ON THE NILPOTENT RANKS OF CERTAIN SEMIGROUPS OF TRANSFORMATIONS

by G. U. GARBA

(Received 5 November, 1991)

1. Introduction. Let P_n be the semigroup of all partial transformations on the set $X_n = \{1, \ldots, n\}$. As usual, we shall say that an element α in P_n is of type (k, r) or belongs to the set $[k, r]$ if $|\text{dom } \alpha| = k$ and $|\text{im } \alpha| = r$. The *completion* α^* of an element $\alpha \in [n-1, n-1]$ is an element in $[n, n]$ defined by

$$i\alpha^* = j, \qquad x\alpha^* = x\alpha \text{ otherwise},$$

where $\{i\} = X_n \backslash \text{dom } \alpha$ and $\{j\} = X_n \backslash \text{im } \alpha$.

For n even, the subsemigroup SI_n of P_n consisting of all strictly partial one–one transformations was proved to be nilpotent-generated by Gomes and Howie [**2**]. If n is odd, they showed that the nilpotents in SI_n generate $\text{SL}_n \backslash W_{n-1}$, where W_{n-1} consists of all $\alpha \in [n-1, n-1]$ whose completions are odd permutations.

Simultaneously and independently, Sullivan [**7**] showed that the subsemigroup SP_n of P_n consisting of all strictly partial transformations of X_n is nilpotent-generated if n is even. If n is odd, the nilpotents in SP_n generate $\text{SP}_n \backslash W_{n-1}$.

The *rank* of a semigroup S is the cardinality of any subset A of minimal order in S such that $\langle A \rangle = S$. If the generating set A consists of nilpotent elements only, then we shall refer to the cardinality of A as the *nilpotent rank* of S. Since one of the semigroups we will be considering is an inverse semigroup, we would like to clarify the notion of a generating set in an inverse semigroup. Given a subset A in an inverse semigroup S, we shall always want to consider the smallest inverse subsemigroup containing A. In effect this is the set of all finite products of elements of A and their inverses. Following [**3**], we shall use the notation $\langle\langle A \rangle\rangle$ for this inverse subsemigroup. Accordingly, by the rank of an inverse semigroup S we shall mean the cardinality of any subset A of minimal order in S such that $\langle\langle A \rangle\rangle = S$.

Let N and M be the sets of all nilpotent elements in SI_n and SP_n respectively. In [**3**], Gomes and Howie proved that the rank and the nilpotent rank of $\langle\langle N \rangle\rangle$ are both equal to $n + 1$ for all n, and in [**1**], Garba showed that the rank and the nilpotent rank of $\langle M \rangle$ are both equal to $n + 2$ for all n. In Section 2 we generalize the results of Gomes and Howie [**3**] (in line with Howie and McFadden [**6**]) by showing that if $1 \le r \le n - 2$ then the rank and the nilpotent rank of the inverse semigroup

$$U(n, r) = \{\alpha \in \text{SI}_n : |\text{im } \alpha| \le r\}$$

are both equal to $\binom{n}{r} + 1$. In Section 3 we generalize the results of Garba [**1**] by showing that if $1 \le r \le n - 2$ then the rank and the nilpotent rank of the semigroup

$$V(n, r) = \{\alpha \in \text{SP}_n : |\text{im } \alpha| \le r\}$$

are both equal to $(r + 1)S(n, r + 1)$, where $S(n, r + 1)$ is the Stirling number of the second kind, defined by

$$S(n, 1) = S(n, n) = 1, \qquad S(n, r) = S(n - 1, r - 1) + rS(n - 1, r).$$

For standard terms in semigroup theory see [**4**]. In all that follows, we consider $n \ge 3$.

https://doi.org/10.1017/S0017089500030482 Published online by Cambridge University Press

2. One–one partial transformations.

LEMMA 2.1. *For all $r \le n-2$, we have*

$$J_r \subseteq (N \cap J_r)^2,$$

where $J_r = \{\alpha \in \mathrm{SI}_n : |\mathrm{im}\ \alpha| = r\}$ is the $\mathcal{J}$-class of all elements in SI_n with rank r.

Proof. The result is trivial for $r \le 1$. If $r \ge 2$ then the result follows from Remark 3.16 in [**2**], where the authors prove that $J_{n-2} \subseteq (N \cap J_{n-2})^2$, and from Lemma 4.1 in the same paper, which states that if $J_r \subseteq (N \cap J_r)^k$ then $J_{r-1} \subseteq (N \cap J_{r-1})^k$ for $2 \le r \le n-1$.

It follows from this lemma that the nilpotents in J_r generate $U(n, r)$.

Denote by P_r the principal factor $U(n, r)/U(n, r-1)$. Then P_r may be thought of in the usual way as $J_r \cup \{0\}$. Also, P_r has $\binom{n}{r}$ non-null $\mathcal{R}$-classes corresponding to the $\binom{n}{r}$ possible domains of cardinality r, and $\binom{n}{r}$ non-null $\mathcal{L}$-classes corresponding to the $\binom{n}{r}$ possible images. It is a Brandt semigroup isomorphic to $B(S_r, \{1, \ldots, m\})$, where S_r is the symmetric group on X_r and $m = \binom{n}{r}$. Hence, since the rank of S_r is known to be 2, it follows by Theorem 3.3 in [**3**] that P_r has inverse semigroup rank $\binom{n}{r} + 1$.

From [**2**], we borrow the notation $\|a_1 a_2 \ldots a_{r+1}\|$ $(1 \le r \le n-1)$ for the nilpotent α with domain $\{a_1, \ldots, a_r\}$ and image $\{a_2, \ldots, a_{r+1}\}$ for which $a_i\alpha = a_{i+1}$ $(i = 1, \ldots, r)$. We shall refer to these type of nilpotents as *primitive* in the next section.

THEOREM 2.2. *Let $n \ge 3$ and let $r \le n-2$. Then*

$$\mathrm{rank}\langle\langle U(n,r)\rangle\rangle = \mathrm{nilrank}\langle\langle U(n,r)\rangle\rangle = \begin{cases} \binom{n}{r}+1 & \text{if } r \ge 3, \\ \binom{n}{r} & \text{if } r = 2, \\ n-1 & \text{if } r = 1. \end{cases}$$

Proof. From the fact that P_r (as an inverse semigroup) has rank $\binom{n}{r} + 1$ it follows that rank $\langle\langle U(n,r)\rangle\rangle \ge \binom{n}{r} + 1$. To complete the proof we must find a generating set of $\langle\langle U(n,r)\rangle\rangle$ consisting of $\binom{n}{r} + 1$ nilpotents.

Let $A_1, A_2, \ldots, A_m$ be a list of the subsets of X_n of cardinality r. Thus $m = \binom{n}{r}$. Let H_{A_i,A_j} denote the $\mathcal{H}$-class in J_r consisting of all the elements whose domain is A_i and image A_j $(i, j = 1, 2, \ldots, m)$. Suppose that $A_1 = \{1, 2, \ldots, r\}$. Then the $\mathcal{H}$-class H_{A_1,A_1} is the symmetric group on $\{1, 2, \ldots, r\}$, and if $r \ge 3$ then it is generated by the elements σ, τ, where

$$\sigma = (1\,2), \qquad \tau = (1\,2 \ldots r).$$

We now show that each of σ, τ can be expressed as a product of nilpotents. For this purpose, we will suppose that $A_2 = \{2, \ldots, r, r+1\}$, $A_3 = \{1, \ldots, r-1, r+1\}$ and $A_4 = \{2, \ldots, r-1, r+1, r+2\}$. The proof depends on whether r is odd or even. For r odd we have

$$\sigma = \alpha_2^{-1}\beta\alpha_3 \quad \text{and} \quad \tau = \gamma_2^{-1}\alpha_2,$$

where

$$\begin{aligned} \alpha_2 &= \|r+1\ r\ r-1 \ldots 2\ 1\| \in H_{A_2,A_1}, \\ \beta &= \|r\ r-2\ r-4 \ldots 3\ r+1\ r-1 \ldots 4\ 2\ 1\| \in H_{A_2,A_3}, \\ \alpha_3 &= \|r+1\ 1\ 2 \ldots r\| \in H_{A_3,A_1}, \\ \gamma_2 &= \|r+1\ r-1 \ldots 2\ r\ r-2 \ldots 3\ 1\| \in H_{A_2,A_1}. \end{aligned}$$

If for this case we now choose a nilpotent $\alpha_i \in H_{A_i,A_1}$ for $i = 4, \ldots, m$ in an arbitrary way, we see that

$$\sigma, \tau, \alpha_2, \ldots, \alpha_m \in \langle\langle \alpha_2, \ldots, \alpha_m, \beta, \gamma_2 \rangle\rangle.$$

By the remark before Theorem 3.3 in **[3]**, the elements σ, τ, $\alpha_2, \ldots, \alpha_m$ generate P_r. It follows that P_r, and hence also $U(n, r)$ is generated by the $m+1$ nilpotents $\alpha_2, \ldots, \alpha_m$, β, γ_2 provided r is odd.

For r even we have

$$\sigma = \alpha_3^{-1}\beta\alpha_4 \quad \text{and} \quad \tau = \gamma_4^{-1}\alpha_4,$$

where

$$\alpha_3 = \|r+1\,2\,3 \ldots r-2\;r-1\;1\;r\| \in H_{A_3,A_1},$$
$$\beta = \|1\;r-2\;r+1\;3\,2\,5\,4 \ldots r-5\;r-6\;r-3\;r-4\;r-1\;r+2\| \in H_{A_3,A_4},$$
$$\alpha_4 = \|r+2\,2\,4 \ldots r\| \cup \|r+1\;r-1 \ldots 3\;1\| \in H_{A_4,A_1},$$
$$\gamma_4 = \|r+1\;r-2\;r-1\;r-4\;r-3\;r-6\;r-5 \ldots 9\,6\,7\,4\,5\,2\,3\,r\| \cup \|r+2\;1\| \in H_{A_4,A_1}.$$

In this case P_r and hence $U(n, r)$ is generated by the $m+1$ nilpotents $\alpha_2, \ldots, \alpha_m$, β, γ_4, where $\alpha_i \in H_{A_i,A_1}$ are chosen arbitrarily for $i = 2, 5, 6, \ldots, m$.

It now remains to show that the result is true for $r = 2$ and $r = 1$.

If $r = 2$, S_2 is cyclic and thus has only one generator. For this case we will suppose that $A_1 = \{1, 2\}$ and $A_m = \{n-1, n\}$. The $\mathscr{H}$-class H_{A_1,A_1} is the symmetric group on A_1 and is generated by

$$\sigma = (1\,2).$$

Now,

$$\sigma = \gamma_m^{-1}\alpha_m,$$

where

$$\alpha_m = \|n-1\;2\| \cup \|n1\| \in H_{A_m,A_1},$$
$$\gamma_m = \|n-1\;1\| \cup \|n2\| \in H_{A_m,A_1}.$$

So, if we choose nilpotents $\alpha_2, \ldots, \alpha_{m-1}$ as in the above cases, we see that $\alpha_2, \ldots, \alpha_m$, γ_m generate $U(n, r)$. Thus $U(n, r)$ has rank $1 + m - 1 = m$.

If $r = 1$, the symmetric group S_1 has rank 0, and it is easy to verify that the following $n-1$ nilpotents generate $U(n, r)$:

$$\|2\,1\|, \|3\,1\|, \|4\,1\|, \ldots, \|n\,1\|.$$

3. Partial transformations. The semigroup $V(n, r)$ has $r+1$ $\mathscr{J}$-classes, namely J_r, $J_{r+1}, \ldots, J_0$ (where J_0 consists of the empty map). For each t such that $1 \le t \le r$ we have

$$J_t = \bigcup_{k=t}^{n-1} [k, t].$$

The number of $\mathscr{L}$-classes in the $\mathscr{J}$-class J_r of $V(n, r)$ is the number of image sets in X_n of cardinality r, namely $\binom{n}{r}$, and the number of $\mathscr{R}$-classes in J_r is the number of equivalence relations ρ on each of the subsets A of cardinality k (where $n-1 \ge k \ge r$) for which $|A/\rho| = r$, and this number is

$$\sum_{k=r}^{n-1} \binom{n}{k} S(k,r) = \sum_{k=r}^{n} \binom{n}{k} S(k,r) - S(n,r)$$
$$= S(n+1, r+1) - S(n,r)$$
$$= (r+1)S(n, r+1).$$

Like $U(n,r)$, the semigroup $V(n,r)$ is generated by the nilpotent elements in J_r (see Lemma 2.3 in [**1**]). We also have from Lemma 3 in [**6**] that for $2 \le r \le n-2$,

$$\operatorname{rank}(V(n,r)) \ge (r+1)S(n, r+1).$$

THEOREM 3.1. *For $n \ge 3$ and $2 \le r \le n-2$, we have*

$$\operatorname{rank}(V(n,r)) = \operatorname{nilrank}(V(n,r)) = (r+1)S(n,r+1).$$

The proof depends on the following lemma.

LEMMA 3.2. *Suppose that we can arrange the subsets $A_1, \ldots, A_m$ (where $m = \binom{n}{r}$ and $2 \le r \le n-2$) of X_n of cardinality r in such a way that $|A_i \cap A_{i-1}| = r-1$ for $i = 1, \ldots, m-1$ and $|A_m \cap A_1| = r-1$. Then there exist nilpotents $\alpha_1, \ldots, \alpha_p$ (where $p = (r+1)S(n, r+1)$) such that $\{\alpha_1, \ldots, \alpha_p\}$ is a set of generators for $V(n,r)$.*

Proof. Notice first that every element $\alpha \in [k,r]$, $r < k \le n-1$, is expressible as a product of a nilpotent in its own $\mathcal{R}$-class and an element in $[r,r]$. For

$$\alpha = \begin{pmatrix} A_1 & A_2 & \ldots & A_{r-1} & A_r \\ a_2 & a_3 & \ldots & a_r & x \end{pmatrix} \begin{pmatrix} a_2 & a_3 & \ldots & a_r & x \\ b_1 & b_2 & \ldots & b_{r-1} & b_r \end{pmatrix},$$

where

$$\alpha = \begin{pmatrix} A_1 & \ldots & A_r \\ b_1 & \ldots & b_r \end{pmatrix},$$

$a_i \in A_i$ for all $i \in \{2, \ldots, r\}$ and $x \in X_n \backslash \operatorname{dom} \alpha$.

In the arrangement of our subsets $A_1, \ldots, A_m$ we shall assume that $A_1 = \{n-r+1, n-r+2, \ldots, n\}$, $A_2 = \{n-r, \ldots, n-1\}$ and $A_m = \{1, n-r+2, \ldots, n\}$. We shall also represent any two adjacent subsets A_i, A_{i+1} by the two subsets $\{x_1, \ldots, x_{r-1}, y_i\}$ and $\{x_1, \ldots, x_{r-1}, z_{i+1}\}$, where $z_{i+1} \ne y_i$, and $z_{i+1}, y_i \ne x_i$ for any i. Define H_{A_i, A_j} to consist of all elements $\alpha \in [r,r]$ for which $\operatorname{dom} \alpha = A_i$ and $\operatorname{im} \alpha = A_j$. For $i = 1, \ldots, m$ define a mapping $\xi_i \in H_{A_i, A_m}$ as follows:

$$\xi_1 = \begin{pmatrix} n-r+1 & n-r+2 & \ldots & n \\ 1 & n-r+2 & \ldots & n \end{pmatrix},$$

$$\xi_2 = \begin{pmatrix} n-r & n-r+1 & n-r+2 & \ldots & n-1 \\ n-r+2 & 1 & n-r+3 & \ldots & n \end{pmatrix},$$

and for $i = 2, \ldots, m-1$ if

$$\xi_i = \begin{pmatrix} x_1 & x_2 & \ldots & x_{r-1} & y_i \\ t_1 & t_2 & \ldots & t_{r-1} & t_r \end{pmatrix}$$

https://doi.org/10.1017/S0017089500030482 Published online by Cambridge University Press

define

$$\xi_{i+1}=\begin{pmatrix} x_1 & x_2 & \dots & x_{r-1} & z \\ t_2 & t_3 & \dots & t_r & t_1 \end{pmatrix}.$$

Then it is easy to see that the mapping

$$\phi : B(S_r, \{1, \dots, m\}) \to Q_r$$

defined by $(i, \eta, j)\phi = \xi_i \eta \xi_j^{-1}$ is an isomorphism. Here S_r is the symmetric group on $\{1, n-r+2, \dots, n\}$, Q_r is the principal factor

$$[r, r] \Big/ \bigcup_{l=0}^{r-1} [l, l] \simeq [r, r] \cup \{0\}.$$

From Proposition 2.4 in [**1**], the set

$$T = \{(1, g_1, 1), (1, g_2, 2), (2, e, 3), \dots, (m-1, e, m), (m, e, 1)\},$$

where $g_1 = (1\ n-r+2 \dots n)$, $g_2 = (1\ n-r+2)$ and e is the identity permutation in S_r, generates $B(S_r, \{1, \dots, m\})$. Thus $T\phi$ generates Q_r and hence $[r, r]$. If we now define

$$\alpha_1 = \xi_1 g_2 \xi_2^{-1},\ \alpha_i = \xi_i \xi_{i+1}^{-1} \qquad \text{for } i = 2, \dots, m-1$$

and

$$\beta = \xi_m \xi_1^{-1}, \qquad \delta = \xi_1 g_1 \xi_1^{-1},$$

we obtain a generating set $\{\beta, \delta, \alpha_1, \dots, \alpha_{m-1}\}$ of $[r, r]$, where

$$\alpha_1 = \|n\ n-1 \dots n-r+1\ n-r\|,$$

$$\alpha_i = \|y_i x_{r-1} \dots x_1 z_{i+1}\| \qquad \text{for } i = 2, \dots, m-1$$

are all nilpotents. On the other hand,

$$\delta = \begin{pmatrix} n-r+1 & n-r+2 & \dots & n-1 & n \\ n-r+2 & n-r+3 & \dots & n & n-r+1 \end{pmatrix}$$

is clearly non-nilpotent. However if r is odd we have

$$\delta = \alpha_1 \lambda_1, \tag{3.3}$$

where

$$\lambda_1 = \|n-r\ n-r+2 \cdots n-1\ n-r+1\ n-r+3 \cdots n-2\ n\|.$$

If r is even, and is of the form $4q+2 (q \geq 0)$, then

$$\delta = \alpha_1 \eta_1 \eta_2, \tag{3.4}$$

where

$$\eta_1 = \|n-r+1\ n-r+5 \dots n-1\ n-r+3\ n-r+7 \dots n-3\ n-r\ n-r+4 \dots n-2\ 1\| \\ \bigcup \|n-r+2\ n-r+6 \dots n\|$$

and

$$\eta_2 = \|n-r\ n-1\ n-3 \dots n-r+3\ n-r+1\| \bigcup \|1\ n\ n-2\ n-4 \dots n-r+2\|.$$

If r is even and of the form $4q (q \geq 1)$ then

$$\delta = \alpha_1 \psi_1 \psi_2, \tag{3.5}$$

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where

$$\psi_1 = \|n-r+1\; n-r\; n-r+3\; n-r+2\; n-r+5\; n-r+4 \ldots n-1\; n-2\; 1\|$$

and

$$\psi_2 = \|n-r\; n-r+3\; n-r+2\; n-r+5 \ldots n-2\; n-r+1\| \cup \|1\; n\|.$$

Next, β may or may not be nilpotent. However, as $\beta \in [r, r]$, if β is non-nilpotent, then by Lemma 2.1 it is expressible as a product of two nilpotents in $[r, r]$, say

$$\beta = \zeta_1 \zeta_2. \tag{3.6}$$

It is clear that $\beta \mathscr{R} \zeta_1$ and $\beta \mathscr{L} \zeta_2$, that is $R_{\zeta_1} = A_m$ and $L_{\zeta_2} = A_1$.

We now define λ_1', η_1', η_2', ψ_1', ψ_2' and ζ_2' as follows:

$$\lambda_1' = \lambda_1 \cup (1, n),$$

$$\eta_1' = \eta_1 \cup (n, 1), \qquad \eta_2' = \eta_2 \cup (n-r+1, n),$$

$$\psi_1' = \psi_1 \cup (n, n-2), \qquad \psi_2' = \psi_2 \cup (n, n-r+1).$$

Before we define ζ_2', we note that from Theorem 2.8 in [**2**], ζ_2 can be expressed as a disjoint union of k primitive nilpotents, say

$$\zeta_2 = \mu_1 \cup \mu_2 \cup \ldots \cup \mu_k.$$

If $k \geq 2$, then assume

$$\mu_1 = \|x_1 \ldots x_s\| \quad \text{and} \quad \mu_2 = \|y_1 \ldots y_t\|$$

and define ζ_2' as

$$x\zeta_2' = x\zeta_2 \quad \text{if} \quad x \in \operatorname{dom} \zeta_2$$

and

$$x_s \zeta_2' = y_t.$$

On the other hand if $k = 1$ then $|\operatorname{dom} \zeta_2 \cup \operatorname{im} \zeta_2| = r+1$, and since $r \leq n-2$ we have $X_n \backslash (\operatorname{dom} \zeta_2 \cup \operatorname{im} \zeta_2)$ to be non-empty. Then define ζ_2' as

$$\zeta_2' = \zeta_2 \cup (x, n-r+1),$$

where $x \in X_n \backslash (\operatorname{dom} \zeta_2 \cup \operatorname{im} \zeta_2)$.

Note that λ_1', η_1', η_2', ψ_1', ψ_2' and ζ_2' are distinct, and belong to $[r+1, r]$. If we now replace λ_1, η_1, η_2, ψ_1 and ψ_2 by λ_1', η_1', η_2', ψ_1' and ψ_2' respectively in equations (3.3)–(3.5) then it is easy to see that the equations remain unaltered. Since β, ζ_1, ζ_2 are all one–one and of the same height, we must have

$$\operatorname{dom} \beta = \operatorname{dom} \zeta_1, \qquad \operatorname{im} \zeta_1 = \operatorname{dom} \zeta_2,$$

and since $x_s, x \notin \operatorname{dom} \zeta_2 = \operatorname{im} \zeta_1$ we conclude that

$$\zeta_1 \zeta_2 = \zeta_1 \zeta_2'.$$

Now, if β is nilpotent then $V(n, r)$ is generated by

$$\{\beta, \lambda_1', \alpha_1, \ldots, \alpha_{p-2}\}, \qquad \{\beta, \eta_1', \eta_2', \alpha_1, \ldots, \alpha_{p-3}\}$$

or

$$\{\beta, \psi_1', \psi_2', \alpha_1, \ldots, \alpha_{p-3}\}$$

according to whether r is odd, even and of the form $4q+2$ $(q \geq 0)$ or even and of the form $4q$ $(q \geq 1)$, and $\alpha_m, \ldots, \alpha_{p-k}$ $(k=2, 3)$ are chosen arbitrarily to cover all the $\mathcal{R}$-classes in J_r.

If β is non-nilpotent, then $V(n,r)$ is generated by

$$\{\zeta_1, \zeta_2', \lambda_1', \alpha_1, \ldots, \alpha_{p-3}\}, \qquad \{\zeta_1, \zeta_2', \eta_1', \eta_2', \alpha_1, \ldots, \alpha_{p-4}\}$$

or

$$\{\zeta_1, \zeta_2', \psi_1', \psi_2', \alpha_1, \ldots, \alpha_{p-4}\}$$

according to whether r is odd, even and of the form $4q+2$ $(q \geq 0)$ or even and of the form $4q$ $(q \geq 1)$, and $\alpha_m, \ldots, \alpha_{p-k}$ $(k=3, 4)$ are chosen arbitrarily to cover all the $\mathcal{R}$-classes in J_r.

To conclude the proof of Theorem 3.1, it remains to prove that the listing of the subsets of X_n of cardinality r as postulated in the statement of Lemma 3.2 can actually be carried out. Let $n \geq 4$ and $2 \leq r \leq n-2$, and consider the following proposition.

($\mathbf{P}(n,r)$): *there is a way of listing the subsets of* X_n *of cardinality* r *as* $A_1, A_2, \ldots, A_m$ (*with* $m = \binom{n}{r}$), $A_1 = \{n-r+1, \ldots, n\}$, $A_2 = \{n-r, \ldots, n-1\}$, $A_m = \{1, n-r+2, \ldots, n\}$) *such that* $|A_i \cap A_{i+1}| = r-1$ *for* $i = 1, \ldots, m-1$ *and* $|A_m \cap A_1| = r-1$.

We shall prove this by a double induction on n and r, the key step being a kind of Pascal's Triangle implication.

$$\mathbf{P}(n-1, r-1) \quad \text{and} \quad \mathbf{P}(n-1, r) \Rightarrow \mathbf{P}(n, r).$$

First, however, we anchor the induction with two lemmas.

LEMMA 3.7. $\mathbf{P}(n,2)$ *holds for every* $n \geq 4$.

Proof. Consider the following arrangement of the subsets of X_n of cardinality 2.

$$\begin{array}{llllll}
\{1,2\}, & \{1,3\}, & \ldots, & & \{1,n-1\}, & \{1,n\}, \\
 & \{2,3\}, & \ldots, & & \{2,n-1\}, & \{2,n\}, \\
 & & \ddots & & \vdots & \vdots \\
 & & & & \{n-2,n-1\}, & \{n-2,n\}, \\
 & & & & & \{n-1,n\}.
\end{array}$$

If we denote the first row by R_1, second row by R_2, etc., then we note that the first entry in R_i is $\{i, i+1\}$ and the last entry is $\{i, n\}$. Thus the number of elements in R_i is $n-i$, and the total number of subsets in all the rows is

$$\sum_{i=1}^{n-1} (n-i) = \frac{n}{2}(n-1) = \binom{n}{2}.$$

Hence above is a complete list of the subsets of X_n of cardinality 2.

Note that for any two subsets A_s, A_r in R_i, $A_s \cup A_r = \{i\}$, and the intersection of the last entry in R_{i+1} with the first entry in R_i is $\{i+1\}$. Hence the following arrangement satisfies $\mathbf{P}(n,2)$:

$$R_{n-1}, R_{n-2}, \ldots, R_{i+1}, R_i, \ldots, R_2, R_1.$$

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That is, the list begins with all the subsets in R_{n-1}, followed by the subsets in R_{n-2}, followed by the subsets in R_{n-3}, and so on, until R_1 is reached.

LEMMA 3.8. *$\mathbf{P}(n, n-2)$ holds for every $n \geq 4$.*

Proof. Note that $\mathbf{P}(4, 2)$ follows from Lemma 3.7. So we will assume that $n \geq 5$. Let R_i' be the list of the complements of the subsets in R_i (defined in the proof of Lemma 3.7) arranged in the same order as in R_i. Let $(R_i')^{-1}$ be R_i' arranged in the reverse order. For example

$$R_{n-2} = \{n-2, n-1\}, \{n-2, n\},$$
$$R'_{n-2} = \{1, \ldots, n-3, n\}, \{1, \ldots, n-3, n-1\},$$
$$(R'_{n-2})^{-1} = \{1, \ldots, n-3, n-1\}, \{1, \ldots, n-3, n\}.$$

Let $T = \{1, 3\}, \{1, 4\}, \ldots, \{1, n-1\}$ and $T' = R_1' \backslash (\{1, 2\}', \{1, n\}')$.

It is clear that, for any two subsets A_s', A_r' in R_i', we have $|A_s' \cap A_r'| = n-3$, and the intersection of the last subset in R'_{i+1} and the first subset in R_i' also contains $n-3$ elements. We also have $n-3$ elements in the intersection of the last subset in R_3' with the first subset in $(R_2')^{-1}$, and the same number of elements in the intersection of the last subset in T' with the subset in R'_{n-1}. We now have the following arrangement satisfying $\mathbf{P}(n, n-2)$:

$$A_1', A_2', T', R'_{n-1}, R'_{n-2}, \ldots, R_3', (R_2')^{-1},$$

where $A_1' = \{1, 2\}'$ and $A_2' = \{1, n\}'$.

LEMMA 3.9. *Let $n \geq 6$ and $3 \leq r \leq n-3$. Then $\mathbf{P}(n-1, r-1)$ and $\mathbf{P}(n-1, r)$ together imply $\mathbf{P}(n, r)$.*

Proof. From the assumption $\mathbf{P}(n-1, r)$ we have a list $A_1, \ldots, A_m$ (where $m = \binom{n-1}{r}$) of the subsets of X_{n-1} with cardinality r such that $|A_1 \cap A_{i+1}| = r-1$ for $i = 1, \ldots, m-1$, and

$$A_1 = \{n-r, \ldots, n-1\}, A_2 = \{n-r-1, \ldots, n-2\}, A_m = \{1, n-r+1, \ldots, n-1\}.$$

From the assumption $\mathbf{P}(n-1, r-1)$, we have a list $B_1, \ldots, B_t$ (where $t = \binom{n-1}{r-1}$) of subsets of X_{n-1} of cardinality $r-1$ such that $|B_i \cap B_{i+1}| = r-2$ for $i = 1, \ldots, r-1$, and

$$B_1 = \{n-r+1, \ldots, n-1\}, B_2 = \{n-r, \ldots, n-2\}, B_t = \{1, n-r+2, \ldots, n-1\}.$$

Let $B_i' = B_i \cup \{n\}$. Then

$$A_1, \ldots, A_m, B_1', \ldots, B_t'$$

is a complete list of the subsets of X_n of cardinality r. (Notice that $t + m = \binom{n}{r}$.) Now, arrange the above subsets as follows:

$$B_1', A_1, A_m, \ldots, A_2, B_2', \ldots, B_t'.$$

Then it is easy to verify that this arrangement satisfies $\mathbf{P}(n, r)$. Hence the induction is complete and we may deduce that $\mathbf{P}(n, r)$ is true for all $n \geq 4$ and all r such that $2 \leq r \leq n-2$.

ACKNOWLEDGEMENT. My sincere thanks are due to my research supervisor, Professor J. M. Howie, for his helpful suggestions and encouragement.

REFERENCES

1. G. U. Garba, Nilpotents in semigroups of partial one-one order-preserving mappings, *Semigroup Forum*, to appear.

2. G. M. S. Gomes and J. M. Howie, Nilpotents in finite symmetric inverse semigroups, *Proc. Edinburgh Math. Soc.* (2) **30** (1987), 383–395.

3. G. M. S. Gomes and J. M. Howie, On the ranks of certain finite semigroups of transformations, *Math. Proc. Cambridge Philos. Soc.* **101** (1987), 395–403.

4. G. M. S. Gomes and J. M. Howie, On the ranks of certain semigroups of order-preserving transformations, *Semigroup Forum* **45** (1992), 272–282.

5. J. M. Howie, *An introduction to semigroup theory* (Academic Press, 1976).

6. J. M. Howie and R. B. McFadden, Idempotent rank in finite full transformation semigroups, *Proc. Roy. Soc. Edinburgh Sect. A* **114** (1990), 161–167.

7. R. P. Sullivan, Semigroups generated by nilpotent transformations, *J. Algebra* **110** (1987), 324–345.

DEPARTMENT OF MATHEMATICAL AND COMPUTATIONAL SCIENCES
UNIVERSITY OF ST ANDREWS
SCOTLAND

Present address:

DEPARTMENT OF MATHEMATICS
AHMADU BELLO UNIVERSITY ZARIA
NIGERIA