Šimerka must have had access to *Crelle's Journal* while he was teaching in Budweis.

Šimerka's article [22] contains other ideas that we do not discuss here. In particular, in [22, Article 12] he tries to get to grips with decompositions of noncyclic class groups into 'periods' (cyclic subgroups); in this connection he gives the example $\Delta = -2184499$ with class group of type $(5, 5, 11)$; as observed by Šimerka, this is a 'remarkably rare case'. In fact, the smallest discriminant with a noncyclic 5-class group is $\Delta = -11199$, and the minimal m with $\Delta = -4m$ and noncyclic 5-class group is $m = 4486$. In [22, Article 18], Šimerka solves diophantine equations of the form $pz^m = ax^2 + bxy + cy^2$.

For more on Šimerka, see [5, 13, 16].

2. *The Šimerka map*

Let us now present Šimerka's ideas from [22] in a modern form. At the end of this section, we will explain Šimerka's language. Let Q be a positive definite binary quadratic form with discriminant Δ. If Q primitively represents a (necessarily positive) integer a, then Q is equivalent to a unique form (a, B, C) with $-a < B \leqslant a$. Let

$$a = p_1^{a_1} \dots p_r^{a_r}$$

denote the prime factorization of a. For each prime $p_j \mid a$, fix an integer $-p_j < b_j \leqslant p_j$ with $B \equiv b_j \bmod p_j$ and set

$$s_j = \begin{cases} +1 & \text{if } b_j \geqslant 0, \\ -1 & \text{if } b_j < 0. \end{cases}$$

Thus if $a = Q(x, y)$, then we can define

$$\check{s}(Q, a) = \prod p_j^{s_j a_j}.$$

Example. The principal form $Q_0 = (1, 0, 5)$ with discriminant -20 represents the following values.

a	1	5	6	9	14	21	21
Q	$(1, 0, 5)$	$(5, 0, 1)$	$(6, 2, 1)$	$(9, 4, 1)$	$(14, 6, 1)$	$(21, 8, 1)$	$(21, 20, 5)$
$\check{s}(a, Q_0)$	1	5	$2 \cdot 3$	3^2	$2 \cdot 7$	$3 \cdot 7$	$3^{-1} \cdot 7$

Forms equivalent to $Q = (2, 2, 3)$ give us the following values.

a	2	3	7	87	87
Q	$(2, 2, 3)$	$(3, -2, 2)$	$(7, 6, 2)$	$(87, 26, 2)$	$(87, 32, 3)$
$\check{s}(a, Q_0)$	2	3^{-1}	7	$3 \cdot 29$	$3 \cdot 29^{-1}$

The ideal theoretic interpretation of the Šimerka map is the following: there is a correspondence between binary quadratic forms Q with discriminant $\Delta < 0$ and ideals $\mathfrak{a}(Q)$ in a suitable order of the quadratic number field $\mathbb{Q}(\sqrt{\Delta})$. Equivalent forms correspond to equivalent ideals, and integers a represented by Q, say $Q(x, y) = a$, correspond to norms of elements $\alpha\mathfrak{a}(Q)$ via $a = N\alpha / N\mathfrak{a}(Q)$. Integers represented primitively by Q are characterized by the fact that $\alpha \in \mathfrak{a}(Q)$ is not divisible by a rational prime number. If we fix prime ideals $\mathfrak{p}_j = \mathfrak{a}(Q_j)$ by $\mathfrak{a}(Q_j)$ for $Q_j = (p_j, B_j, C)$, with $0 \leqslant B_j \leqslant p_j$, and formally set $\mathfrak{p}_j^{-1} = \mathfrak{a}(Q_j')$, with $Q_j' = (p_j, -B_j, C)$, then $\check{s}(a, Q) = p_1^{a_1} \dots p_r^{a_r}$ is equivalent to $(\alpha) = \mathfrak{p}_1^{a_1} \dots \mathfrak{p}_r^{a_r} \mathfrak{a}(Q)$.

Assume that $a = p_1 \dots p_r$, and that $Q = (a, B, C)$. Then Dirichlet composition shows that

$$(a, B, C) = (p_1, B, p_2 \dots p_r C) \cdot (p_2, B, p_1 p_3 \dots p_r C) \dots (p_r, B, p_1 \dots p_{r-1} C).$$

If we write $b_j \equiv B \bmod 2p_j$ with $-p_j < b_j \leqslant p_j$, then

$$\check{s}(a, Q) = \check{s}(p_1, Q_1) \dots \check{s}(p_r, Q_r)$$

by definition of $\check{s}$.

We start by showing that the value set of $\check{s}$ is closed with respect to inversion. To this end we use the notation $(A, B, C)^{-1} = (A, -B, C)$. Then it follows right from the definition of $\check{s}$ that if $\check{s}(a, Q) = r$, then $\check{s}(a, Q^{-1}) = r^{-1}$.

Now we make the following claim.

LEMMA 2.1. *Let Δ be a fundamental discriminant. Assume that $Q_1(x_1, y_1) = a_1$ and $Q_2(x_2, y_2) = a_2$, and that $Q_3 \sim Q_1 Q_2$. Then there exist integers a_3, x_3, y_3 such that $Q_3(x_3, y_3) = a_3$ and $\check{s}(a_3, Q_3) = \check{s}(a_1, Q_1) \cdot \check{s}(a_2, Q_2)$.*

Proof. Writing $Q_1 = (a_1, B_1, C_1) = (p_1, B_1, a_1 C_1/p_1) \dots (p_r, B_1, a_1 C_1/p_r)$ and $Q_2 = (a_2, B_2, C_2) = (q_1, B_2, a_2 C_2/q_1) \dots (q_s, B_2, a_2 C_2/q_s)$, where $a_1 = p_1 \dots p_r$ and $a_2 = q_1 \dots q_s$ are the prime factorizations of a_1 and a_2, we see that it is sufficient to prove the result for prime values of a_1 and a_2. There are several cases.

(1) The case where $Q_1 = (p, b_1, c_1)$, $Q_2 = (q, b_2, c_2)$ with $p \neq q$: for composing these forms using Dirichlet's method, we choose an integer b satisfying the congruences

$$b \equiv b_1 \mod 2p \quad \text{and} \quad b \equiv b_2 \mod 2q.$$

Then $Q_1 \sim (p, b, qc')$ and $Q_2 \sim (q, b, pc')$, and we find $Q_1 Q_2 = (pq, b, c')$ as well as $\check{s}(pq, Q_1 Q_2) = \check{s}(p, Q_1)\, \check{s}(q, Q_2)$ by the definition of $\check{s}$.

(2) The case where $Q_1 = (p, b_1, c_1)$, $Q_2 = (p, -b_1, c_1) = Q^{-1}$: here Dirichlet composition shows $Q_1 Q_2 = (1, b_1, pc_1) \sim Q_0$, and since $\check{s}(Q_2) = \check{s}(Q_1)^{-1}$ we also have $1 = \check{s}(1, Q_1 Q_2) = \check{s}(p, Q_1)\, \check{s}(p, Q_2)$.

(3) The case where $Q_1 = (p, b_1, c_1) = Q_2$: if $p \nmid \Delta$, then $p \nmid b_1$, and we can easily find an integer $b \equiv b_1 \bmod 2p$ with $b^2 \equiv \Delta \bmod 2p_1^2$. But then $Q_1 \sim (p, b, pc')$ and, by Dirichlet composition, $Q_1^2 = (p^2, b, c')$. As before, the definition of $\check{s}$ immediately shows that $\check{s}(p^2, Q_1^2) = \check{s}(p, Q_1)^2$.

If $p \mid \Delta$ and p is odd, on the other hand, then $p \mid b_1$. Since Δ is fundamental, the form Q_1 is ambiguous, hence $Q_1^2 \sim Q_0$. Since $\check{s}(Q_1) = 1$, the multiplicativity is clear.

This completes the proof. □

PROPOSITION 2.2. *Let Q_0 denote the principal form with discriminant $\Delta < 0$. Then the elements $\check{s}(a, Q_0)$ form a subgroup $\mathcal{R}$ of $\mathbb{Q}^\times$.*

Proof. It remains to show that if Q represents a and b, then it represents ab in such a way that $\check{s}(ab, Q_0) = \check{s}(a, Q_0)\, \check{s}(b, Q_0)$. Again we can reduce this to the case of prime values of a and b, and in this case the claim follows from the proof of Lemma 2.1. □

PROPOSITION 2.3. *Assume that a is represented properly by Q, and that a' is represented properly by Q'. If $Q \sim Q'$, then*

$$\check{s}(a, Q) \equiv \check{s}(a', Q') \mod \mathcal{R}.$$

Proof. Since equivalent forms represent the same integers it is sufficient to show that if a form Q properly represents numbers a and b, then $\check{s}(a, Q) \equiv \check{s}(b, Q) \bmod \mathcal{R}$.

Assume that $Q = (A, B, C)$, and set $\check{s}(a, Q) = r$ and $\check{s}(b, Q) = s$. If a and b are coprime, then $\check{s}(ab, Q_0) = r \cdot s^{-1} \in \mathcal{R}$, where Q_0 is the composition of Q and Q^{-1}. This implies the claim.

If a and b have a factor in common, then there is an integer c such that $n = ab/c^2$ is represented by Q_0 in such a way that $\check{s}(n, Q_0) = r \cdot s^{-1} \in \mathcal{R}$, and the claim follows as above. □

These propositions show that š induces a homomorphism

$$\check{s} : \mathrm{Cl}(\Delta) \longrightarrow \mathbb{Q}^\times/\mathcal{R}$$

from the class group $\mathrm{Cl}(\Delta)$ to $\mathbb{Q}^\times/\mathcal{R}$, which we will also denote by š, and which will be called the Šimerka map.

THEOREM 2.4. *Let $\Delta < 0$ be a fundamental discriminant. Then the Šimerka map is an injective homomorphism of abelian groups.*

Proof. We have to show that š is injective. To this end, let $[Q]$ denote a class with $a = \check{s}(Q) \in \mathcal{R}$. Then there is a form $Q_0' = (A, B, C) \sim Q_0$ with $\check{s}(A, Q_0) = a$. But then $Q_1 = Q \cdot (A, -B, C)$ is a form equivalent to Q with $\check{s}(Q_1) = 1$. This in turn implies that Q_1 represents 1, hence is equivalent to the principal form by the classical theory of binary quadratic forms. □

Šimerka's idea is to use a set of small prime numbers $S = \{p_1, \dots, p_r\}$ which are smaller than $\sqrt{-\Delta/3}$ (and a subset of these if $|\Delta|$ is large), find integers a_j primitively represented by Q whose prime factors are all in S, and use linear combinations to find a relation in $\mathcal{R}$, which gives him an integer h such that $Q^h \sim 1$. It is then easy to determine the exact order of Q.

Šimerka's language. Šimerka denotes binary quadratic forms $Ax^2 + Bxy + Cy^2$ by (A, B, C) and considers forms with even as well as odd middle coefficients. The principal form with discriminant Δ is called an end form[†] (Endform, Schlussform), and ambiguous[‡] forms are called middle forms (Mittelformen).

The subgroup generated by a form Q is called its period, the exponent of a form Q in the class group is called the length of its period. Šimerka represents a form $f = (A, B, C)$ by a small prime number p represented by f; the powers $f1 = f$, $f2$, $f3$ of f then represent p, p^2, p^3 and so on, and the exponent m of the mth power fm is called the pointer (Zeiger[§]) of f. What we denote by $\check{s}(Q^m) \equiv a \bmod \mathcal{R}$, Šimerka wrote as $fm = a$.

Šimerka introduced this notation in [**22**, Article 10]; instead of $\check{s}(Q) = 2$ for $Q = (2, 0, C)$ he simply wrote $(2, 0, C) = 2$. He explained the general case as follows:

> So ist z.B. $(180, -17, 193) = \frac{3^2 \times 5}{2^2}$ weil $180 = 2^2 \times 3^2 \times 5$ und $-17 \equiv -1 \pmod 4$, $-17 \equiv 1 \pmod 6$, $-17 \equiv 3 \pmod{10}$[¶].

One of the tricks he used over and over again is the following:

$$(A, B, C) \sim (A, B \pm 2A, A \pm B + C) \sim (A \pm B + C, -B \mp 2A, A) \tag{2}$$

shows that if $Q = (A, B, C)$ represents an integer $m = Q(1, -1) = A \pm B + C$, then $\check{s}(Q)$ can be computed from $Q \sim (m, \mp 2A - B, A)$. Similarly, we have

$$(A, B, C) \sim (A \pm B + C, B \pm 2C, C).$$

[†]Computing the powers of a form Q, one finds $Q, Q^2, \dots, Q^h \sim Q_0$ before everything repeats. The last form in such a 'period' of reduced forms is thus always the principal form.

[‡]The word ambiguous was coined by Poullet-Deslisle in the French translation of Gauss's Disquisitiones Arithmeticae; it became popular after Kummer had used it in his work on higher reciprocity laws. Šimerka knew Legendre's 'diviseurs quadratiques bifides' as well as Gauss's 'forma anceps'.

[§]This word is apparently borrowed from the book [**8**] on combinatorial analysis by Andreas von Ettinghausen, professor of mathematics at the University of Vienna. Ettinghausen used the word 'Zeiger' (see [**8**, p. 2]) as the German translation of the Latin word 'index'. Šimerka refers to [**8**] in [**22**, p. 55].

[¶]Thus we have, for example, $(180, -17, 193) = \frac{3^2 \times 5}{2^2}$ because $180 = 2^2 \times 3^2 \times 5$ and $-17 \equiv -1 \pmod 4$, $-17 \equiv 1 \pmod 6$, $-17 \equiv 3 \pmod{10}$.

3. *Šimerka's calculations*

In this section we will reconstruct a few of Šimerka's calculations of (factors of) class numbers and factorizations.

When $\Delta = -10079$. Šimerka first considers a simple example (see [22, p. 58]): he picks a discriminant Δ for which $\Delta - 1$ is divisible by 2, 3, 5 and 7 (which allows him to use a small 'factor base'), namely $\Delta = -10079$. Consider the form $Q = (5, 1, 504)$ with discriminant Δ. The small powers of Q provide us with the following factorizations.

n	Q^n	$\check{s}(Q^n)$
1	$\sim(504, -1, 5)$	$2^{-3} \cdot 3^{-2} \cdot 7^{-1}$
3	$(36, 17, 72)$	$2^2 \cdot 3^{-2}$
	$\sim(72, -17, 36)$	$2^{-3} \cdot 3^2$

This implies

$$\check{s}(Q^6) \equiv \check{s}(Q^3)\,\check{s}(Q^3) \equiv 2^2 \cdot 3^{-2} \cdot 2^{-3} \cdot 3^2 \equiv 2^{-1},$$
$$\check{s}(Q^{15}) \equiv \check{s}(Q^3)^3\,\check{s}(Q^3)^2 \equiv 2^6 \cdot 3^{-6} \cdot 2^{-6} \cdot 3^4 \equiv 3^{-2},$$
$$\check{s}(Q^{32}) \equiv \check{s}(Q^{-1})\,\check{s}(Q^{-3})\,\check{s}(Q^6)^6 \equiv 7.$$

Now $7 = \check{s}(R)$ for $R = (7, 1, 360)$: this is easily deduced from $\Delta \equiv 1 \equiv 1^2 \bmod 7$. From $R^2 \sim (49, -41, 60)$ Šimerka reads off $\check{s}(Q^{64}) \equiv 2^2 \cdot 3^{-1} \cdot 5$. But then $\check{s}(Q^{63}) \equiv 2^2 \cdot 3^{-1}$ and therefore

$$\check{s}(Q^{75}) \equiv \check{s}(Q^{63}) \cdot \check{s}(Q^6)^2 \equiv 2^2 \cdot 3^{-1} \cdot 2^{-2} \equiv 3 \mod \mathcal{R}.$$

This implies $\check{s}(Q^{150}) \equiv \check{s}(Q^{15})$ and therefore $\check{s}(Q^{135}) \equiv 1 \bmod \mathcal{R}$. Since neither Q^{45} nor Q^{27} are principal, the class of Q has order 135.

For showing that $h(\Delta) = 135$, Šimerka would have to determine the pointers of all primes $p < \sqrt{-\Delta/3} \approx 100.3$. The fact that h is odd would then also show that Δ is a prime number.

When $\Delta = -121271$. For larger discriminants, Šimerka suggests the following method.

> Bei grossen Determinanten, oder wo die vorige Methode nicht zum Ziele führt, nimmt man die Zeiger einiger kleiner Primzahlen als unbekannt an, scheidet dann jene Grössen aus den Producten der Bestimmungsgleichungen aus, und sucht die anderen Primzahlen in Bestimmungsgleichungen durch jene unbekannten Zeiger darzustellen[†].

Šimerka chooses the discriminant $\Delta = -121271$; in the course of the calculation it becomes clear that $\Delta = 99^2 - 2^{17}$, and quite likely the discriminant was constructed in this way. This is supported by Šimerka's remark on [22, p. 64] that if $D = a^m - b^2$ is a (positive) determinant and if a is odd, then the exponent of the form $(a, 2b, a^{m-1})$ is divisible by m, as can be seen from the 'period'

$$(a, 2b, a^{m-1}), (a^2, 2b, a^{m-2}), \ldots, (a^m, 2b, 1).$$

Observe that this statement only holds under the additional assumption that these forms be reduced, that is, that $0 < 2b \leqslant a$. Examples are $D = 3^3 - 1 = 26$ and $h(-4 \cdot 26) = 6$, or $D = 3^5 - 4 = 239$ and $h(-4 \cdot 239) = 15$. A similar observation was made by Joubert [12] just a few years after Šimerka. The connection between classes of order n and solutions of the diophantine equation $a^m - Dc^2 = b^2$ was investigated recently in [10].

Let us write $Q_2 = (2, 1, 15159)$ and $Q_3 = (3, 1, 10106)$. Then $Q_2^2 \sim (4, 5, 7581)$ and $\check{s}(Q_2^2) \equiv 3 \cdot 7^{-1} \cdot 19^{-2}$. Since $\check{s}(Q_3) \equiv 3$, we find $\check{s}(Q_2^{-2} Q_3) \equiv 7 \cdot 19$.

Also, $Q_2^3 \sim (8, 13, 3795)$ gives $\check{s}(Q_2^3) \equiv 3^{-1} \cdot 5 \cdot 11^{-1} \cdot 23$ and $\check{s}(Q_2^3 Q_3) \equiv 5 \cdot 11^{-1} \cdot 23$.

[†]For large determinants, or in cases where the preceding method is not successful, we take the pointers of some small primes as unknowns, eliminate those numbers from the products of the determination equations, and seek to represent these unknown pointers by the other primes in these determination equations.

We can summarize Šimerka's calculations as follows.

n	$Q_2^n \sim$	$\check{s}(Q_2^n) \bmod \mathcal{R}$	n	$Q_2^n \sim$	$\check{s}(Q_2^n) \bmod \mathcal{R}$
2	$(4, 5, 7581)$		6	$(64, 29, 477)$	
	$(7581, -5, 4)$	$3 \cdot 7^{-1} \cdot 19^{-2}$		$(477, -29, 64)$	$3^2 \cdot 53$
3	$(8, 13, 3795)$			$(675, 227, 64)$	$3^{-3} \cdot 5^{-2}$
	$(3795, -13, 8)$	$3^{-1} \cdot 5 \cdot 11^{-1} \cdot 23$	7	$(128, 157, 285)$	
4	$(16, 29, 1908)$			$(285, -157, 128)$	$3^{-1} \cdot 5 \cdot 19^{-1}$
	$(1953, -61, 16)$	$3^{-2} \cdot 7^{-1} \cdot 31$		$(483, 355, 128)$	$3^{-1} \cdot 7 \cdot 23^{-1}$
5	$(32, 29, 954)$				
	$(957, 35, 32)$	$3^{-1} \cdot 11 \cdot 29$			
	$(1015, -93, 32)$	$5^{-1} \cdot 7 \cdot 29$			

Note that if $\check{s}(Q_2^n) \equiv 2^{-1}u$ for some odd number u, then $\check{s}(Q_2^{n+1}) \equiv u$. Thus $\check{s}(Q_2^4) \equiv 2^{-2} \cdot 3^2 \cdot 53$ implies $\check{s}(Q_2^6) \equiv 3^2 \cdot 53$, and in such cases we have listed only the relation that does not involve a power of 2.

The computation of Q_2^7 reveals $\Delta = 99^2 - 2^{17}$, and shows that $\check{s}(Q_2^7) \equiv 2^{-8}$, which gives $\check{s}(Q_2^{15}) \equiv 1$.

Now Šimerka continues as follows: the relations

$$\check{s}(Q_2^2) \equiv 3 \cdot 7^{-1} \cdot 19^{-2} \quad \text{and} \quad \check{s}(Q_2^7) \equiv 3^{-1} \cdot 5 \cdot 19^{-1}$$

give

$$\check{s}(Q_2^{12}) \equiv \check{s}((Q_2^7)^2 Q_2^{-2}) \equiv 3^{-2} \cdot 5^2 \cdot 19^{-2} \cdot 3^{-1} \cdot 7 \cdot 19^2 = 3^{-3} \cdot 5^2 \cdot 7.$$

Using the relations

$$\check{s}(Q_2^{12} Q_3^3) \equiv 5^2 \cdot 7 \quad \text{and} \quad \check{s}(Q_2^6 Q_3^3) \equiv 5^{-2},$$

Šimerka deduces

$$\check{s}(Q_2^3 Q_3^6) \equiv \check{s}(Q_2^{18} Q_3^6) \equiv 7. \tag{3}$$

This allows him to eliminate the sevens from his relations, which gives

$$\check{s}(Q_2^{-4} Q_3^7) \equiv \check{s}(Q_2^{-7})\, \check{s}(Q_3)\, \check{s}(Q_2^3 Q_3^6) \equiv 23,$$
$$\check{s}(Q_2^7 Q_3^8) \equiv \check{s}(Q_2^4)\, \check{s}(Q_3^2)\, \check{s}(Q_2^3 Q_3^6) \equiv 31.$$

For the actual computation of the order of Q_3, only the relation (3) will be needed.

Šimerka also investigates the powers of Q_3 and finds the following.

n	$Q_3^n \sim$	$\check{s}(Q_3^n) \bmod \mathcal{R}$	n	$Q_3^n \sim$	$\check{s}(Q_3^n) \bmod \mathcal{R}$
1	$(3, 1, 10106)$		5	$(243, 205, 168)$	
	$(10108, 2, 3)$	$2^2 \cdot 7 \cdot 19^2$		$(616, 541, 168)$	$2^3 \cdot 7^{-1} \cdot 11^{-1}$
3	$(27, 43, 1140)$		6	$(729, 205, 56)$	
	$(1210, -97, 27)$	$2^{-1} \cdot 5 \cdot 11^{-2}$		$(56, -205, 729)$	$2^{-3} \cdot 7$
	$(1162, 65, 27)$	$2 \cdot 7^{-1} \cdot 83$			
4	$(81, 43, 380)$				
	$(380, -43, 81)$	$2^2 \cdot 5^{-1} \cdot 19^{-1}$			
	$(418, 119, 81)$	$2^{-1} \cdot 11 \cdot 19$			

Šimerka observes

$$\check{s}(Q_2^2Q_3^9) \equiv \check{s}(Q_3^3)\,\check{s}(Q_2^{-1})\,\check{s}(Q_2^3Q_3^6) \equiv 83,$$

but does not use this relation in the following. He continues with

$$\check{s}(Q_2Q_3^4) \equiv 11\cdot 19, \quad \check{s}(Q_2^3Q_3^{-5}) \equiv 7\cdot 11,$$

from which he derives the following relations:

$$\check{s}(Q_3^{-11}) \equiv \check{s}(Q_2^3Q_3^{-5})\,\check{s}(Q_2^{-3}Q_3^{-6}) \equiv 11, \quad \check{s}(Q_2Q_3^{15}) \equiv \check{s}(Q_2Q_3^4)\,\check{s}(Q_3^{11}) \equiv 19,$$
$$\check{s}(Q_2^8Q_3^{16}) \equiv \check{s}(Q_2^7)\,\check{s}(Q_3)\,\check{s}(Q_2Q_3^{15}) \equiv 5, \quad \check{s}(Q_2^{22}Q_3^{35}) \equiv \check{s}(Q_2^{16}Q_3^{32})\,\check{s}(Q_2^6Q_3^3) \equiv 1.$$

Raising the last relation to the 15th power yields $\check{s}(Q_3^{525}) \equiv 1$. Checking that Q_3^{75}, Q_3^{105} and Q_3^{175} are not principal then shows that Q_3 has order $h = 525 = 3\cdot 5^2\cdot 7$. In fact, `pari` tells us that this is the class number of $\Delta = -121271$.

4. *Class number calculations*

Let us remark first that Šimerka does not compute class numbers but rather the order of a given form in the class group. Note that this is sufficient for factoring the discriminant. Šimerka is well aware of the fact that his method only produces divisors of the class number: in [**22**, Article 13], he writes the following.

> Was die Länge θ anbelangt, sucht man $fm = 1$ zu erhalten, wo dann entweder $\theta = m$ oder ein Theiler von m ist. Die wichtigsten Glieder der Perioden sind die zu kleinen Primzahlen gehörigen Formen. Welches die grösste Primzahl wäre, deren Zeiger man kennen müsse, um vor Irrthum sicher zu sein, konnte ich bis jetzt nicht ermitteln, jedenfalls ist sie kleiner als $\sqrt{D/3}$ bei den unpaaren, und als $2\sqrt{D/3}$ bei den paaren Formen, wahrscheinlich aber reichen dazu nur wenige Primzahlen hin[†].

In the example $\Delta = -121271$ above we have seen that the powers of Q_2 only give a subgroup of order 15 in the class group, whereas the powers of 3 include all forms representing the primes

$$p = 2, 3, 5, 7, 11, 19, 23, 29, 31, 53, 83.$$

For verifying that $h(-121271) = 525$, one would have to find the pointers for the other primes p with $(\Delta/p) = +1$ and $\Delta < 202$ as well, namely those of

$$p = 47, 61, 73, 79, 89, \ldots, 197.$$

Since the pointers of all small primes are known, this is only a little additional work. The fact that the class number is odd then implies that $-\Delta = 121271$ is a prime.

When $\Delta = -4\cdot 265371653$. Consider the forms

$$Q_3 = (3, 2, 88457218), \quad Q_{11} = (11, 10, 24124698) \quad \text{and} \quad Q_{13} = (13, 10, 20413206).$$

Using a computer it is easily checked that $Q_3 \sim Q_{11}^5 Q_{13}^{-3}$, but this relation was apparently not noticed by Šimerka. It would follow easily from

$$Q = Q_{11}^5 = (6591, -6568, 41899), \qquad Q(0,1) = 11\cdot 13\cdot 293,$$
$$Q = Q_{13}^3 = (2197, -2174, 121326), \quad Q(1,-1) = 3\cdot 11\cdot 13\cdot 293,$$

but perhaps the prime 293 was not an element of Šimerka's factor base.

[†]As for the length θ of the period, one tries to find $fm = 1$, and then either $\theta = m$, or θ is a divisor of m. The most important members of the period are those belonging to small prime numbers. I have not yet found what the smallest prime number is whose pointer must be known in order not to commit an error; in any case it is smaller than $\sqrt{D/3}$ for odd forms, and than $2\sqrt{D/3}$ for the even forms, but most likely just a few prime numbers are sufficient.

A computer also finds the following relations among the small powers of these three forms:

$$Q_{11}^{13}Q_{13}^{11} = (1058, 918, 251023); \qquad \check{s}(Q_{11}^{13}Q_{13}^{11}) \equiv 2 \cdot 23^{-2},$$
$$Q_3^{14}Q_{11}^{12}Q_{13} = (529, -140, 501657); \quad \check{s}(Q_3^{14}Q_{11}^{12}Q_{13}) \equiv 23^{-2}.$$

Composition shows that

$$\begin{aligned} Q_3^{-14}Q_{11}Q_{13}^{10} &\equiv Q_{11}^{13}Q_{13}^{11}Q_3^{-14}Q_{11}^{-12}Q_{13}^{-1} \\ &= (1058, 918, 251023)(529, 140, 501657) = (2, 918, 132791167), \end{aligned}$$

and squaring yields

$$Q_3^{-28}Q_{11}^2Q_{13}^{20} \sim Q_0.$$

Similarly,

$$Q_3^3Q_{11}^{15}Q_{13}^{11} = (16389, -16010, 20102), \quad \check{s}(Q_3^3Q_{11}^{15}Q_{13}^{11}) \equiv 2 \cdot 19 \cdot 23^2,$$
$$Q_3^{12}Q_{11}^{15}Q_{13}^{8} = (6859, 5028, 39611), \qquad \check{s}(Q_3^{12}Q_{11}^{15}Q_{13}^{8}) \equiv 19^3,$$

which implies

$$Q_3^3Q_{11}^{15}Q_{13}^{11} \cdot Q_{11}^{13}Q_{13}^{11} \sim (19, 12, 13966931), \quad \check{s}(Q_3^3Q_{11}^{28}Q_{13}^{22}) \equiv 19,$$

and so

$$1 \equiv \check{s}(Q_3^3Q_{11}^{28}Q_{13}^{22})^3 / \check{s}(Q_3^{12}Q_{11}^{15}Q_{13}^{8}) \equiv \check{s}(Q_3^{-3}Q_{11}^{69}Q_{13}^{58}).$$

Eliminating $Q_3 \sim Q_{11}^5Q_{13}^{-3}$ from the relations

$$Q_3^{-28}Q_{11}^2Q_{13}^{20} \sim Q_3^{-3}Q_{11}^{69}Q_{13}^{58} \sim Q_0$$

then implies

$$Q_{11}^{-138}Q_{13}^{104} \sim Q_0 \quad \text{and} \quad Q_{11}^{54}Q_{13}^{67} \sim Q_0,$$

hence

$$Q_{11}^{14862} \sim Q_0.$$

It is then easily checked that Q_3 and Q_{11} have exponent 14862 in the class group, whereas Q_{13} is a sixth power and has order 2477. A quick calculation with `pari` reveals that $h(\Delta) = 14862$.

Šimerka must have proceeded differently, as he records the relations

$$Q_3^{119}Q_{11}^{11}Q_{13}^{8} \sim Q_0, \quad Q_3^{1276}Q_{11}^{94}Q_{13}^{26} \sim Q_0, \quad Q_3^{385}Q_{11}^{31}Q_{13}^{4} \sim Q_0.$$

It is not impossible that by playing around with small powers of Q_3, Q_{11} and Q_{13}, Šimerka's calculations can be reconstructed. It is more difficult to reconstruct Šimerka's factorization of $N = \frac{1}{9}(10^{17} - 1)$, since he left no intermediate results at all (apparently he was forced to shorten his manuscript drastically before publication).

Šimerka knew that it is often not necessary to determine the class number for factoring integers; in [**22**, Article 17] he observed the following.

> Bei Zahlenzerlegungen nach dieser Methode findet man oft $f2a = m^2$, oder es lässt sich aus den Bestimmungsgleichungen eine solche Form ableiten; dann hat man $(f2a/m^2) = (fa/m)^2 = 1$, und es kann $fa : m$ blos eine Schluss- oder Mittelform sein. Gewöhnlich ist das letztere der Fall[†].

To illustrate this idea we present an example that cannot be found in Šimerka's article. Let $\Delta = -32137459$ and consider the form $Q = (5, 1, 1606873)$ with discriminant Δ. It is quickly seen that $Q^{26}(1, 0) = 11^2$. This observation immediately leads to a factorization

[†]In factorizations with this method one often finds $fa = m^2$, or such a form can be derived from certain determination equations; then we have $(f2a/m^2) = (fa/m)^2 = 1$, and $fa : m$ can only be an end or a middle form. Most often, the latter possibility occurs.

of Δ: the form Q^{26} represents 11^2, hence Q^{13} represents 11, as does $Q_{11} = (11, 3, 730397)$. Thus $(Q^{13}R^{-1})^2$ represents 1, which implies that $Q^{13}R^{-1}$ is ambiguous (see [22, p. 36]). In fact, $Q^{13}R^{-1} = (1511, 1511, 5695)$, which gives the factorization $\Delta = -1511 \cdot 21269$.

5. *Shanks*

The factorization method based on the class group of binary quadratic forms was rediscovered by Shanks [21], who, however, used a completely different method for computing the class group: he estimated the class number h using truncated Dirichlet L-series and then found the correct value of h with his baby step – giant step method. Attempts to speed up the algorithm led, within just a few years, to Shanks's discovery of the infrastructure and his square form factorization method SQUFOF.

The factorization method described by Šimerka was rediscovered by Schnorr [18]; the Šimerka map is defined in [18, Lemma 4] (see also [20, Theorem 3.1]), although in a slightly different guise: a quadratic form $Q = (a, b, c)$ is factored into 'prime forms' $I_p = (p, b_p, C)$, where $B = b_p$ is the smallest positive solution of $B^2 \equiv \Delta \bmod 4p$ for $\Delta = -N \equiv 1 \bmod 4$. Thus the equation corresponding to our

$$\check{s}(Q) = \prod_{i=1}^{n} p_i^{\pm e_i} \quad \text{looks like} \quad Q = \prod_{i=1}^{n} (I_p)^{\pm e_i}$$

in [20], 'where the plus sign in the exponent e_i holds if and only if $b \equiv b_{p_i} \bmod 2p_i$'. Variations of this method were later introduced by Mc Curley and Atkin.

The main difference between the methods of Šimerka and Schnorr is that Šimerka factors the forms Q_p^n for small prime numbers p and small exponents n, whereas Schnorr factors products $Q_1^{n_1} \dots Q_r^{n_r}$ of forms $Q_j = (p_j, *, *)$ for primes in his factor base and exponent vectors $(n_1, \dots, n_r)$ chosen at random.

The basic idea of combining relations, which is also used in factorization methods based on continued fractions, quadratic sieves or the number field sieve, is not due to Šimerka but rather occurs already in the work of Fermat and played a role in his challenge to the English mathematicians, notably Wallis and Brouncker. In this challenge, Fermat explained that if one adds to the cube $343 = 7^3$ all its proper divisors, then the sum $1 + 7 + 7^2 + 7^3 = 400 = 20^2$ is a square, and asked for another cube with this property.

Fermat's solution is best explained by studying a simpler problem first, namely that of finding a number n with $\sigma(n^2) = m^2$, where $\sigma(n) = \sum_{d|n} d$ is the sum of all divisors of a number. Making a table of $\sigma(p)$ for small prime powers p one observes that $\sigma(2^4) = \sigma(5^2) = 31$, hence $\sigma(20^2) = 31^2$.

The solution[†] of Fermat's challenge also exploits the multiplicativity of $\sigma(n)$: with little effort one prepares a table for the values of $\sigma(p)$ for small primes p such as the following.

p	$\sigma(p^3)$
2	$3 \cdot 5$
3	$2^3 \cdot 5$
5	$2^2 \cdot 3 \cdot 13$
7	$2^4 \cdot 5^2$
11	$2^3 \cdot 3 \cdot 61$

p	$\sigma(p^3)$
13	$2^2 \cdot 5 \cdot 7 \cdot 17$
17	$2^2 \cdot 3^2 \cdot 5 \cdot 29$
19	$2^3 \cdot 5 \cdot 181$
23	$2^4 \cdot 3 \cdot 5 \cdot 53$
29	$2^2 \cdot 3 \cdot 5 \cdot 421$

p	$\sigma(p^3)$
31	$2^6 \cdot 13 \cdot 37$
37	$2^2 \cdot 5 \cdot 19 \cdot 137$
41	$2^2 \cdot 3 \cdot 7 \cdot 29^2$
43	$2^3 \cdot 5^2 \cdot 11 \cdot 37$
47	$2^5 \cdot 3 \cdot 5 \cdot 13 \cdot 17$

Then it is readily seen that $n = 751530 = 2 \cdot 3 \cdot 5 \cdot 13 \cdot 41 \cdot 47$.

[†]Sufficiently many hints can be found in Frenicle's letter in [29, XXXI], and in subsequent letters by Wallis and Schooten. See also the detailed exposition given by Hofmann [11].

Concluding remarks

Šimerka's contributions to the theory of quadratic forms and the factorization of numbers would have remained unknown if his articles could not be found online. In particular, his memoirs [**22**, **23**, **24**] can be accessed via google books[†], and the articles that appeared in the journal *Časopis* are available on the website of the GDZ[‡] in Göttingen. I would also like to remark that a prerequisite for understanding the importance of [**22**] is a basic familiarity with composition of binary quadratic forms.

Šimerka's question in Section 4 concerning the number of primes p such that the forms (p, B, C) generate the class group was answered under the assumption of the Extended Riemann Hypothesis by Schoof [**19**, Corollary 6.2], who showed that the first $c \log^2 |\Delta|$ prime numbers suffice; Bach [**2**] showed that, for fundamental discriminants Δ, we can take $c = 6$.

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[†]See http://books.google.com.

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