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# ON THE CONVOLUTION OF A BOX SPLINE WITH A COMPACTLY SUPPORTED DISTRIBUTION: LINEAR INDEPENDENCE FOR THE INTEGER TRANSLATES

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ABSTRACT. The problem of linear independence of the integer translates of $\mu * B$, where $\mu$ is a compactly supported distribution and $B$ is an exponential box spline, is considered in this paper. The main result relates the linear independence issue with the distribution of the zeros of the Fourier-Laplace transform, $\hat{\mu}$ of $\mu$ on certain linear manifolds associated with $B$. The proof of our result makes an essential use of the necessary and sufficient condition derived in [12]. Several applications to specific situations are discussed. Particularly, it is shown that if the support of $\mu$ is small enough then linear independence is guaranteed provided that $\hat{\mu}$ does not vanish at a certain finite set of critical points associated with $B$. Also, the results here provide a new proof of the linear independence condition for the translates of $B$ itself.

1. **Introduction.** A very simple model (and surprisingly a very rich one) in multivariate approximate theory is given in terms of a compactly supported function $\phi : \mathbb{R}^s \longrightarrow \mathbb{C}$ and the space $S(\phi)$ spanned by its integer translates. Closely related to such a model is the semi-discrete convolution operator $\phi *$ defined by

$$\phi *: \mathbf{c} \mapsto \phi * \mathbf{c} := \sum_{\alpha \in \mathbb{Z}^s} c_\alpha \phi(\cdot - \alpha), \tag{1.1}$$

where $\mathbf{c}$: $\mathbb{Z}^s$ is a complex-valued sequence. The injectivity of the operator $\phi *$, which is usually referred to as "the (global) linear independence of the integer translates of $\phi$", is certainly one of the most important properties related to $\phi$ and $S(\phi)$, and is intimately connected with the stability of the approximation process by elements from scaled versions of $S(\phi)$.

Exponential box (*EB*) splines, introduced in [10], generalize the well-known polynomial box splines ([2], [3]) and provide a wide selection of choices of the function $\phi$. An essential feature of an EB-spline, which is a piecewise exponential polynomial function, is that it is generated by convolving lower order ones. To introduce a typical EB-spline, let $\Gamma$ be a finite multiset (to be referred later as a *defining set* with cardinality $|\Gamma|$ consisting of the elements of the form

$$\gamma = (\mathbf{x}_\gamma, \lambda_\gamma), \tag{1.2}$$

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where $\mathbf{x}_\gamma \in \mathbb{Z}^s \backslash \{0\}$ and $\lambda_\gamma \in \mathbb{C}$. The EB-spline $B(\Gamma)$, based on $\Gamma$, can be defined via its Fourier transform by

$$(1.3)\qquad \hat{B}(\Gamma|\mathbf{x}) := \prod_{\gamma\in\Gamma} \hat{B}(\{\gamma\}|\mathbf{x}) := \prod_{\gamma\in\Gamma}\left(\frac{e^{\lambda_\gamma - i\mathbf{x}_\gamma\cdot\mathbf{x}} - 1}{\lambda_\gamma - i\mathbf{x}_\gamma\cdot\mathbf{x}}\right).$$

It should be noted that if

$$(1.4)\qquad \langle\Gamma\rangle := \operatorname{span}\{\mathbf{x}_\gamma\}_{\gamma\in\Gamma} = \mathbb{R}^s,$$

then $B(\Gamma)$ gives rise to a compactly supported function $B(\Gamma|\cdot)$; otherwise the EB-spline is merely a distribution (actually a measure) supported in $\langle\Gamma\rangle$. For more information about EB-splines we refer the reader to [10], [11], [7], [5] and [8]. Specifically, the question of linear independence of the integer translates of an EB-spline $B(\Gamma)$ was settled in the (stronger) local sense in [11] and [7].

Given an EB-spline $B(\Gamma)$, we examine in this paper the convolution $\mu * B(\Gamma)$ of the box with an arbitrary compactly supported distribution $\mu$ and the question of the global linear independence of the integer translates of $\mu * B(\Gamma)$. Our motivation to study this problem was based on the observation that bivariate splines of minimal support and quasi-minimal support are obtained in such a way ([4], [6]). In particular we conjectured that convolution with $\mu$ preserves the injectivity of $B(\Gamma)*$ whenever $\mu$ is the support function of a "small enough" domain. A precise statement of this kind is indeed proved in Section 3. Surprisingly, we found that the results here are also applicable to the analysis of the translates of $B(\Gamma)$: in addition to providing a new treatment of the case of integer set of directions, we discuss a bivariate example where the restriction $\{\mathbf{x}_\gamma\}_{\gamma\in\Gamma} \subset \mathbb{Z}^s$ is removed.

The approach we choose makes an essential use of the following necessary and sufficient condition for the linear independence of integer translates of a compactly supported distribution.

RESULT 1.1 ([12]). *Let $\psi$ be a compactly supported distribution and $\hat{\psi}$ its Fourier-Laplace transform. Then the integer translates of $\psi$ are globally linearly dependent if and only if for some $\theta \in \mathbb{C}^s$*

$$(1.5)\qquad \hat{\psi}(\theta + 2\pi\alpha) = 0, \quad \forall\alpha \in \mathbb{Z}^s.$$

The result we obtain in this paper is characterized in tems of the correspondence between the defining set $\Gamma$ and the distribution of the zeros of the Fourier transform $\hat{\mu}$ of $\mu$, and is proved to be applicable to situations where $\mu$ is defined by using geometrical means. The following examples serve as typical illustrations.

EXAMPLE 1.1. Let $B(\Gamma)$ be a three-directional polynomial box spline; that is, $s = 2$, $\lambda_\gamma = 0$ and $\mathbf{x}_\gamma \in \{(1,0),(0,1),(1,1)\}$ for all $\gamma \in \Gamma$. As mentioned above, $\mu$ is

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assumed to be a bivariate compactly supported distribution. Note that in this case the defining set $\Gamma$ consists of three distinct elements $\gamma_1, \gamma_2, \gamma_3$ with (possible) multiplicities. Here, a straighforward application of the results in [12] shows that the integer translates of $\mu * B(\Gamma)$ are globally linearly independent if and only if the same holds for $\mu * B(\{\gamma_1, \gamma_2, \gamma_3\})$. In fact a stronger solution for this problem is valid as follows:

*For a three-directional polynomial box spline $B(\Gamma)$, the integer translates of $\mu * B(\Gamma)$ are globally linearly independent if and only if $\hat{\mu}(0) \neq 0$ and the integer translates of each $\mu * B(\{\gamma_i\})$ are globally linearly independent, for $j = 1, 2, 3$.*

We remark that similar results (with a suitable modification of $\hat{\mu}(0) \neq 0$) hold for the (more general) three-directional *exponential* box splines (i.e., when the restriction $\lambda_\gamma = 0, \forall \gamma$, is removed) although in this case no direct application of the results of [12] seems to be available.

The second example below shows that the analysis here may sometimes lead to an explicit geometric characterization:

EXAMPLE 1.2. As in the above example, let $B(\Gamma)$ be a three-directional polynomial box spline. In addition, let $\mu$ be a measure whose support is contained in the unit square, whose total mass is one and is equally distributed on its support, and whose integer translates are linearly independent.† What shape does $\operatorname{supp} \mu$ admit to ensure the linear independence of the integer translates of $\phi := \mu * B(\Gamma)$? One special case is actually well-known: if $\operatorname{supp} \mu$ consists of the north-west south-east diagonal of the unit square, then $\phi$ is a so-called four-directional polynomial box spline, whose integer translates are globally linearly dependent.

*The integer translates of $\phi$ are globally linearly dependent if and only if there exists an $\varepsilon > 0$ such that for every $\mathbf{x}$ with $0 \leq x_1 - x_2 < 1$, the ratio between the one dimensional Lebegue measures‡ of*

$$\{\mathbf{x} + (t, t)\}_{t \in \mathbf{R}} \cap \operatorname{supp} \mu$$

and

$$\{\mathbf{x} + (t, t+1)\}_{t \in \mathbf{R}} \cap \operatorname{supp} \mu$$

*equals $\varepsilon$ (independent of $\mathbf{x}$). Moreover in this case the kernel of the operator $\phi *$ contains the exponential*

$$c_\alpha = (-\varepsilon)^{\alpha_1 - \alpha_2}.$$

The main results of this paper are presented and proved in Section 2. Section 3 contains applications of the main results to the case where $\operatorname{supp} \mu$ is "small" in a suitable sense, and in Section 4 we discuss Example 1.2, bivariate box splines corresponding to a non-integer direction set and another bivariate example.

† An additional mild restriction is needed here. For details see Section 4.

‡ In case $\operatorname{supp} \mu$ has a zero two-dimensional volume one measures the width of these sets by the counting measure.

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We conclude the introduction with some notations and terminology which will be used in the sequel. Given $K \subset \Gamma$, its cardinality is denoted by $|K|$ while $\langle K \rangle$ stands for the real linear span of $\{\mathbf{x}_\gamma\}_{\gamma \in K}$. The set $K$ is referred to as "linearly independent" whenever $\{\mathbf{x}_\gamma\}_{\gamma \in K}$ form a linearly independent set of vectors. Orthogonal relations are always considered here in the complex situation; thus the notation $K^\perp$ stands for the set of all vectors in $\mathbb{C}^s$ which are orthogonal to all of $\{\mathbf{x}_\gamma\}_{\gamma \in K}$. Given a compactly supported distribution $\mu$, we let $\hat{\mu}$ denote its Fourier-Laplace transform; i.e., $\hat{\mu}$ is the entire function obtained as the analytic continuation of the Fourier transform of $\mu$. The exponential function $e^{i\theta \cdot \mathbf{x}}$ is abbreviated as $e_\theta$. Finally, following de Boor [1], we set $\phi *' f$ for the semi-discrete convolution $\phi * (f|_{\mathbb{Z}^s})$. The notation "$*$" is used for either the usual convolution (of functions or distributions) or the semi-discrete one. The appropriate meaning can be easily verified from the context.

2. **Main results.** In this section, we first introduce the notions of a "basis" and a "node", and then prove the main theorem and discuss some of its immediate corollaries.

Several sets and families of sets are associated with the defining set $\Gamma$ and its corresponding box spline $B(\Gamma)$. One of these is the collection of all "bases" $\mathbf{J}(\Gamma)$, defined as

$$\mathbf{J}(\Gamma) = \{J \subset \Gamma \mid |J| = s, \langle J \rangle = \mathbb{R}^s\}. \tag{2.1}$$

Each "basis" $J \in \mathbf{J}(\Gamma)$ induces a set of $s$ linearly independent linear equations in $s$ variables

$$i\mathbf{x}_\gamma \cdot \theta = \lambda_\gamma, \qquad \forall \gamma \in J. \tag{2.2}$$

the unique solution (in $\mathbb{C}^s$) of this system is denoted by $\theta_J$ and will be referred to as a *node* later. We set

$$\Theta(\Gamma) = \{\theta_J \mid J \in \mathbf{J}(\Gamma)\}. \tag{2.3}$$

(This definition slightly differs from the original one in [10], but seems to be somewhat more convenient in the context of the Fourier analysis methods employed in the sequel).

Given a linearly independent set $K \subset \Gamma$ we may associate $K$ with its *node* $\theta_K$ which is defined similarly by

$$i\mathbf{x}_\gamma = \lambda_\gamma, \qquad \forall \gamma \in K, \tag{2.4}$$

$$\theta_K \in \operatorname{span}\{i\mathbf{x}_\gamma\}_{\gamma \in K}, \tag{2.5}$$

where the span in (2.5) is regarded over $\mathbb{C}$ (not as in the definition of $\langle K \rangle$ when the span is regarded to be taken over $\mathbb{R}$).

Basically most of the analysis concerning exponential box splines (and polynomial box splines in particular) is either based on various recurrence relations or makes use of the simple form of the Fourier transform of $B(\Gamma)$. For questions exclusively concerned with box splines, the first approach is usually more effective and more efficient. Yet, in the

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present situation, where arbitrary distributions are involved as well, the Fourier analysis method seems to be the one that leads to more comprehensive results and therefore is the one chosen here.

In the sequel we make a frequent use of the trivial fact that for arbitrary compactly supported distributions $\mu_1$ and $\mu_2$, the global linear independence for the integer translates of $\mu_1 * \mu_2$ always implies linear independence for the integer translates of each of $\mu_1, \mu_2$. Thus, seeking for conditions to guarantee the linear independence of the integer translates of $\mu * B(\Gamma)$, where $B(\Gamma)$ is an exponential box spline and $\mu$ is a compactly supported distribution, it is necessary to assume the linear independence of the integer translates of $B(\Gamma)$.

THEOREM 2.1. *Let $\mu$ be a compactly supported distribution and $\Gamma$ a defining set. Assume that both the integer translates of $B(\Gamma)$ and of $\mu$ are globally linearly independent. Then the following conditions are equivalent:*

*(a) The integer translates of $\mu * B(\Gamma)$ are globally linearly dependent.*

*(b) There exists a linearly independent set $K \subset \Gamma$ such that the integer translates of $\mu * B(K)$ are globally linearly dependent.*

*(c) There exist a linearly independent set $K \subset \Gamma$ and a $\mathbf{z} \in K^{\perp}$ such that*

$$\hat{\mu}(\theta_K + \mathbf{z} + 2\pi\alpha) = 0, \qquad \forall \alpha \in \mathbb{Z}^s \cap K^{\perp}. \tag{2.6}$$

(d) *Either the integer translates of $\mu * B(K)$ are globally linearly dependent, for some linearly independent set $K \subset \Gamma$ of cardinality $< s$*, or

$$\hat{\mu}(\theta) = 0 \quad \text{for some } \theta \in \Theta(\Gamma).$$

PROOF. We first note that, by (1.3), for any $\gamma \in \Gamma$,

$$\hat{B}(\{\gamma\} \mid \mathbf{x}) = 0 \iff \lambda_\gamma - i\mathbf{x}_\gamma \cdot \mathbf{x} \in 2\pi i \mathbb{Z} \setminus \{0\}. \tag{2.7}$$

We start the proof with the following lemma, which establishes the implication $(c) \Longrightarrow (b)$:

LEMMA 2.1. *Let $K$ be a linearly independent subset of $\Gamma$ that satisfies (2.6). Then*

$$\hat{B}(K \mid \theta_K + \mathbf{z} + 2\pi\alpha) = 0, \quad \forall \alpha \in \mathbb{Z}^s \setminus K^{\perp}, \tag{2.8}$$

*and the integer translates of $\mu * B(K)$ are linearly dependent.*

PROOF. Let $\alpha \in \mathbb{Z}^s \setminus K^{\perp}$. Then, there exists $\gamma \in K$ such that $\mathbf{x}_\gamma \cdot \alpha \neq 0$. Therefore, by the definition of $\theta_K$ (and since $\mathbf{z} \in K^{\perp}$),

$$\lambda_\gamma - i\mathbf{x}_\gamma \cdot (\theta_K + \mathbf{z} + 2\pi\alpha) = -2\pi i \mathbf{x}_\gamma \cdot \alpha \in 2\pi i \mathbb{Z} \setminus \{0\},$$

and (2.8) follows from (2.7). Combining (2.8) with (2.6) yields

$$(\mu * B(K))^{\wedge}(\theta_K + \mathbf{z} + 2\pi\alpha) = 0, \quad \forall \alpha \in \mathbb{Z}^s, \tag{2.9}$$

which, in view of Result 1.1, yields the desired result. ∎

The next lemma is the crux in the proof of the theorem, and may be found to be of independent interest:

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LEMMA 2.2. *Assume that for some $K \subset \Gamma$ the integer translates of $\mu * (B(K)$ are linearly dependent and $K$ is minimal with respect to this property. Then*

*(a) $K$ is linearly independent,*

*(b) there exists a $\mathbf{z} \in K^{\perp}$ such that (2.6) is satisfied, and*

*(c) $\hat{B}(K|\theta_K + \mathbf{z} + 2\pi\alpha) = 0, \quad \forall\alpha \in \mathbb{Z}^s \backslash K^{\perp}$.*

PROOF. By Result 1.1, there exists $\theta^1 \in \mathbb{C}^s$ such that

$$(\mu * B(K))^{\wedge}(\theta^1 + 2\pi\alpha) = 0, \quad \forall\alpha \in \mathbb{Z}^s. \tag{2.10}$$

Furthermore, since $K$ is minimal, it follows that for each $\gamma \in K$ there is an $\alpha_\gamma \in \mathbb{Z}^s$ satisfying $\hat{B}(\{\gamma\}|\theta^1 + 2\pi\alpha_\gamma) = 0$, which implies by (2.7) that

$$\lambda_\gamma - i\mathbf{x}_\gamma \cdot (\theta^1 + 2\pi\alpha_\gamma) \in 2\pi i\mathbb{Z}\backslash\{0\},$$

and in particular, since $\mathbf{x}_\gamma$ and $\alpha_\gamma$ are integers,

$$\lambda_\gamma - i\mathbf{x}_\gamma \cdot (\theta^1 + 2\pi\alpha) \in 2\pi i\mathbb{Z}, \quad \forall\gamma \in K,\ \alpha \in \mathbb{Z}^s. \tag{2.11}$$

On the other hand, we assume that the integer translates of $B(\Gamma)$, hence of $B(K)$, are linearly independent, which means in view of Result 1.1 that for some $\beta \in \mathbb{Z}^s$

$$\hat{B}(\{\gamma\}|\theta^1 + 2\pi\beta) \neq 0, \quad \forall\gamma \in K. \tag{2.12}$$

Then setting $\theta = \theta^1 + 2\pi\beta$, we may combine (2.11), (2.12), and (2.7) to deduce that

$$\lambda_\gamma - i\mathbf{x}_\gamma \cdot \theta = 0, \quad \forall\gamma \in K, \tag{2.13}$$

and therefore for every $\alpha \in \mathbb{Z}^s \cap K^{\perp}$, we have

$$\lambda_\gamma - i\mathbf{x}_\gamma \cdot (\theta + 2\pi\alpha) = 0, \quad \forall\gamma \in K.$$

Using (2.7), we can now conclude that for such $\alpha, \hat{B}(K|\theta + 2\pi\alpha) \neq 0$ and (2.10) thus yields

$$\hat{\mu}(\theta + 2\pi\alpha) = 0, \quad \forall\alpha \in \mathbb{Z}^s \cap K^{\perp}. \tag{2.14}$$

Now let $K_1 \subset K$ be a "basis" for $\langle K \rangle$, (which means, as in (2.1), that the vectors $\{\mathbf{x}_\gamma\}_{\gamma \in K_1}$ form a basis of $\langle K \rangle$). By the definition of $\theta_{K_1}$, (2.13) is satisfied also with $\theta$ replaced by $\theta_{K_1}$ at least for every $\gamma \in K_1$, so that

$$\mathbf{z} := \theta - \theta_{K_1} \in K_1^{\perp} = K^{\perp},$$

and hence (2.14) implies (b). With (b) in hand, we appeal to Lemma 2.1, to obtain that the integer translates of $\mu * B(K_1)$ are linearly dependent. The minimality of $K$ thus shows $K_1 = K$ and (a) follows, while (c) becomes a restatement of (2.8). ∎

Now, whenever the integer translates of $\mu * B(\Gamma)$ are linearly dependent, we may choose $K \subset \Gamma$ to be a minimal set for which the linear dependence of the translates of

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$\mu * B(K)$ still holds and apply the above lemma to conclude that $K$ is linearly independent. Hence (a)$\Rightarrow$(b), and since the converse implication is trivial, these two conditions are equivalent. The implication (b)$\Rightarrow$(c) is a direct consequence of Lemma 2.2(b), since we may assume without loss that the set $K$ in (b) is minimal. Combining the above with Lemma 2.1, we conclude that (a)$\Leftrightarrow$(b)$\Leftrightarrow$(c).

To complete the proof of the theorem we show now that (b) is equivalent to (d). Assuming (b) there is nothing to prove in (d) in case the minimal $K$ that satisfies (b) is of cardinality $< s$. Otherwise, the $K$ in (b) has exactly $s$ elements (i.e., it is a basis from $\mathbf{J}(\Gamma)$ and thus $\theta_K \in \Theta(\Gamma)$) and is minimal and hence Lemma 2.2(b) provides $\mathbf{z} \in K^{\perp}$ such that $\hat{\mu}(\theta_K + \mathbf{z}) = 0$. Since $\#K = s$, we must have $\langle K \rangle = \mathbb{R}^s$, which forces $\mathbf{z} = 0$ and (d) is obtained.

For the converse, it is sufficient to verify (b) only for the case when

$$\mu(\theta) = 0, \qquad \text{for some } \theta \in \Theta(\Gamma) \tag{2.15}$$

(For the other case, (b) is trivially satisfied.) To do so, choose $K \in \mathbf{J}(\Gamma)$ such that $\theta = \theta_K$, and let $z = 0$. Then since $K^{\perp} = 0$, (2.15) implies (2.6) and (c), and hence (b) follows. ∎

As a first illustration, we show now how the known necessary and sufficient condition for the linear independence of the integer translates of an exponential box spline can be derived from Theorem 2.1.

RESULT 2.1 ([11],[7]). *The integer translates of an exponential box spline $B(\Gamma)$ are linearly independent if and only if the following two conditions are satisfied*

$$\hat{B}(\Gamma|\theta) \neq 0, \qquad \forall \theta \in \Theta(\Gamma); \tag{2.16}$$

$$|\det X_J| = 1, \qquad \forall J \in \mathbf{J}(\Gamma). \tag{2.17}$$

*Here, $X_J$ is the matrix whose columns are $\{\mathbf{x}_\gamma\}_{\gamma \in J}$.*

PROOF. The harder implication in the result is to prove that the two conditions (2.16) and (2.17) imply the linear independence of the translates. The proof of the converse is straighforward and will not be given here (cf. [10;§5]).

Let us assume that (2.16) and (2.17) are satisfied, but on the contrary, that the integer translates of $B(\Gamma)$ are linearly dependent. First, we consider the case when $\Gamma$ consists of only $s$ elements. In this case it easily follows from (2.17) that the supports of the integer translates of $B(\Gamma)$ are pairwise disjoint (up to a set of measure zero) and therefore the linear dependence cannot hold.

Knowing therefore that $\Gamma$ contains at least $s + 1$ elements, we pick $\gamma \in \Gamma$ such that $\langle \Gamma \backslash \gamma \rangle = \mathbb{R}^s$, and consider the following two possibilities:

(a) The integer translates of $B(\Gamma\backslash\gamma)$ are still linearly dependent.

(b) The integer translates of $B(\Gamma\backslash\gamma)$ are already linearly independent.

In case (a) there are at least $s+1$ elements in the remaining set, so that we may proceed to delete another element and hence this case is reduced to (b).

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Note that in any case the integer translates of $B(\{\gamma\})$ are linearly independent (otherwise we can extend $\gamma$ to a basis $J \in \mathbf{J}(\Gamma)$, conclude that the translates of $B(J)$ are linearly dependent and arrive as in the preceding paragraph at a contradiciton to (2.17)). Now we apply the equivalence of (a) and (b) in Theorem 2.1 (with $\Gamma$ replaced by $\Gamma\backslash\{\gamma\}$ and $\mu = B(\{\gamma\})$) to conclude that the integer translates of $B(K) * B(\{\gamma\}) = B(K \cup \{\gamma\})$ are linearly dependent for some linearly independent set $K \subset \Gamma\backslash\{\gamma\}$.

If $K \cup \{\gamma\}$ is still a linearly independent set, we can extend it to a basis $J \in \mathbf{J}(\Gamma)$, conclude that the integer translates of $B(J)$ are linearly dependent, and obtain again the same contradiction to (2.17) as before.

Otherwise, $\mathbf{x}_\gamma \in \langle K \rangle$. Here we appeal to Theorem 2.1(c) to conclude that

$$\hat{B}(\{\gamma\} \,|\, \theta_K + \mathbf{z}) = 0, \tag{2.18}$$

for *some* $\mathbf{z} \in K^\perp$. The assumption on $\mathbf{x}_\gamma$ guarantees that the Fourier transform of $B(\{\gamma\})$ is constant along lines orthogonal to $K$, and hence (2.18) is actually valid with respect to *all* $\mathbf{z} \in K^\perp$. Let $J$ be an extension of $K$ to a basis. Then, since $\theta_J$ as well as $\theta_K$ satisfies (2.13), it follows that $\mathbf{z} := \theta_J - \theta_K \in K^\perp$ and thus, by (2.18), $\hat{B}(\{\gamma\})$, and hence $\hat{B}(\Gamma)$, vanishes at $\theta_J$, a contradiction to (2.16). ∎

The following corollary, which is essentially known in the theory of exponential box splines, follows directly from the proof above:

COROLLARY 2.1. *Suppose that the integer translates of $B(\Gamma)$ are linearly dependent. Then there exists a subset $K \subset \Gamma$ of cardinality $s+1$ such that the linear dependence still holds with respect to $B(K)$.*

In the bivariate situation the most interesting case where the integer translates of a box spline are linearly independent is the three-directional mesh, i.e., when $\mathbf{x}_\gamma \in \{(1,0),(1,1),(0,1)\}$ for each $\gamma \in \Gamma$. For this specific situation we deduce from Theorem 2.1 the following

COROLLARY 2.2. *Let $B(\Gamma)$ be a three-directional exponential box spline and $\mu$ a compactly supported distribution. Then the following conditions are equivalent:*

*(a) The integer translates of $\mu * B(\Gamma)$ are globally linearly independent.*

*(b) Both of the following are satisfied:*

*(b1) $\hat{\mu}(\theta)\hat{B}(\Gamma|\theta) \neq 0$, $\forall \theta \in \Theta(\Gamma)$.*

*(b2) For every $\gamma \in \Gamma$ the integer translates of $\mu * B(\{\gamma\})$ are globally linearly independent.*

PROOF. In the case of a three-directional mesh, condition (2.17) is always satisfied and (b1) above implies (2.16), and hence the claim easily follows from the equivalence of (a) and (d) in Theorem 2.1. ∎

Note that the claim in Example 1.1 is also covered by the above theorem, since $\Theta(\Gamma) = \{0\}$ and $\hat{B}(\Gamma|0) \neq 0$ in the polynomial case.

3. **Applications to distributions with small support.** Here we discuss some of the possible applications of Theorem 2.1. The typical nature of these applications seems to be

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that for "small" enough $\operatorname{supp}\mu$, linear independence is more likely to occur. Throughout this section we will always assume that the integer translates of the box spline $B(\Gamma)$, as well as the integer translates of the distribution $\mu$ are linearly independent.

For later reference we record here the following proposition which is a straightforward extension of [10; Cor. 5.1].

PROPOSITION 3.1. *Let $K$ be a linearly independent subset of $\Gamma$. Then*

$$\sum_{\alpha\in\mathbb{Z}^s\cap\langle K\rangle} e_{\theta_K}(\alpha)B(K|\cdot-\alpha)=e_{\theta_K}\chi_{\langle}K\rangle,$$

*where $\chi_{\langle K\rangle}$ is the support function associated with the linear subspace $\langle K\rangle$.*

For the first application we need a certain restriction on $\operatorname{supp}\mu$ which we find convenient to formalize as follows:

DEFINITION 3.1. We say that $\operatorname{supp}\mu$ is $\Gamma$-small if for every $K\subset\Gamma$ with $\langle K\rangle\neq\mathbb{R}^s$ and every sequence c: $\mathbb{Z}^s\longrightarrow\mathbb{C}$, such that

$$\mu*B(K)*\mathbf{c}\equiv 0,$$

we have

$$\sum_{\alpha\in\mathbb{Z}^s\cap\langle K\rangle}c_\alpha[\mu*B(K)](\cdot-\alpha)\equiv 0.$$

The property of $\Gamma$-smallness is referred to the *support* of $\mu$ (rather than, say, to $\mu$ itself) since we seek for conditions where the $\Gamma$-smallness is guaranteed by the relations between $\operatorname{supp}\mu$ and $\Gamma$, regardless of the specific definition of $\mu$. We will elaborate on this point later on, after stating and proving the main result in this context.

THEOREM 3.1. *Let $B(\Gamma)$ and $\mu$ be as in Theorem 2.1. Assume also that the support of $\mu$ is $\Gamma$-small. Then the integer translates of $\mu*B(\Gamma)$ are globally linearly dependent if and only if*

(3.1) $$\hat{\mu}(\theta_J)=0\quad \textit{for some } J\in\mathbf{J}(\Gamma)$$

PROOF. The "if" implication follows directly from the implication (d)⇒(a) in Theorem 2.1. For the converse, we make use of the equivalence of (a) and (c) in Theorem 2.1. Denoting $\theta_K+\mathbf{z}$ in (2.6) by $\theta$, this equivalence relationship, together with the definition of $\theta_K$, ensures the existence of a linearly independent set $K$ that satisfies:

(3.2) $$\lambda_\gamma-i\mathbf{x}_\gamma\cdot\theta=0,\quad\forall\gamma\in K,$$

(3.3) $$\hat{\mu}(\theta+2\pi\alpha)=0,\quad\alpha\in\mathbb{Z}^s\cap K^\perp.$$

In case $\langle K\rangle=\mathbb{R}^s$, (3.1) becomes equivalent to (3.3) (with $J=K$ and $\theta_J=\theta$) and the desired claim is therefore evident. Otherwise, we appeal to Lemma 2.1 to deduce

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that $(\mu * B(K))^{\wedge}(\theta + 2\pi\alpha) = 0, \forall\alpha \in \mathbb{Z}^s$, which implies (cf., e.g., [12; Lem. 2.1]) that $\mu * B(K) *' e_\theta = 0$. Now the $\Gamma$-smallness assumption ensures that

$$\sum_{\alpha\in\mathbb{Z}^s\cap\langle K\rangle} e_\theta(\alpha)(\mu * B(K))(\cdot - \alpha) = 0. \tag{3.4}$$

In this last equation we can replace $\theta$ by any $\theta + \mathbf{y}$ where $\mathbf{y} \in K^\perp$. Extending $K$ to a basis $J \in \mathbf{J}(\Gamma)$, we know from its definition that $\theta_J$ satisfies (3.2) and hence $\mathbf{x}_\gamma \cdot (\theta_J - \theta) = 0$ for all $\gamma \in K$; i.e., $\theta_J - \theta \in K^\perp$. Thus replacing $\theta$ by $\theta_J$ in (3.4), we may appeal to Proposition 3.1 to obtain

$$\sum_{\alpha\in\mathbb{Z}^s\cap\langle K\rangle} e_{\theta_J}(\alpha)(\mu * B(K))(\cdot - \alpha) = \mu * (e_{\theta_J}\chi_{\langle K\rangle}),$$

where $\chi_{\langle K\rangle}$ is a measure supported on $\langle K\rangle$ with mass equally distributed on its support. Therefore we may conclude that

$$\mu * (e_{\theta_J}\chi_{\langle K\rangle})) = 0,$$

which clearly implies that

$$\hat{\mu}(\theta_J + \mathbf{y}) = 0, \qquad \forall y \in K^\perp,$$

and (3.1) follows. ∎

COROLLARY 3.1. *Let $B(\Gamma)$ and $\mu$ be as in Theorem 3.1. If $\Gamma$ is a real defining set (i.e., $\lambda_\gamma \in \mathbb{R}$ for all $\gamma$) and $\mu$ is a positive distribution. Then the integer translates of $\mu * B(\Gamma)$ are globally linearly independent.*

PROOF. Since $\Gamma$ is real, $\Theta(\Gamma) \subset i\mathbb{R}^s$. On the other hand $\hat{\mu}$, as the Fourier transform of a positive distribution vanishes nowhere on $i\mathbb{R}^s$. Hence the result follows by applying Theorem 3.1. ∎

Next, we aim at describing specific situations where the "$\Gamma$-smallness" of $\operatorname{supp}\mu$ is guaranteed. For this purpose we need the notion of a "$\Gamma$-cell":

DEFINITION 3.2. *A $\Gamma$-cell is a maximal (connected) region in $\mathbb{R}^s$ which is disjoint from $\alpha + \langle K\rangle$ for all $\alpha \in \mathbb{Z}^s$ and all $K \subset \Gamma$ with $\langle K\rangle \neq \mathbb{R}^s$.*

We remark that in the tensor product case, (when every $\mathbf{x}_\gamma$ is taken from the standard basis for $\mathbb{R}^s$), the only $\Gamma$-cell (up to an integer translate) is the open unit cube. In the case of a bivariate three-directional mesh, the two $\Gamma$-cells are the triangles obtained when dividing the unit square along its south-west north-east diagonal. Note also that the notion of a $\Gamma$-cell is independent of the choice of the $\lambda$'s.

COROLLARY 3.2. *Suppose that the support of the distribution $\mu$ is contained in the closure of a $\Gamma$-cell $A$ and also that for every $K \subset \Gamma$ with $\langle K\rangle \neq \mathbb{R}^s$, the support of $\mu$ intersects at most one of the manifolds $\{\alpha + \langle K\rangle\}_{\alpha\in\mathbb{Z}^s}$. Then the integer translates of $\mu * B(\Gamma)$ are globally linearly independent if and only if $\hat{\mu}(\theta) \neq 0$ for all $\theta \in \Theta(\Gamma)$.*

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*Moreover, if in addition $\mu$ is a positive distribution and $\Gamma$ is real, the linear independence is always valid.*

PROOF. In view of Theorem 3.1 and Corollary 3.1, it is sufficient to show that the support of $\mu$ is $\Gamma$-small. Yet, this is evident: the support of $\sum_{\alpha\in\langle K\rangle\cap\mathbb{Z}^s} c_\alpha[\mu * B(K)](\cdot-\alpha)$ (with $K\subset\Gamma$ and $\langle K\rangle\neq\mathbb{R}^s$) lies in the set $\cup_{\mathbf{x}\in\langle K\rangle}\{\mathbf{x}+\operatorname{supp}\mu\}$, while the conditions assumed in the theorem clearly imply that the support of any shift of $\mu * B(K)$ by $\alpha\in\mathbb{Z}^s\setminus\langle K\rangle$ does not intersect that set. ∎

REMARK. Some relaxations on the conditions assumed in Corllary 3.2 are available. For instance, one may assume that a certain translate of $\mu$, rather than $\mu$ itself, is supported in a $\Gamma$-cell, since the linear independence property is invariant under translations and convolution commutes with translation.

The condition described in Corollary 3.2 may appear to be unsatisfactory in certain cases of interest. E.g., when $\mu$ is the characteristic function of a $\Gamma$-cell, its support might intersect with two different hyperplanes of the form $\alpha+\langle K\rangle, K\subset\Gamma, \alpha\in\mathbb{Z}^s$. In the following we modify Corollary 3.2 to cover such cases as well.

COROLLARY 3.3. *Let $\mu$ be a compactly supported (Radon) measure. Assume that* $\operatorname{supp}\mu$ *is contained in the closure of a $\Gamma$-cell, and that for every $K\subset\Gamma$ with $\langle K\rangle\neq\mathbb{R}^s$, at most one of the manifolds $\{\alpha+\langle K\rangle\}_{\alpha\in\mathbb{Z}^s}$ intersects* $\operatorname{supp}\mu$ *in a set carrying a non-zero mass of $\mu$. Then the integer translates of $\mu * B(\Gamma)$ are globally linearly independent if and only if $\hat{\mu}(\theta)\neq 0, \forall\theta\in\Theta(\Gamma)$.*

The proof of Corollary 3.3 is straightforward. Indeed, whenever the support of $\mu$ intersects a set of the form $\alpha+\langle K\rangle$ at a set which carries zero mass of $\mu$ we may simply change the definition of $\mu$ to be 0 on that set. Since the total number of such intersections is finite these modifications do not alter $\mu$ as a distribution, and preserve the proof of Corollary 3.2.

COROLLARY 3.4. *Let $\mu$ be a compactly supported measurable function whose support is contained in the closure of a $\Gamma$-cell. Then the integer translates of $\mu * B(\Gamma)$ are globally linearly independent if and only if $\hat{\mu}(\theta)\neq 0, \forall\theta\in\Theta(\Gamma)$. In particular, if $\Gamma$ is real and $\mu$ is positive the linear independence of the translates of $\mu * B(\Gamma)$ is guaranteed.*†

Again the proof is evident: the argument in Corollary 3.3 ensures the $\Gamma$-smallness of $\operatorname{supp}\mu$ and therefore Theorem 3.1 and Corollary 3.1 yield the desired results.

4. **Examples.** We present here three bivariate examples. The first one is devoted to box splines associated with a real direction set and $\mu$ is chosen to be a factor of $B(\Gamma)$ (as in the proof of Result 2.1). In the other examples $B(\Gamma)$ is a three-directional polynomial

† Recall the assumption in the beginning of the section about the linear independence of the integer translates of $B(\Gamma)$ and of $\mu$.

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box spline and $\mu$ is a measure whose mass is equally distributed on its support. (e.g., if $\mathrm{vol}_{\mathbf{R}^2}(\operatorname{supp}\mu) > 0$, $\mu$ is the characteristic function of its support.)

EXAMPLE 4.1. Let $\Gamma$ be a bivariate defining set with $\lambda_\gamma = 0$ for all $\gamma$ and real direction set $\{\mathbf{x}_\gamma\}_{\gamma\in\Gamma}$. Define

$$\Gamma_I := \{\gamma \in \Gamma : \mathbf{x}_\gamma \in \mathbb{Z}^2\},$$

and

$$\Gamma_R := \Gamma \backslash \Gamma_I.$$

PROPOSITION 4.1. *The integer translates of $B(\Gamma)$ are linearly independent if and only if the integer translates of $B(K)$ are linearly independent for every $K \in \mathbf{J}(\Gamma) \cup \{\Gamma_R\}$.*

PROOF. The "only if" implication is trivial. For the converse, assume that the integer translates of $B(K)$ are linearly independent for every $K \in \mathbf{J}(\Gamma)\cup\{\Gamma_R\}$, while the integer translates of $B(\Gamma)$ are linearly dependent. We will show that this leads to a contradiction.

First, by Result 2.1, it follows that the integer translates of $B(\Gamma_I)$ are linearly independent, and hence we may apply Theorem 2.1 (with $\mu = B(\Gamma_R)$ and $\Gamma = \Gamma_I$) to conclude that the integer translates of $B(K) * B(\Gamma_R) = B(K \cup \Gamma_R)$ are linearly dependent for some linearly independent $K \subset \Gamma_I$. Furthermore, since we are in the polynomial case, $\Theta(\Gamma_I) = 0$ and $\hat{B}(\Gamma_R|0) \neq 0$, and hence Lemma 2.2 implies that $\#K = 1$; i.e., $K = \gamma_0$ for some $\gamma_0 \in \Gamma$. Observing here that $\theta_K = 0$, this lemma also provides a $\theta \in K^\perp \backslash \{0\}$ such that

$$\hat{B}(\Gamma_R|\theta + 2\pi\alpha) = 0, \forall\alpha \in \mathbb{Z}^2 \cap K^\perp. \tag{4.1}$$

Since the one-dimensional space $K^\perp$ is spanned by $\theta$, (4.1) implies that

$$\hat{B}(\Gamma_R|(1 + ja)\theta) = 0, \forall j \in \mathbb{Z} \tag{4.2}$$

for some non-zero $a$, and hence, by (2.7), for every $j \in \mathbb{Z}$ there exists $\gamma \in \Gamma_R$ such that

$$(1 + ja)\theta \cdot x_\gamma \in 2\pi\mathbb{Z}\backslash\{0\}. \tag{4.3}$$

The argument used in the proof of [9; Thm. 2.1] thus shows that there exists $\gamma_1 \in \Gamma_R$ that satisfies (4.3) for *every* $j \in \mathbb{Z}$, and hence for the same $\gamma_1$

$$\hat{B}(\{\gamma_1\}|\theta + 2\pi\alpha) = 0, \forall\alpha \in \mathbb{Z}^2 \cap K^\perp. \tag{4.4}$$

It follows that, with $J := \{\gamma_0, \gamma_1\}$, we have $\hat{B}(J|\theta + 2\pi\alpha) = 0, \forall\alpha \in \mathbb{Z}^2$, and consequently (by Result 1.1) the integer translates of $B(J)$ are linearly dependent. Furthermore, $\theta \cdot \mathbf{x}_{\gamma_1} \neq 0$, and since $\theta$ is perpendicular to $\mathbf{x}_{\gamma_0}$, we conclude that $\mathbf{x}_{\gamma_0}$ and $\mathbf{x}_{\gamma_1}$ are linearly independent and thus $J \in \mathbf{J}(\Gamma)$. This contradicts the assumption that for every $J \in \mathbf{J}(\Gamma)$ the integer translates of $B(J)$ are linearly independent, completing thereby the proof of the proposition. ∎

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From Proposition 4.1 it follows that whenever there are no more than two distinct directions in $\Gamma_R$, the independence of the translates of $B(J)$ for every $J \in \mathbf{J}(\Gamma)$ implies the independence of the translates of $B(\Gamma)$. We mention that [9; Ex. 3.2] shows that an analogous result is not longer valid in case $\Gamma_R$ contains at least four distinct vectors.

EXAMPLE 4.2. Let $B(\Gamma)$ be a three-directional box spline. Assume that $\mu$ is the support function of a subset of the closed triangle with vertices $(0,0)$, $(1,0)$, $(1,1)$. (As mentioned before, this triangle is the closure of a $\Gamma$-cell for the case of a three-directional mesh.) Here we discuss three different possibilities:

(a) The support of $\mu$ has a two-dimensional positive volume. In this case $\mu$ is a function and thus the linear independence for the integer translates of $\mu * B(\Gamma)$ is guaranteed by Corollary 3.4. If we choose $\mu$ to be the characteristic function of the above triangle, the functions $\mu * B(\Gamma)$ so obtained are certain functions of minimal supports (see [4],[6]). The linear independence for this special case was already proved in [12].

(b) The support of $\mu$ has a zero two-dimensional volume but has a positive one-dimensional volume. In this case the vertices of the triangle still carry zero mass; thus Corollary 3.3 is easily shown to be applicable to this case, namely: the integer translates of $\mu * B(\Gamma)$ are necessarily linearly independent.

(c) The last case is when $\mu$ is a finite sum of translates of the $\delta$ distribution; i.e., a certain difference operator. Here, if at least two of the point-masses are located at the vertices, then the condition needed for the application of Corollary 3.3 is violated. It can be shown that in this case linear dependence is obtained if and only if the support of $\mu$ lies on the union of a vertex and the edge opposite to that vertex; (both must contain some of the support).

We turn now to the discussion of Example 1.2. As mentioned in the introduction a certain constraint should be imposed: for the case $\mathrm{vol}_{\mathbf{R}^2}(\operatorname{supp}\mu) = 0$, we exclude the situation when this support intersects two parallel edges of the boundary of the unit square at sets carrying non-zero mass from $\mu$.

First note that $B(\Gamma)$, as a three-directional polynomial box spline, is generated by repeated convolution of $B(\{\gamma_1\}), B(\{\gamma_2\})$ and $B(\{\gamma_3\})$, where $\mathbf{x}_{\gamma_1} = (1,0)$, $\mathbf{x}_{\gamma_2} = (0,1)$, and $\mathbf{x}_{\gamma_3} = (1,1)$, and $\lambda_\gamma = 0$ for all $\gamma$'s. Also, $\Theta(\Gamma) = \{0\}$ and thus the Fourier transform of a positive measure $\mu$ vanishes nowhere on $\Theta(\Gamma)$. We can now appeal to Corollary 2.2 to conclude that the integer translates of $\mu * B(\Gamma)$ are linearly dependent if and only if the same holds for $\mu * B(\{\gamma_j\})$ for some $1 \leq j \leq 3$. On the other hand, Corollary 3.2, together with the fact that $\operatorname{supp}\mu$ lies in the unit square, shows that the integer translates of $\mu * B(\{\gamma_j\})$ are linearly independent for $j = 1,2$; (to see this we simply apply Corollary 3.2 to a bivariate tensor case). Consequently, the solution to our problem can be derived from the behaviour of $\phi := \mu * B(\{\gamma_3\})$.

To analyze the latter case, let $\mathbf{c} : \mathbb{Z}^s \longrightarrow \mathbb{C}$ and assume that

$$\phi * \mathbf{c} = 0 \tag{4.5}$$

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Now let $D$ be the directional derivative in the $(1,1)$ direction, and $\nabla$ the difference operator defined by

$$\nabla f = f - f(\cdot - (1,1)). \tag{4.6}$$

Then it can be easily verified (cf. [10]), that

$$DB(\{\gamma_3\}) = \nabla\delta, \tag{4.7}$$

where $\delta$ is the Dirac distribution. Hence, by (4.5) we have

$$0 = D(\phi * \mathbf{c}) = (\nabla\mu * \mathbf{c}) = \mu * \nabla\mathbf{c}.$$

Since the integer translates of $\mu$ are linearly independent, we have $\nabla\mathbf{c} = 0$, which means that $\mathbf{c}$ is constant along lines in the direction (1,1). Let $S$ be the strip located between $\alpha_1 - \alpha_2 = 0$ and $\alpha_1 - \alpha_2 = 1$. We can assume without loss of generality that $c_{(j,j)} = 1$ and $c_{(j+1,j)} = a$ for some complex-valued constant $a$. Since we know that the support of $\mu * B(\{\gamma_3\})$ is entirely located between the lines $\alpha_1 - \alpha_2 = 1$ and $\alpha_1 - \alpha_2 = -1$, the restriction of equation (4.5) to $S$ reads as follows:

$$\sum_{j=-\infty}^{\infty} c_{(j,j)}\phi(\cdot - (j,j)) + \sum_{j=-\infty}^{\infty} c_{(j+1,j)}\phi(\cdot - (j+1,j)) = 0. \tag{4.8}$$

Now, since here $\theta_{\gamma_3} = 0$, Proposition 3.1 shows that

$$\sum_{j=-\infty}^{\infty} B(\{\gamma_3\} \mid \cdot - (j,j)) = \chi,$$

where $\chi$ is the support function associated with the line $\{(t,t)\}_{t\in\mathbf{R}}$. Thus (4.8) becomes

$$\mu * \chi + a[(\mu * \chi)(\cdot - (1,0))] = 0, \tag{4.9}$$

and this relation should hold in the strip $S$.

Fixing $\mathbf{x} \in S$, we see that the first term in (4.9) measures the width of the section of $\operatorname{supp}\mu$ that lies on the line $\{\mathbf{x} + (t,t)\}_{t\in\mathbf{R}}$, while the second term in (4.9) is $a$ times the width of the section that lies on $\{\mathbf{x} + (t, t+1)\}_{t\in\mathbf{R}}$. In order for (4.9) to be valid, it is necessary and sufficient that there is a constant ratio $(-a)$ between the two widths. Thus we have established the claim of the example.

We remark that the same approach might be applied to higher-dimensional settings when all the $\mathbf{x}_\gamma$ vectors are either elements of the standard basis or the vector $(1,1,\ldots,1)$ in $\mathbb{R}^s$.

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