Can. J. Math., Vol. XXIII, No. 4, 1971, pp. 653–658

AN INCLUSION THEOREM FOR BOHR-HARDY SUMMABILITY FACTORS

B. THORPE

1. Let A denote a sequence to sequence transformation given by the normal matrix $A = (a_{nk})(n, k = 0, 1, 2, \ldots)$, i.e., a lower triangular matrix with $a_{nn} \neq 0$ for all n. For $B = (b_{nk})$ we write $B \Rightarrow A$ if every B limitable sequence is A limitable to the same limit, and say that B is equivalent to A if $B \Rightarrow A$ and $A \Rightarrow B$. If B is normal, then it is well known that the inverse of B exists (we denote it by B^{-1}) and that $B \Rightarrow A$ if and only if $F = AB^{-1}$ is a regular transformation, i.e., transforms every convergent sequence into a sequence converging to the same limit. We say that a series $\sum a_n$† is summable A if its sequence of partial sums is A-limitable. A sequence $\{\epsilon_n\}$ is a Bohr-Hardy summability factor for A, written $\epsilon_n \in (A; A)$, if, for every series $\sum a_n$ summable A, $\sum a_n \epsilon_n$ is summable A (see [**1**; **4**] in which Hardy and Bohr independently obtained sufficient conditions for $\epsilon_n \in ((C, k); (C, k))$, k a positive integer. For non-integral k and necessity of the conditions, see [**2**] and the other references given there). Jurkat and Peyerimhoff obtained results of a more general character, corresponding to the range $0 \leqq k \leqq 1$, by using normal matrices satisfying a mean value condition.

Definition [**9**]. A normal matrix $A = (a_{nk})$ satisfies the mean value condition $M_K(A)$ if

$$\left|\sum_{k=0}^{m} a_{nk}s_k\right| \leqq K \sup_{\mu \leqq m} \left|\sum_{k=0}^{\mu} a_{\mu k}s_k\right|, \tag{1}$$

for $m \leqq n$ and K independent of m, n and $\{s_n\}$.

The following result has been proved by Jurkat and Peyerimhoff (see [**6**] where also earlier references are given).

THEOREM 1. *Suppose that $A = (a_{nk})$ is a normal, regular sequence to sequence matrix. A necessary condition for $\epsilon_n \in (A; A)$ is that*

$$\epsilon_n = \alpha + \sum_{k=n}^{\infty} \alpha_k \bar{a}_{kn}, \tag{2}$$

where $\sum |\alpha_k| < \infty$, α is a constant, and

$$\bar{a}_{kn} = \sum_{\nu=n}^{k} a_{k\nu}.$$

†$\sum$ without limits denotes $\sum_0^\infty$.

Received October 14, 1970 and in revised form, April 23, 1971.

If A satisfies the mean value condition $M_K(A)$, then (2) is also sufficient for $\epsilon_n \in (A; A)$.

It is clear that if (2) holds then

$$\epsilon_n - \epsilon_{n+1} = \Delta\epsilon_n = \sum_{k=n}^{\infty} \alpha_k a_{kn}. \tag{3}$$

Conversely, if $\sum \alpha_k$ is an arbitrary absolutely convergent series and (3) holds, then if A satisfies the conditions of Theorem 1, (2) must hold with $\alpha = \lim_{n\to\infty} \epsilon_n$. For, if (3) holds and the series in (2) converges, it follows that (2) holds with some α. Since A is regular, $\bar{a}_{kn}$ is bounded so (2) converges and because $\sum_{k=n}^{\infty} |\alpha_k|$ tends to 0 as $n \to \infty$, it is clear that $\alpha = \lim_{n\to\infty} \epsilon_n$.

The relationship between (2) and (3) was investigated in [**7**; **8**], and we shall apply Theorem 1 with (2) replaced by (3).

For the proof of Theorem 4 we shall need the following result of Hahn [**3**].

THEOREM 2. *In order that the series to sequence transformation,*

$$w_n = \sum_{k=0}^{\infty} \gamma_{nk} a_k,$$

should transform every absolutely convergent series into a convergent sequence it is necessary and sufficient that

(i) *for each fixed k, $\gamma_{nk} \to \gamma_k$ say, as $n \to \infty$, and*

(ii) $|\gamma_{n,k}| < M$ *all n, k.*

If these conditions are satisfied, then $w_n \to \sum \gamma_k a_k$ as $n \to \infty$.

2. The object of this note is to obtain necessary and sufficient conditions on the matrices A, B such that every Bohr-Hardy factor for A is a Bohr-Hardy factor for B.

The proof of this result makes use of an observation made in [**10**] which we state as

THEOREM 3. *Suppose that $F = (f_{nk})$ is an arbitrary matrix transformation such that*

$$\sum_{k=0}^{\infty} f_{nk} = 1 \quad \text{for all } n. \tag{4}$$

(i) *If $F = (f_{nk})$ is regular as a sequence to sequence transformation, then the transpose matrix $F' = (f_{nk}') = (f_{kn})$ is absolutely regular as a series to series transformation, i.e., takes absolutely convergent series into absolutely convergent series preserving their sums.*

(ii) *Conversely, if F' is absolutely regular as a series to series transformation, F is regular as a sequence to sequence transformation if and only if*

$$f_{nk} \to 0 \quad \text{as } n \to \infty \text{ for each fixed } k.$$

This is used with $F = AB^{-1}$ in the "only if" part of Theorem 4, where it is shown that, under the hypotheses on A and B, $f_{nk} \to 0$ as $n \to \infty$ and so Theorem 3(ii) applies. It is worth pointing out that if (4) holds, F' is absolutely regular if and only if F' is absolutely conservative (i.e., preserves absolute convergence) and a necessary and sufficient condition for this is

$$\sum_{n=0}^{\infty} |f'_{nk}| = \sum_{n=0}^{\infty} |f_{kn}| = O(1), \tag{5}$$

by Knopp and Lorentz's Theorem [**5**].

LEMMA. *Suppose that $A = (a_{nk})$ satisfies the conditions of Theorem 1 (including the mean value condition) and that (3) holds for some $\{\alpha_n\}$ with $\sum |\alpha_n| < \infty$. Then*

$$\alpha_n = \sum_{\nu=n}^{\infty} a_{\nu n}^{-1} \Delta\epsilon_\nu, \tag{6}$$

where $A^{-1} = (a_{nk}^{-1})$ and A is perfect.†

Proof. Take a fixed $N > n$. Then

$$\begin{aligned} \sum_{\nu=n}^{N} a_{\nu n}^{-1} \Delta\epsilon_\nu &= \sum_{\nu=n}^{N} a_{\nu n}^{-1} \sum_{k=\nu}^{\infty} \alpha_k a_{k\nu} \\ &= \sum_{k=n}^{N} \alpha_k \sum_{\nu=n}^{k} a_{k\nu} a_{\nu n}^{-1} + \sum_{k=N+1}^{\infty} \alpha_k \sum_{\nu=n}^{N} a_{k\nu} a_{\nu n}^{-1} \\ &= \alpha_n + \sum_{k=N+1}^{\infty} \alpha_k \sum_{\nu=n}^{N} a_{k\nu} a_{\nu n}^{-1}. \end{aligned} \tag{7}$$

Since A satisfies $M_K(A)$,

$$\left| \sum_{\nu=n}^{N} a_{k\nu} a_{\nu n}^{-1} \right| \leqq K \max_{n \leqq \mu < N} \left| \sum_{\nu=n}^{\mu} a_{\mu\nu} a_{\nu n}^{-1} \right| = K,$$

and hence the second term in (7) is $o(1)$ as $N \to \infty$, because $\sum \alpha_k$ is absolutely convergent, and so (6) must hold.

In particular if $\Delta\epsilon_n = 0$ for all n in (3), where $\sum |\alpha_k| < \infty$, then (6) shows that $\alpha_n = 0$ for all n, and so A is perfect. (This case is just [**9**, Lemma II.4.)

THEOREM 4. *Let A, B be normal, regular sequence to sequence matrices satisfying the mean value conditions $M_K(A)$ and $M_L(B)$ respectively. Then*

$$\epsilon_n \in (A; A) \Rightarrow \epsilon_n \in (B; B)$$

for every $\epsilon_n \in (A; A)$ if and only if $B \Rightarrow A$.

Proof. If we suppose that $B \Rightarrow A$, then it follows from the results of Jurkat and Peyerimhoff that if (2) holds, then $\exists \{\beta_k\}$ with $\sum |\beta_k| < \infty$ such that

$$\epsilon_n = \alpha + \sum_{k=n}^{\infty} \beta_k \bar{b}_{kn},$$

†A regular, normal matrix $A = (a_{n\nu})$ is called perfect if $\sum_{n=\nu}^{\infty} \alpha_n a_{n\nu} = 0$ for $\nu = 0, 1, 2, \ldots$, and $\sum |\alpha_n| < \infty$ implies that $\alpha_n = 0$ for all n (see [**9**]).

where $\bar{b}_{kn} = \sum_{\nu=n}^{k} b_{k\nu}$. (See, for example, [**8**, (14)].) But now using Theorem 1 (the necessity part for A and the sufficiency part for B) we see that $\epsilon_n \in (A;A) \Rightarrow \epsilon_n \in (B;B)$ and so this half of the result is proved. Although Jurkat and Peyerimhoff did not state this result explicitly in [**8**], it is implicit in their paper.

Any regular normal matrix A is equivalent to a regular normal matrix A^* with row sums equal to 1; for only a finite number of rows of A can have row sums equal to 0, so we can replace these by rows with row sums equal to 1 and diagonal terms $\neq 0$ without altering the summability properties of A, and in the remaining rows define

$$a^*_{nk} = \frac{a_{nk}}{\sum_{\mu=0}^{n} a_{n\mu}}.$$

It follows from [**9**, Theorem II.18] that if we replace A by such a regular normal matrix then this matrix will also satisfy a mean value condition. Also, the class of $\epsilon_n \in (A;A)$ is not altered by this change (although the $\{\alpha_n\}$'s in (3) may change). Thus, we can assume throughout the rest of the proof that both A and B have row sums 1. Because A and B are normal, $B \Rightarrow A$ if and only if $F = AB^{-1}$ is regular where $B^{-1} = (b_{nk}^{-1})$, $BB^{-1} = B^{-1}B = I$, $F = (f_{n\mu})$. Hence,

$$f_{n\mu} = \sum_{k=\mu}^{n} a_{nk} b_{k\mu}^{-1} \quad \text{and} \quad \sum_{k=\mu}^{n} b_{nk} b_{k\mu}^{-1} = \delta_{n\mu},$$

where $\delta_{n\mu}$ is the Kronecker delta. Since A, B have row sums equal to 1, it follows that for all n, (4) must hold.

Now suppose that $\epsilon_n \in (A;A) \Rightarrow \epsilon_n \in (B;B)$. By Theorem 1, and the remarks after it, if $\epsilon_n \in (A;A)$, (3) holds and by the Lemma, (6) holds. Analogous equations to (3) and (6) hold for B, namely, $\epsilon_n \in (B;B)$ if and only if $\Delta\epsilon_n = \sum_{k=n}^{\infty} \beta_k b_{kn}$ where $\sum |\beta_k| < \infty$. If this holds then

$$\beta_n = \sum_{\nu=n}^{\infty} b_{\nu n}^{-1} \Delta\epsilon_\nu.$$

Replacing $\Delta\epsilon_\nu$ by (3) in this equation, we see that

$$\beta_n = \sum_{\nu=n}^{\infty} b_{\nu n}^{-1}\left(\sum_{k=\nu}^{\infty} \alpha_k a_{k\nu}\right). \tag{8}$$

Since $\sum \alpha_k$ is an arbitrary absolutely convergent series, for

$$\epsilon_n \in (A;A) \Rightarrow \epsilon_n \in (B;B)$$

it is necessary that (8) should transform every absolutely convergent series into an absolutely convergent series, i.e., (8) has to be absolutely conservative as an iterated series to series transformation. In fact, (8) is absolutely conservative if and only if

$$\beta_n = \sum_{k=n}^{\infty} f_{kn} \alpha_k \tag{9}$$

is an absolutely conservative series to series matrix transformation. To prove this, take the outer sum in (8) from $\nu = n$ to N say, so

$$\sum_{\nu=n}^{N} b_{\nu n}^{-1}\left(\sum_{k=\nu}^{\infty} \alpha_k a_{k\nu}\right) = \sum_{k=n}^{N} \alpha_k f_{kn} + \sum_{k=N+1}^{\infty} \alpha_k \sum_{\nu=n}^{N} a_{k\nu} b_{\nu n}^{-1}, \tag{10}$$

and since A satisfies $M_K(A)$,

$$\left|\sum_{\nu=n}^{N} a_{k\nu} b_{\nu n}^{-1}\right| \leqq K \max_{n \leqq \mu \leqq N} \left|\sum_{\nu=n}^{\mu} a_{\mu\nu} b_{\nu n}^{-1}\right|$$

$$= K \max_{n \leqq \mu < N} |f_{\mu n}|.$$

If we assume that (9) is absolutely conservative, it follows from Theorem 2 that f_{kn} is bounded, so the second term on the right in (10) is $o(1)$ as $N \to \infty$. Hence (8) and (9) are identical in this case and thus (8) is absolutely conservative.

Conversely, if (8) is absolutely conservative as an iterated transformation, β_n (in (8)) must exist for every fixed n, so (8) can be written as

$$\beta_n = \lim_{N\to\infty} \sum_{k=n}^{\infty} \gamma_{N,k} \alpha_k, \tag{11}$$

where

$$\gamma_{N,k} = f_{nk} \qquad (k \leqq N)$$

$$= \sum_{\nu=n}^{N} a_{k\nu} b_{\nu n}^{-1} \quad (k > N).$$

Thus we require that (11) should transform every absolutely convergent series to a convergent sequence. By Theorem 2, since

$$\gamma_{Nk} \to f_{kn} \text{ as } N \to \infty,\ k \text{ fixed},$$

(11) becomes

$$\beta_n = \sum_{k=0}^{\infty} \gamma_k \alpha_k = \sum_{k=n}^{\infty} f_{kn} \alpha_k,$$

i.e., (11) becomes (9), so (9) is absolutely conservative if and only if (8) is absolutely conservative as claimed.

Using this, $\epsilon_n \in (A; A) \Rightarrow \epsilon_n \in (B; B)$ implies that (5) holds and so $F' = (f_{nk}') = (f_{kn})$ is an absolutely regular series to series transformation.

By Theorem 3(ii), we have only to prove that $f_{nk} \to 0$ as $n \to \infty$ for fixed k to deduce that $F = (f_{nk})$ is a regular sequence to sequence transformation.

Since (5) holds, F takes bounded sequences (and hence null sequences) to bounded sequences and because A, B and F are row finite,

$$Ax = (FB)x = F(Bx), \tag{12}$$

for any sequence x. Thus $Bx = o(1) \Rightarrow Ax = O(1)$. By the Lemma, B is perfect, so we can now appeal to [**9**, Theorem II.8] to conclude that

$Bx = o(1) \Rightarrow Ax = o(1)$. Since B is normal, it follows from (12) (with the aid of (5)) that F is regular for null sequences and hence $f_{nk} \to 0$ as $n \to \infty$.

Thus $\epsilon_n \in (A;A) \Rightarrow \epsilon_n \in (B;B)$ implies that F is regular (sequence to sequence) and hence $B \Rightarrow A$.

Acknowledgement. In conclusion, I should like to thank Professor B. Kuttner for his help in writing this paper and Professor L. S. Bosanquet for his comments.

References

1. H. Bohr, *Sur la série de Dirichlet*, Comptes Rendus *148* (1909) 75–80.
2. L. S. Bosanquet, *Note on the Bohr-Hardy Theorem*, J. London Math. Soc. *17* (1942) 166–173.
3. H. Hahn, *Über Folgen linearer Operationen*, Monatsch. für Math. und Phys. *32* (1922) 3–88.
4. G. H. Hardy, *Generalisation of a theorem in the theory of divergent series*, Proc. London Math. Soc. (2) *6* (1908) 255–264.
5. K. Knopp and G. G. Lorentz, *Beträge zur absoluten Limitierung*, Arch. Math. *2* (1949) 10–16.
6. W. Jurkat and A. Peyerimhoff, *Mittelwertsätze bei Matrix und Integraltransformationen*, Math. Z. *55* (1951) 92–108.
7. ——— *Summierbarkeitsfaktoren*, Math. Z. *58* (1953) 186–203.
8. ——— *Über Sätze vom Bohr-Hardyschen Typ*, Tôhoku Math. J. (2) *17* (1965), 55–71.
9. A. Peyerimhoff, *Lectures on Summability*, Lecture Notes in Mathematics No. 107 (Springer-Verlag, Berlin, 1969).
10. P. Vermes, *The transpose of a summability matrix*, Colloque sur la Théorie des Suites, Centre Belge de recherches Math. (1958), 60–86.

University of Birmingham,
Birmingham, England;
University of Western Ontario,
London, Ontario