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BONGARTZ τ-COMPLEMENTS OVER SPLIT-BY-NILPOTENT EXTENSIONS

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Abstract. Let C be a finite dimensional algebra with B a split extension by a nilpotent bimodule E, and let M be a τ_C-rigid module with U its Bongartz τ_C-complement. If the induced module, $M \otimes_C B$, is τ_B-rigid, we give a necessary and sufficient condition for $U \otimes_C B$ to be its Bongartz τ_B-complement. If M is τ_B-rigid, we again provide a necessary and sufficient condition for $U \otimes_C B$ to be its Bongartz τ_B-complement.

1. Introduction. Let C be a finite dimensional algebra over an algebraically closed field k. By module is meant throughout a finitely generated right C-module and mod C denotes the category of finitely generated right C-modules. Let add M denote the full subcategory of mod C whose objects are direct sums of direct summands of M. Following [**1**], we call a C-module M τ_C-*rigid* if $\operatorname{Hom}_C(M, \tau_C M) = 0$ and τ_C-*tilting* if M is τ_C-rigid and the number of pairwise non-isomorphic indecomposable summands of M equals the number of pairwise non-isomorphic simple modules of C. We say M is *almost complete* τ_C-*tilting* if M is τ_C-rigid and $|M| = |C| - 1$. It was shown in [**1**] that, given any τ_C-rigid module, there exists a τ_C-rigid module U such that $M \oplus U$ is a τ_C-tilting module. This module U is called the *Bongartz* τ_C-*complement* of M. In this paper, we are interested in the problem of extending Bongartz τ-complements. More precisely, let C and B be two finite dimensional k-algebras such that there exists a split surjective algebra morphism $B \to C$, whose kernel E is contained in the radical of B. We then say B is a split extension of A by the nilpotent bimodule E.

Our first main result is the following theorem.

THEOREM 1.1 (Theorem 2.2). *Let B be a split extension of C by a nilpotent bimodule E, and let M be a τ_C-rigid module with U its Bongartz τ_C-complement. If $M \otimes_C B$ is τ_B-rigid, then $U \otimes_C B$ is the Bongartz τ_B-complement if and only if* $\operatorname{Hom}_C(U \otimes_C E, \tau_C M) = 0$.

Our second main result concerns M as a τ_B-rigid module and its Bongartz τ_B-complement. Here, $(\tau_B M)_C$ denotes the C-module structure of $\tau_B M$.

THEOREM 1.2 (Theorem 3.3). *Let B be a split extension of C by a nilpotent bimodule E, and let M be a τ_C-rigid module with U its Bongartz τ_C-complement. If M is τ_B-rigid, then $U \otimes_C B$ is the Bongartz τ_B-complement if and only if* $\operatorname{Hom}_C(U, (\tau_B M)_C) = 0$.

We use freely and without further reference properties of the module categories and Auslander–Reiten translations as can be found in [**3**]. For an algebra C, we denote by τ_C the Auslander–Reiten translation in mod C.

1.1. Split extensions and extensions of scalars. We begin this section with the formal definition of a split extension.

DEFINITION 1.3. Let B and C be two algebras. We say B is a *split extension* of C by a nilpotent bimodule E if there exists a short exact sequence of B-modules

$$0 \to E \to B \underset{\sigma}{\overset{\pi}{\rightleftarrows}} C \to 0,$$

where π and σ are algebra morphisms, such that $\pi \circ \sigma = 1_C$, and $E = \ker \pi$ is nilpotent.

A useful way to study the module categories of C and B is a general construction via the tensor product, also known as *extension of scalars*, that sends a C-module to a particular B-module. Here, D denotes the standard duality functor.

DEFINITION 1.4. Let C be a subalgebra of B such that $1_C = 1_B$, then

$$- \otimes_C B : \operatorname{mod} C \to \operatorname{mod} B,$$

is called the *induction functor*, and dually

$$D(B \otimes_C D-) : \operatorname{mod} C \to \operatorname{mod} B,$$

is called the *coinduction functor*. Moreover, given $M \in \operatorname{mod} C$, the corresponding induced module is defined to be $M \otimes_C B$, and the coinduced module is defined to be $D(B \otimes_C DM)$.

It was shown in [**6**, 3.6] that, as a C-module, $M \otimes_C B \cong M \oplus (M \otimes_C E)$. Next, we state a result that gives information about $\operatorname{Hom}_B(-, \tau_B(M \otimes_C B))$ and $\operatorname{Hom}_B(M \otimes_C B, -)$.

LEMMA 1.5. *Let M be a C-module, $M \otimes_C B$ the induced module, and let X be any B-module. Then, we have*

$$\operatorname{Hom}_B(X, \tau_B(M \otimes_C B)) \cong \operatorname{Hom}_B(X, \operatorname{Hom}_C({}_BB_C, \tau_C M)) \cong \operatorname{Hom}_C(X \otimes_B B_C, \tau_C M)$$

and

$$\operatorname{Hom}_B(M \otimes_C B, X) \cong \operatorname{Hom}_C(M, \operatorname{Hom}_B({}_CB_B, X)).$$

Proof. These isomorphisms follow from [**2**, Lemma 2.1] and the adjunction isomorphism. □

We note that $- \otimes_B B_C$ and $\operatorname{Hom}_B({}_CB_B, -)$ are two expressions for the forgetful functor $\operatorname{mod} B \to \operatorname{mod} C$.

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1.2. τ-rigid modules and Bongartz τ-complements. We start with a definition.

DEFINITION 1.6. Let M be a C-module. We define *Gen* M to be the class of all modules X in mod C generated by M, that is, the modules X such that there exists an integer $d \geq 0$ and an epimorphism $M^d \to X$ of C-modules. Here, M^d is the direct sum of d copies of M. Dually, we define *Cogen* M to be the class of all modules Y in mod C cogenerated by M, that is, the modules Y such that there exist an integer $d \geq 0$ and a monomorphism $Y \to M^d$ of C-modules.

To describe Bongartz τ-complements, we need the notion of a torsion class and torsion pair.

DEFINITION 1.7. A pair of full subcategories $(\mathcal{T}, \mathcal{F})$ of mod C is called a *torsion pair* if the following conditions are satisfied:

(a) $\mathrm{Hom}_C(M, N) = 0$ for all $M \in \mathcal{T}$, $N \in \mathcal{F}$.
(b) $\mathrm{Hom}_C(M, -)|_{\mathcal{F}} = 0$ implies $M \in \mathcal{T}$.
(c) $\mathrm{Hom}_C(-, N)|_{\mathcal{T}} = 0$ implies $N \in \mathcal{F}$.

We call $\mathcal{T}$ and $\mathcal{F}$ a *torsion class* and *torsionfree class*, respectively.

DEFINITION 1.8. Let $\mathcal{T}$ be a full subcategory of mod C and $X \in \mathcal{T}$. We say a C-module X is Ext-*projective* in $\mathcal{T}$ if $\mathrm{Ext}^1_C(X, \mathcal{T}) = 0$. We denote by $P(\mathcal{T})$ the direct sum of one copy of each indecomposable Ext-projective module in $\mathcal{T}$ up to isomorphism.

It was shown in [**1**, 2.10] that, for every τ_C-rigid module M, there exists a module U such that $M \oplus U$ is τ_C-tilting. This module is called the Bongartz τ_C-complement of M. To give an explicit construction, we define

$$^{\perp}(\tau_C M) = \{X \in \mathrm{mod}\, C \mid \mathrm{Hom}_C(X, \tau_C M) = 0\}.$$

It was also shown in [**1**](2.11) that $^{\perp}(\tau_C M)$ forms a torsion class, the corresponding torsionfree class is $\mathrm{Cogen}(\tau_C M)$, and $(^{\perp}(\tau_C M), \mathrm{Cogen}(\tau_C M))$ is a torsion pair.

Then, $P(^{\perp}(\tau_C M))$ is a τ_C-tilting module satisfying $M \in \mathrm{add}(P(^{\perp}(\tau_C M)))$. Let U be the direct sum of one copy of each indecomposable Ext-projective module in $^{\perp}(\tau_C M)$ up to isomorphism that does not belong to add M. Then, $M \oplus U$ is τ_C-tilting and U is the Bongartz τ_C-complement of M.

2. Main results and corollaries. Throughout this section, B is a split extension of C by a nilpotent bimodule E. We begin with a result proved in [**2**] that shows precisely when an induced module, $M \otimes_C B$, is τ_B-rigid (τ_B-tilting).

THEOREM 2.1 ([**2**, Theorem A]). *Let M be a C-module. Then, $M \otimes_C B$ is τ_B-rigid (τ_B-tilting) if and only if M is τ_C-rigid (τ_C-tilting) and* $\mathrm{Hom}_C(M \otimes_C E, \tau_C M) = 0$.

We are now ready for our main result. We assume throughout that M is τ_C-rigid with U its Bongartz τ_C-complement.

THEOREM 2.2. *Suppose $M \otimes_C B$ is τ_B-rigid. Then, $U \otimes_C B$ is the Bongartz τ_B-complement if and only if* $\mathrm{Hom}_C(U \otimes_C E, \tau_C M) = 0$.

Proof. Suppose $U \otimes_C B$ is the Bongartz τ_B-complement of $M \otimes_C B$. This implies $\mathrm{Hom}_B(U \otimes_C B, \tau_B(M \otimes_C B)) = 0$. Using Lemma 1.5 and [**6**, 3.6], we have the following

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isomorphisms

$$\mathrm{Hom}_B(U\otimes_C B, \tau_B(M\otimes_C B)) \cong \mathrm{Hom}_B(U\otimes_C B, \mathrm{Hom}_C({}_BB_C, \tau_C M)) \cong$$

$$\mathrm{Hom}_C(U\otimes_C B\otimes_B B_C, \tau_C M) \cong \mathrm{Hom}_C(U\otimes_C B_C, \tau_C M) \cong$$

$$\mathrm{Hom}_C(U\otimes_C (C\oplus E)_C, \tau_C M) \cong \mathrm{Hom}_C(U\oplus (U\otimes_C E), \tau_C M) \cong$$

$$\mathrm{Hom}_C(U, \tau_C M)\oplus \mathrm{Hom}_C(U\otimes_C E, \tau_C M).$$

We conclude that $\mathrm{Hom}_C(U\otimes_C E, \tau_C M) = 0$.

Conversely, suppose $\mathrm{Hom}_C(U\otimes_C E, \tau_C M) = 0$. Then, $\mathrm{Hom}_C(U\otimes_C E, \tau_C U) = 0$ because U is Ext-projective in ${}^{\perp}(\tau_C M)$ and proposition [**3**, VI, 1.11] shows $\tau_C U$ is cogenerated by $\tau_C M$ since $\mathrm{Cogen}(\tau_C M)$ is the corresponding torsionfree class by [**1**, 2.11]. Thus, Theorem 2.1 says $U\otimes_C B$ is τ_B-rigid. Using the above vector space isomorphisms, we see $\mathrm{Hom}_B(U\otimes_C B, \tau_B(M\otimes_C B)) = 0$. Next, we will show $U\otimes_C B$ is Ext-projective in ${}^{\perp}(\tau_B(M\otimes_C B))$. By proposition [**1**, 2.9], we need to show that

$$\mathrm{Gen}(U\otimes_C B) \subseteq {}^{\perp}(\tau_B(M\otimes_C B)) \subseteq {}^{\perp}(\tau_B(U\otimes_C B)).$$

The first containment is clear so let $X\in {}^{\perp}(\tau_B(M\otimes_C B))$ but $X\notin {}^{\perp}(\tau_B(U\otimes_C B))$. Using the above vector space isomorphisms, $\mathrm{Hom}_C(X_C, \tau_C M) = 0$ and $\mathrm{Hom}_C(X_C, \tau_C U)\neq 0$, where X_C denotes the C-module structure of X. Since proposition [**3**, VI, 1.11] says $\tau_C U$ is cogenerated by $\tau_C M$, we have a contradiction. Thus, $U\otimes_C B$ is Ext-projective in ${}^{\perp}(\tau_B(M\otimes_C B))$.

Finally, we need to show $U\otimes_C B$ comprises all the indecomposable Ext-projective modules in ${}^{\perp}(\tau_B(M\otimes_C B))$ up to isomorphism not in $\mathrm{add}(M\otimes_C B)$. Suppose not and let Y be the direct sum of all remaining Ext-projective modules in ${}^{\perp}(\tau_B(M\otimes_C B))$ up to isomorphism not in $\mathrm{add}(M\otimes_C B)$. Then, $(U\otimes_C B)\oplus Y$ is the Bongartz τ_B-complement of $M\otimes_C B$. Thus, $(M\otimes_C B)\oplus(U\otimes_C B)\oplus Y$ is a τ_B-tilting module such that the number of pairwise non-isomorphic indecomposable summands equals the number of pairwise non-isomorphic simple modules of B. However, [**6**, 3.4] implies the number of pairwise non-isomorphic simple modules of C and B are equal. Thus, we have the inequality $|(M\otimes_C B)\oplus(U\otimes_C B)\oplus Y| > |B|$ but this contradicts [**1**, 1.3]. We conclude Y must be 0 and $U\otimes_C B$ is the Bongartz τ_B-complement of $M\otimes_C B$. □

Next, we present three corollaries. If $M\in \mathrm{Gen}\,U$, then $\mathrm{Hom}_C(U\otimes_C E, \tau_C M) = 0$ guarantees $M\otimes_C B$ is τ_B-rigid with $U\otimes_C B$ the Bongartz τ_B-complement.

Corollary 2.3. *Suppose $M\in \mathrm{Gen}\,U$. Then, $M\otimes_C B$ is τ_B-rigid with $U\otimes_C B$ its Bongartz τ_B-complement if and only if* $\mathrm{Hom}_C(U\otimes_C E, \tau_C M) = 0$.

Proof. We only need to show $M\otimes_C B$ being τ_B-rigid follows from the assumption $\mathrm{Hom}_C(U\otimes_C E, \tau_C M) = 0$. The rest follows from Theorem 2.2. Since $M\in \mathrm{Gen}\,U$, there exists an epimorphism $f : U^d \to M$ where $d\geq 0$. The functor $\_\otimes_C E$ is right exact and applying to f yields an epimorphism $f\otimes_C 1_E : (U\otimes_C E)^d \to M\otimes_C E$. Thus, $\mathrm{Hom}_C(U\otimes_C E, \tau_C M) = 0$ implies $\mathrm{Hom}_C(M\otimes_C E, \tau_C M) = 0$ that further implies $M\otimes_C B$ is τ_B-rigid by Theorem 2.1. □

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In the special case, where M is indecomposable and non-projective, we always have $M \in \operatorname{Gen} U$.

COROLLARY 2.4. *Let M be indecomposable and non-projective. Then, $M \otimes_C B$ is τ_B-rigid with $U \otimes_C B$ its Bongartz τ_B-complement if and only if* $\operatorname{Hom}_C(U \otimes_C E, \tau_C M) = 0$.

Proof. We need to show $M \in \operatorname{Gen} U$ and the result will follow from corollary 2.3. By [**1**, 2.22], either $M \in \operatorname{Gen} U$ or ${}^{\perp}(\tau_C U) \subseteq {}^{\perp}(\tau_C M)$. Assume ${}^{\perp}(\tau_C U) \subseteq {}^{\perp}(\tau_C M)$ is true. Since U is the Bongartz τ_C-complement, we have ${}^{\perp}(\tau_C M) \subseteq {}^{\perp}(\tau_C U)$ by [**1**, 2.9]. Thus, ${}^{\perp}(\tau_C U) = {}^{\perp}(\tau_C M)$. Again, since U is the Bongartz τ-complement of M, we know $\tau_C U \in \operatorname{Cogen}(\tau_C M)$. Now, $\operatorname{Gen} M \subseteq {}^{\perp}(\tau_C M) = {}^{\perp}(\tau_C U)$ and [**1**, 2.9] implies M is Ext-projective in ${}^{\perp}(\tau_C U)$. [**3**, VI, 1.11] gives $\tau_C M \in \operatorname{Cogen}(\tau_C U)$. Since $\tau_C U$ and $\tau_C M$ cogenerate each other, we conclude $\tau_C M \cong \tau_C U$. This is only possible if both $\tau_C M$ and $\tau_C U$ are 0 that implies M and U are projective. But we assumed M is not projective and thus a contradiction. We conclude $M \in \operatorname{Gen} U$. □

Next, we assume that $E \in \operatorname{Gen} M$ when E is viewed as a right C-module.

COROLLARY 2.5. *Let $E \in \operatorname{Gen} M$. Then, $M \otimes_C B$ is τ_B-rigid with $U \otimes_C B$ its Bongartz τ_B-complement.*

Proof. Since $E \in \operatorname{Gen} M$, we have $\operatorname{Hom}_C(E, \tau_C M) = 0$. Since $\tau_C U$ is cogenerated by $\tau_C M$ by [**3**, VI, 1.11], we also have $\operatorname{Hom}_C(E, \tau_C U) = 0$. Using the adjunction isomorphism,

$$0 = \operatorname{Hom}_C(M, \operatorname{Hom}_C(E, \tau_C M)) \cong \operatorname{Hom}_C(M \otimes_C E, \tau_C M).$$

By Theorem 2.1, $M \otimes_C B$ is τ_B-rigid. By the same reasoning, $\operatorname{Hom}_C(U \otimes_C E, \tau_C M)$ and $\operatorname{Hom}_C(U \otimes_C E, \tau_C U)$ are equal to 0. The result now follows from Theorem 2.2. □

Our next proposition concerns almost complete τ-tilting modules.

PROPOSITION 2.6. *Suppose M is an almost complete τ_C-titling module such that $M \oplus Y$ is τ_C-tilting and Y is not the Bongartz τ_C-complement for some indecomposable C-module Y. Suppose $M \otimes_C B$ is τ_B-tilting. Then, $(M \otimes_C B) \oplus (Y \otimes_C B)$ is τ_B-tilting if and only if* $\operatorname{Hom}_C(M \otimes_C E, \tau_C Y) = 0$.

Proof. Since Y is indecomposable and not the Bongartz τ_C-complement, we have $Y \in \operatorname{Gen} M$ by [**1**, 2.22]. Thus, there exists an epimorphism $f : M^d \to Y$ where $d \geq 0$. The functor $\_ \otimes_C B$ is right exact and applying to f yields an epimorphism $f \otimes_C 1_E : (M \otimes_C B)^d \to Y \otimes_C B$. Since $M \otimes_C B$ is τ_B-rigid and $Y \otimes_C B \in \operatorname{Gen}(M \otimes_C B)$, we have $\operatorname{Hom}_B(Y \otimes_C B, \tau_B(M \otimes_C B)) = 0$. Using Lemma 1.5 and [**6**, 3.6], we have

$$\operatorname{Hom}_B(M \otimes_C B, \tau_B(Y \otimes_C B)) \cong \operatorname{Hom}_C((M \otimes_C B)_C, \tau_C Y) \cong$$

$$\operatorname{Hom}_C(M, \tau_C Y) \oplus \operatorname{Hom}_C(M \otimes_C E, \tau_C Y).$$

Thus, $\operatorname{Hom}_C(M \otimes_C E, \tau_C Y) = 0$ if and only if $\operatorname{Hom}_B(M \otimes_C B, \tau_B(Y \otimes_C B)) = 0$ and our statement follows. □

3. M as a τ-rigid B-module. In this section, we present several results concerning a C-module M which is τ_B-rigid. Throughout, we assume B is a split extension of C

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by a nilpotent bimodule E and M is τ_C-rigid. We begin with a sufficient condition for M to be τ_B-rigid.

PROPOSITION 3.1. *If* $\mathrm{Hom}_C(M \otimes_C E, \mathrm{Gen}M) = 0$, *then* M *is* τ_B*-rigid.*

Proof. By [**6**, 3.6], we have the following short exact sequence in mod B

$$0 \to M \otimes_C E \to M \otimes_C B \to M \to 0.$$

Applying $\mathrm{Hom}_B(-, \mathrm{Gen}M)$, we obtain an exact sequence

$$\mathrm{Hom}_B(M \otimes_C E, \mathrm{Gen}M) \to \mathrm{Ext}^1_B(M, \mathrm{Gen}M) \to \mathrm{Ext}^1_B(M \otimes_C B, \mathrm{Gen}M).$$

First, we wish to show $\mathrm{Ext}^1_B(M \otimes_C B, \mathrm{Gen}M) = 0$. We know from [**5**, 5.8] this is equivalent to $\mathrm{Hom}_B(M, \tau_B(M \otimes_C B)) = 0$. By Lemma 1.5 and the assumption that M is τ_C-rigid, $\mathrm{Hom}_B(M, \tau(M \otimes_C B)) \cong \mathrm{Hom}_C(M, \tau_C M) = 0$. Next, we want to show $\mathrm{Hom}_B(M \otimes_C E, \mathrm{Gen}M) = 0$. By restriction of scalars, any non-zero morphism from $M \otimes_C E$ to $\mathrm{Gen}M$ in mod B would give a non-zero morphism in mod C, contrary to our assumption. Thus, $\mathrm{Hom}_B(M \otimes_C E, \mathrm{Gen}M) = 0$. We conclude $\mathrm{Ext}^1_B(M, \mathrm{Gen}M) = 0$ and [**5**, 5.8] implies M is τ_B-rigid. □

The following determines precisely when $M \otimes_C B$ is Ext-projective in ${}^\perp(\tau_B M)$. Recall, we denote the C-module structure of $\tau_B M$ by $(\tau_B M)_C$.

PROPOSITION 3.2. *Suppose* M *is* τ_B*-rigid. Then* $M \otimes_C B \in P({}^\perp(\tau_B M))$ *if and only if* $\mathrm{Hom}_C(M, (\tau_B M)_C) = 0$.

Proof. Assume $M \otimes_C B \in P({}^\perp(\tau_B M))$. Then $\mathrm{Hom}_B(M \otimes_C B, \tau_B M) = 0$. Using Lemma 1.5, we have $\mathrm{Hom}_B(M \otimes_C B, \tau_B M) \cong \mathrm{Hom}_C(M, (\tau_B M)_C) = 0$. Next, assume $\mathrm{Hom}_C(M, (\tau_B M)_C) = 0$. Again, Lemma 1.5 gives $\mathrm{Hom}_B(M \otimes_C B, \tau_B M) = 0$. Thus, $M \otimes_C B \in {}^\perp(\tau_B M)$ and we need to show $M \otimes_C B \in P^\perp(\tau_B M)$. We have $\tau_B(M \otimes_C B) \in \mathrm{Cogen}(\tau_B M)$ by [**4**, 1.2] and [**3**, VI, 1.11] gives $M \otimes_C B$ is Ext-projective in ${}^\perp(\tau_B M)$. □

Suppose U is the Bongartz τ_C-complement of M. If M is τ_B-rigid, our main result gives a necessary and sufficient condition for $U \otimes_C B$ to be the Bongartz τ_B-complement.

THEOREM 3.3. *Suppose* M *is* τ_B*-rigid. Then* $U \otimes_C B$ *is the Bongartz* τ_B*-complement if and only if* $\mathrm{Hom}_C(U, (\tau_B M)_C) = 0$.

Proof. Assume $U \otimes_C B$ is the Bongartz τ_B-complement. Then $\mathrm{Hom}_B(U \otimes_C B, \tau_B M) = 0$ and Lemma 1.5 gives $\mathrm{Hom}_B(U \otimes_C B, \tau_B M) \cong \mathrm{Hom}_C(U, (\tau_B M)_C) = 0$. Next, assume $\mathrm{Hom}_C(U, (\tau_B M)_C) = 0$. Again, Lemma 1.5 gives $\mathrm{Hom}_B(U \otimes_C B, \tau_B M) = 0$. Thus, $U \otimes_C B \in {}^\perp(\tau_B M)$ and we need to show $U \otimes_C B \in P^\perp(\tau_B M)$. Using [**1**, 2.9], we need to show the following containments

$$\mathrm{Gen}(U \otimes_C B) \subseteq {}^\perp(\tau_B M) \subseteq {}^\perp(\tau_B(U \otimes_C B)).$$

The first is clear so let $X \in {}^\perp(\tau_B M)$. We need to show $X \in {}^\perp(\tau_B(U \otimes_C B))$. If $X \notin {}^\perp(\tau_B(U \otimes_C B))$, then Lemma 1.5 implies $\mathrm{Hom}_B(X, \tau_B(U \otimes_C B)) \cong \mathrm{Hom}_C(X_C, \tau_C U) \neq 0$. Since $\tau_C U \in \mathrm{Cogen}(\tau_C M)$, we would have $\mathrm{Hom}_C(X_C, \tau_C M) \neq 0$. Since we assumed $X \in {}^\perp(\tau_B M)$ and $\tau_B(M \otimes_C B) \in \mathrm{Cogen}(\tau_B M)$ by [**4**, 1.2], we must have $\mathrm{Hom}_B(X, \tau_B(M \otimes_C B)) = 0$. However, using Lemma 1.5, we see

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$\mathrm{Hom}_B(X, \tau_B(M \otimes_C B)) \cong \mathrm{Hom}_C(X_C, \tau_C M) = 0$, a contradiction. Thus, we must have $X \in {}^{\perp}(\tau_B(U \otimes_C B))$ and conclude by proposition [**1**, 2.9] that $U \otimes_C B \in P^{\perp}(\tau_B M)$. Finally, to show $U \otimes_C B$ comprises all the indecomposable Ext-projective modules in ${}^{\perp}(\tau_B M)$ up to isomorphism not in add$M$, we apply the same reasoning used in the conclusion of Theorem 2.2. □

Our next result shows that $(M \otimes_C B) \oplus (U \otimes_C B)$ and $M \oplus U$ are both τ_B-tilting if and only if they are isomorphic to each other.

PROPOSITION 3.4. *$M \oplus U$ and $(M \otimes_C B) \oplus (U \otimes_C B)$ are both τ_B-tilting if and only if $M \otimes_C E = 0$ and $U \otimes_C E = 0$.*

Proof. Assume $M \oplus U$ and $(M \otimes_C B) \oplus (U \otimes_C B)$ are both τ_B-tilting. Since $M \otimes_B U$ is τ_B-tilting, we know $\mathrm{Ext}^1_B(M \oplus U, \mathrm{Gen}(M \oplus U)) = 0$ by [**5**, 5.8]. Since $(M \otimes_C B) \oplus (U \otimes_C B)$ is τ_B-tilting, we know $\mathrm{Hom}_C((M \otimes_C E) \oplus (U \otimes_C E), \tau_C(M \oplus U)) = 0$ by Theorems 2.1 and 2.2. Thus, $(M \otimes_C E) \oplus (U \otimes_C E) \in \mathrm{Gen}(M \oplus U)$ by [**1**, 2.12]. However, we know $\mathrm{Ext}^1_B(M \oplus U, (M \otimes_C E) \oplus (U \otimes_C E)) \neq 0$ by [**6**, 3.6]. This contradicts the fact that $\mathrm{Ext}^1_B(M \oplus U, \mathrm{Gen}(M \oplus U)) = 0$ unless $M \otimes_C E$ and $U \otimes_C E$ are equal to 0.

Assume $M \otimes_C E$ and $U \otimes_C E$ are equal to 0. [**6**, 3.6] implies $(M \otimes_C B) \oplus (U \otimes_C B) \cong (M \oplus U)$. Also, $\mathrm{Hom}_C((M \otimes_C E) \oplus (U \otimes_C E), \tau_C(M \oplus U)) = 0$ implies $(M \otimes_C B) \oplus (U \otimes_C B)$ is τ_B-tilting by Theorems 2.1 and 2.2 and our statement follows. □

If we don't assume M is τ_C-rigid (τ_C-tilting), our last result shows M being τ_B-rigid (τ_B-tilting) guarantees M being τ_C-rigid (τ_C-tilting).

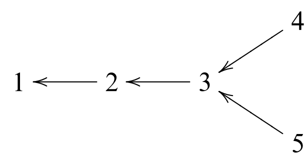
PROPOSITION 3.5. *Suppose M is τ_B-rigid (τ_B-tilting), then M is τ_C-rigid (τ_C-tilting).*

Proof. Since M is τ_B-rigid (τ_B-tilting), $\mathrm{Hom}_B(M, \tau_B M) = 0$. Since $\tau_C(M \otimes_C B)$ is a submodule of $\tau_B M$ by [**4**, 1.2], we must have $\mathrm{Hom}_B(M, \tau_B(M \otimes_C B)) = 0$. Using Lemma 1.5 and the fact M is also a C-module, we have

$$\mathrm{Hom}_B(M, \tau_B(M \otimes_C B)) \cong \mathrm{Hom}_C(M \otimes_B B_C, \tau_C M) \cong \mathrm{Hom}_C(M, \tau_C M).$$

Thus, we have $\mathrm{Hom}_C(M, \tau_C M) = 0$ and conclude M is τ_C-rigid (τ_C-tilting). □

4. Examples. In this section, we give two examples illustrating our results. We will construct a cluster-tilted algebra from a tilted algebra. Such a construction is an example of a split extension. Let A be the path algebra of the following quiver:



Since A is a hereditary algebra, we may construct a tilted algebra. To do this, we need an A-module which is tilting. Consider the Auslander–Reiten quiver of A which

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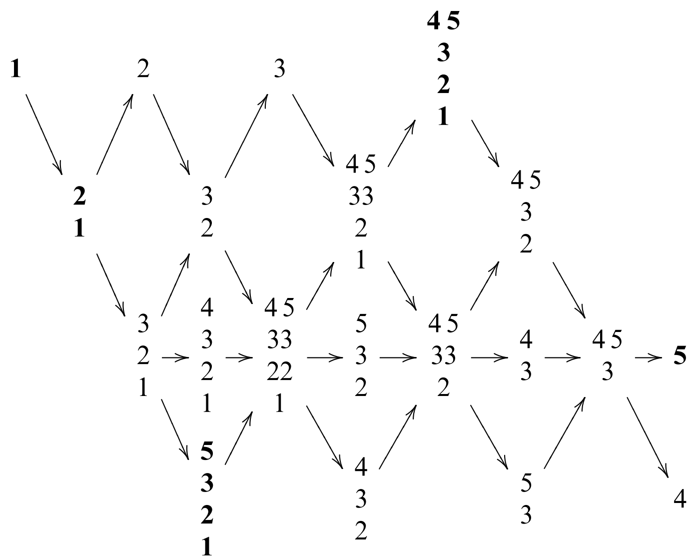
is given by



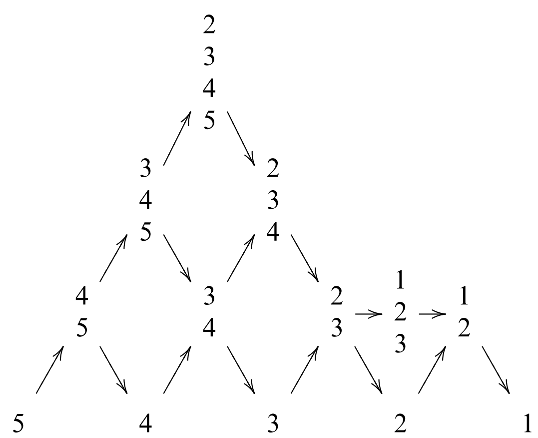
Let T be the tilting A-module

$$T = 5 \oplus \begin{matrix}4\,5\\3\\2\\1\end{matrix} \oplus \begin{matrix}5\\3\\2\\1\end{matrix} \oplus \begin{matrix}2\\1\end{matrix} \oplus 1.$$

The corresponding titled algebra $C = \mathrm{End}_A T$ is given by the bound quiver

$$1 \xrightarrow{\alpha} 2 \xrightarrow{\beta} 3 \xrightarrow{\gamma} 4 \longrightarrow 5 \qquad \alpha\beta\gamma = 0.$$

Then, the Auslander–Reiten quiver of C is given by



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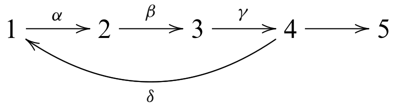
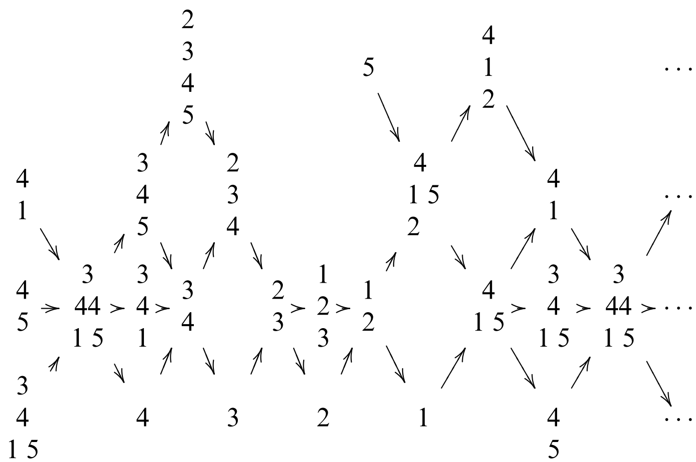
The corresponding cluster-tilted algebra $B = C \ltimes \mathrm{Ext}^2_C(DC, C)$ is given by the bound quiver



$$\alpha\beta\gamma = \beta\gamma\delta = \gamma\delta\alpha = \delta\alpha\beta = 0.$$

Then, the Auslander–Retien quiver of B is given by



EXAMPLE 4.1. In $\operatorname{mod} C$, consider $M = \begin{smallmatrix}2\\3\\4\\5\end{smallmatrix} \oplus \begin{smallmatrix}2\\3\\4\end{smallmatrix} \oplus 3$. M is a τ_C-rigid module with Bongartz τ_C-complement $U = \begin{smallmatrix}1\\2\\3\end{smallmatrix} \oplus \begin{smallmatrix}3\\4\end{smallmatrix}$. In this case, we have $M \otimes_C B \cong M$ that implies $M \otimes_C E = 0$. Thus, $M \otimes_C B \cong M$ is τ_B-rigid and the induced module of U, $U \otimes_C B = \begin{smallmatrix}1\\2\\3\end{smallmatrix} \oplus \begin{smallmatrix}3\\4\\1\end{smallmatrix}$, is the Bongartz τ_B-complement. Notice, we have $\tau_C M = \begin{smallmatrix}3\\4\\5\end{smallmatrix} \oplus 4$, $U \otimes_C E = 1$, and $\mathrm{Hom}_C(U \otimes_C E, \tau_C M) = 0$, in accordance with Theorem 2.2.

EXAMPLE 4.2. In $\operatorname{mod} C$, consider $M = \begin{smallmatrix}3\\4\\5\end{smallmatrix}$. M is projective with Bongartz τ_C-complement $U = 5 \oplus \begin{smallmatrix}4\\5\end{smallmatrix} \oplus \begin{smallmatrix}2\\3\\4\\5\end{smallmatrix} \oplus \begin{smallmatrix}1\\2\\3\end{smallmatrix}$. We have $M \otimes_C E = 1$ and it is clear to see $\mathrm{Hom}_C(M \otimes_E C, \mathrm{Gen} M) = 0$. Thus, M is τ_B-rigid by proposition 3.1 with

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$\tau_B M = \begin{matrix}4\\1\end{matrix}$. Since $M \otimes_C B = \begin{matrix}3\\4\\1\ 5\end{matrix}$, Proposition 3.2 says $M \otimes_C B \in P({}^{\perp}(\tau_B M))$ because $\mathrm{Hom}_C(M, (\tau_B M)_C) = \mathrm{Hom}_C(M, 4 \oplus 1) = 0$.

We have $U \otimes_C B = 5 \oplus \begin{matrix}4\\1\ 5\\2\end{matrix} \oplus \begin{matrix}2\\3\\4\\5\end{matrix} \oplus \begin{matrix}1\\2\\3\end{matrix}$. Here, not every summand of $U \otimes_C B$ is a summand of the Bongartz τ_B-complement because $\mathrm{Hom}_B\left(\begin{matrix}1\\2\\3\end{matrix} \oplus \begin{matrix}4\\1\ 5\\2\end{matrix}, \begin{matrix}4\\1\end{matrix}\right) \neq 0$. Notice, $(\tau_B M)_C = 4 \oplus 1$ and $\mathrm{Hom}_C\left(\begin{matrix}1\\2\\3\end{matrix}\oplus, 4 \oplus 1\right) \neq 0$ in accordance with Theorem 3.3. However, Theorem 3.3 guarantees $5 \oplus \begin{matrix}2\\3\\4\\5\end{matrix}$ are summands of the Bongartz τ_B-complement since $\mathrm{Hom}_C\left(5 \oplus \begin{matrix}2\\3\\4\\5\end{matrix}, 4 \oplus 1\right) = 0$.

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