ON COMMUTATIVITY OF C*-ALGEBRAS

by C.-S. LIN

(Received 7 October, 1985)

1. Two numerical characterizations of commutativity for C*-algebra $\mathscr{A}$ (acting on the Hilbert space H) were given in [**1**]; one used the norms of self-adjoint operators in $\mathscr{A}$ (Theorem 2), and the other the numerical index of $\mathscr{A}$ (Theorem 3). In both cases the proofs were based on the result of Kaplansky which states that if the only nilpotent operator in $\mathscr{A}$ is 0, then $\mathscr{A}$ is commutative ([**2**] 2.12.21, p. 68). Of course the converse also holds.

We shall apply in this note both Kaplansky's result and Holbrook's operator radii [**3**] to give two types of characterizations; one is by means of operator radii, and the other in terms of C_p-classes of operators in $\mathscr{A}$. These also enable us to generalize Theorem 2 and 3 in [**1**]. Finally, a particular case of our Theorem 5 shows that $\mathscr{A}$ is commutative if and only if every $T \in \mathscr{A}$ satisfies the first order growth condition (G_1).

2. First we need some notation, definitions and well known results. Let $T \in \mathscr{A}$. We recall that T is in the class $C_p(p>0)$, operators having unitary p-dilation, if

$$(ph, h) - 2\mathrm{Re}(z(p-1)Th, h) + |z|^2((p-2)Th, Th) \geqslant 0$$

holds for all $h \in H$ and $|z| \leqslant 1$ ([**4** p. 45]). Since the inequality can be rewritten as $\mathrm{Re}(p - 2z(p-1)T + |z|^2(p-2)T^*T) \geqslant 0$, it follows easily that $T \in C_p$ if and only if for all $|z| \leqslant 1$ we have

$$(p-2)(I-zT)^*(I-zT) + (I-zT) + (I-zT)^* \geqslant 0. \qquad (*)$$

We need the following properties from [**3**].

(1) $T \in C_p$ if and only if $w_p(T) \leqslant 1$, where

$$w_p(T) = \inf\{u : u > 0,\ T/u \in C_p\},$$

the operator radius of T.

(2) $w_1(T) = \|T\|$, and $w_2(T)$ is the numerical radius of T.

(3) $w_p(uT) = |u|\, w_p(T)$, and $w_p(T) \geqslant \|T\|/p$.

(4) $w_p(.)$ is a norm on $\mathscr{A}$ whenever $0 < p \leqslant 2$.

THEOREM 1 (Theorem 4.4, 4.5 and 5.5 [**3**]). *Let* $T \in \mathscr{A}$,

(1) *If* $0 < p < p'$, *then* $w_{p'}(T) \leqslant w_p(T)$ *and* $w_p(T) \leqslant \left(\dfrac{2p'}{p} - 1\right) w_{p'}(T)$.

(2) *If* $\|T\| = 1$ *and* $T^2 = 0$, *then* $w_p(T) = 1/p$ *for every* $p > 0$.

https://doi.org/10.1017/S0017089500006704 Published online by Cambridge University Press

(3) *If T is normaloid, that is $\|T\|$ is the spectral radius of T, then*

$$w_p(T)=\begin{cases}\left(\frac{2}{p}-1\right)\|T\| & \text{if } 0<p<1,\\ \|T\| & \text{if } p\geq 1.\end{cases}$$

Note that a normal operator is normaloid.

3. Now we are ready to give characterizations.

THEOREM 2. *For $p>0$ and $p\neq 1$, $\mathscr{A}$ is commutative if and only if*

$$w_p(T)=\begin{cases}\left(\frac{2}{p}-1\right)\|T\| & \text{if } 0<p<1,\\ \|T\| & \text{if } p>1,\end{cases}$$

for every $T\in\mathscr{A}$.

Proof. ($\Rightarrow$). The commutativity implies that every $T\in\mathscr{A}$ is normal and so we may apply (3) in Theorem 1.

($\Leftarrow$). If $\mathscr{A}$ is not commutative, then there exists a $T\in\mathscr{A}$, $T\neq 0$ and $T^2=0$ such that $w_p(T)=\|T\|/p$, $p>0$ by (2) in Theorem 1. Hence the equality in the statement does not hold.

COROLLARY 1. *For $p>0$ and $p\neq 1$, $\mathscr{A}$ is not commutative if and only if $w_p(T)=\|T\|/p$* for some $T\in\mathscr{A}$.

From the well known results in section two we see that

$$\|T\|/p\leq w_p(T)\leq\left(\frac{2-p}{p}\right)\|T\|$$

holds for $0<p<1$, and $\|T\|/p\leq w_p(T)\leq\|T\|$ for $p\geq 1$. Let us define

$$n_p(\mathscr{A})=\inf\{w_p(T)\colon T\in\mathscr{A},\ \|T\|=1\}.$$

Clearly, $1/p\leq n_p(\mathscr{A})\leq\dfrac{2-p}{p}$ for $0<p<1$, and $1/p\leq n_p(\mathscr{A})\leq 1$ for $p\geq 1$. Note that $n_2(\mathscr{A})$ is called the *numerical index* of $\mathscr{A}$.

COROLLARY 2. (Theorem 3 **[1]** when $p=2$). (1) *For $p>1$, $\mathscr{A}$ is commutative or not commutative according to $n_p(\mathscr{A})$ is* 1 *or* $1/p$.

(2) *For $0<p<1$, $\mathscr{A}$ is commutative or not commutative according to $n_p(\mathscr{A})$ is* $\dfrac{2-p}{p}$ *or* $1/p$.

Proof. This is a simple consequence of Theorem 2 and Corollary 1.

https://doi.org/10.1017/S0017089500006704 Published online by Cambridge University Press

THEOREM 3. *The following statements are equivalent.*

(1) $\mathcal{A}$ *is commutative.*

(2) $cT \notin C_p$ *for all* $p \geq 1$, *all* T *with* $0 \neq T \in \mathcal{A}$ *and any* $c > \|T\|^{-1}$.

(3) $\left(\frac{cp}{2-p}\right)T \notin C_p$ *for all* p *with* $0 < p < 1$, *all* T *with* $0 \neq T \in \mathcal{A}$ *and any* $c > \|T\|^{-1}$.

Proof. (1)⇒(2). Since cT is normal, $w_p(cT) = c\,\|T\| > 1$ for every $p \geq 1$ by (3) in Theorem 1. Hence, $cT \notin C_p$.

(2)⇒(1). If $\mathcal{A}$ is not commutative, then $w_p(cT) = c\,\|T\|/p$ for some $T \in \mathcal{A}$, $T \neq 0$ and $T^2 = 0$. We may select a suitable $p > 1$ so that $p \geq c\,\|T\|$. This implies that $cT \in C_p$.

(1)⇒(3). $w_p\left(\frac{cp}{2-p}T\right) = \left(\frac{cp}{2-p}\right)\left(\frac{2-p}{p}\right)\|T\| = c\,\|T\| > 1$ by (3) in Theorem 1 and so $\left(\frac{cp}{2-p}\right)T \notin C_p$ for every p with $0 < p < 1$.

(3)⇒(1). If $\mathcal{A}$ is not commutative, then $w_p\left(\frac{cp}{2-p}T\right) = \frac{c\,\|T\|}{2-p}$ for some $T \in \mathcal{A}$, $T \neq 0$ and $T^2 = 0$. By choosing suitable c and $p(\leq 2 - c\,\|T\|)$ we may conclude that $\left(\frac{cp}{2-p}\right)T \in C_p$.

THEOREM 4 (Theorem 2 [**1**] when $p = 1$). *The following statements are equivalent.*

(1) $\mathcal{A}$ *is commutative.*

(2) *For any* $p \geq 1$, $w_p(A+B) \leq 1 + w_p(AB)$ *for all self-adjoint operators* A *and* $B \in \mathcal{A}$ *with* $\|A\| = \|B\| = 1$.

(3) *For any* p *with* $0 < p < 1$, $w_p(A+B) \leq \frac{2-p}{p} + w_p(AB)$ *for all self-adjoint operators* A *and* $B \in \mathcal{A}$ *with* $\|A\| = \|B\| = 1$.

Proof. We shall adapt the original result in [**1**]; $\mathcal{A}$ is commutative if and only if $\|A+B\| \leq 1 + \|AB\|$ for all self-adjoint operators A and $B \in \mathcal{A}$ with $\|A\| = \|B\| = 1$.

(1)⇒(2). $w_p(A+B) = \|A+B\| \leq 1 + \|AB\| = 1 + w_p(AB)$ by (3) in Theorem 1.

(2)⇒(1). If $\mathcal{A}$ is not commutative, then for some self-adjoint operators A and B, $w_p(A+B) = \|A+B\| > 1 + \|AB\| \geq 1 + w_p(AB)$ by (1) in Theorem 1 for the last inequality.

(1)⇒(3). $\left(\frac{p}{2-p}\right)w_p(A+B) = \|A+B\| \leq 1 + \|AB\| = 1 + \left(\frac{p}{2-p}\right)w_p(AB)$ by (3) in Theorem 1.

(3)⇒(1). If $\mathcal{A}$ is not commutative, for some self-adjoint operators A and B,

$$\left(\frac{p}{2-p}\right)w_p(A+B) = \|A+B\| > 1 + \|AB\| \geq 1 + \left(\frac{p}{2-p}\right)w_p(AB)$$

by (1) in Theorem 1 for the last inequality.

https://doi.org/10.1017/S0017089500006704 Published online by Cambridge University Press

We recall that $T \in \mathscr{A}$ satisfies the first order growth condition (G_1) if $\|(u-T)^{-1}\| = 1/d(u)$ for all $u \notin \sigma(T)$, the spectrum of T, where $d(u)$ denotes the distance from u to $\sigma(T)$. It is known that $\|(u-T)^{-1}\| \geqslant 1/d(u)$ holds for any $T \in \mathscr{A}$, and a normal operator satisfies condition (G_1). We shall next generalize this to operator radii and prove the following result.

THEOREM 5. *The following statements are equivalent.*

(1) *$\mathscr{A}$ is commutative.*

(2) *For $p \geqslant 1$, $w_p((u-T)^{-1}) = 1/d(u)$ for every $T \in \mathscr{A}$ and $u \notin \sigma(T)$.*

·(3) *For $0<p<1$, $w_p((u-T)^{-1}) = \dfrac{2-p}{pd(u)}$ for every $T \in \mathscr{A}$ and $u \notin \sigma(T)$.*

Proof. (1)$\Rightarrow$(2). Since $(u-T)^{-1}$ is normal,

$$d(u)w_p((u-T)^{-1}) = d(u)\,\|(u-T)^{-1}\| = d(u)(1/d(u)) = 1$$

by (3) in Theorem 1.

(2)$\Rightarrow$(1). Let $T \in \mathscr{A}$, $T^k = 0$, $k \geqslant 2$ and $T^{k-1} \neq 0$; then $\sigma(T) = \{0\}$. We shall show that if T satisfies the condition $w_p((u-T)^{-1}) \leqslant 1/d(u)$, that is $u(u-T)^{-1} \in C_p$ for any complex number $u \neq 0$, then $T=0$. To this end, let $z=1$ in the inequality (*). We obtain

$$(p-2)(I-u(u-T)^{-1})^*(I-u(u-T)^{-1}) + (I-u(u-T)^{-1}) + (I-u(u-T)^{-1})^* \geqslant 0.$$

Let the left hand side in the above be F; then $(u-T)^*F(u-T) \geqslant 0$ and so

$$(p-2)T^*T-(u-T)^*T-T^*(u-T) \geqslant 0.$$

We claim that $N(T) \subseteq N(T^*)$, where $N(\cdot)$ denotes the null space. Let $x \in N(T)$, and Q be the left side in above inequality; then $Q(x) = -T^*(ux)$ so that $\|Q^{1/2}(x)\|^2 = (Q(x), x) = (-T^*(ux), x) = 0$ and hence $T^*(x) = 0$. Now, for any $x \in H$, $0 = T^k(x) = T^*T^{k-1}(x)$ by the claim. It follows that $T^{k-1} = 0$, contrary to our assumption. Thus, $T=0$.

(1)$\Rightarrow$(3). Normality of $(u-T)^{-1}$ implies that

$$\frac{pd(u)}{2-p}\,w_p((u-T)^{-1}) = \frac{pd(u)}{2-p}\,\|(u-T)^{-1}\|\,\frac{2-p}{p} = 1$$

by (3) in Theorem 1.

(3)$\Rightarrow$(1). The proof may be carried out in a manner similar to the one above by showing that $T=0$. To make computation simple let $u=1$, and $z=-1$ in (*). Since $\dfrac{p}{2-p}(I-T)^{-1} \in C_p$, it follows that

$$S = (p-2)\left(\frac{2}{2-p}-T\right)^*\left(\frac{2}{2-p}-T\right) + (I-T)^*\left(\frac{2}{2-p}-T\right) + \left(\frac{2}{2-p}-T\right)^*(I-T) \geqslant 0.$$

https://doi.org/10.1017/S0017089500006704 Published online by Cambridge University Press

Let $x \in N(T)$ and consider $S(x)$; after a few simplifications we have

$$S(x) = \frac{p}{p-2} T^*(x).$$

The remainder of the proof is the same as above.

Finally, we remark that the above conditions on T may be relaxed and the result still holds. For example, $u \in U \backslash \sigma(T)$, where the set U need not contain $\sigma(T)$ but $U \backslash \sigma(T)$ must be non-empty.

REFERENCES

1. M. J. Crabb, J. Duncan and C. M. McGregor, Characterizations of commutativity for C^*-algebras, *Glasgow Math. J.* **15** (1974), 172–175.

2. J. Dixmier, *C*-algebras,* (North-Holland, 1977).

3. J. A. R. Holbrook, On the power-bounded operators of Sz-Nagy and Foias, *Acta Sci. Math.,* **29** (1968), 299–310.

4. B. Sz-Nagy and C. Foias, *Harmonic analysis of operators on Hilbert Space,* (Akadémiai Kiadó, Budapest, 1970).

DEPARTMENT OF MATHEMATICS
BISHOP'S UNIVERSITY
LENNOXVILLE, QUEBEC
CANADA