Note on the Integral Equations for the Lamé Functions.

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§ 1. The Lamé Functions of degree n (where n is a positive integer) may be defined as those solutions of the equation

$$\frac{d^2 u}{dx^2} + \{a - n(n+1) k^2 \operatorname{sn}^2 x\} u = 0$$

which are polynomials in the elliptic functions $\operatorname{sn} x$, $\operatorname{cn} x$, $\operatorname{dn} x$ of real modulus k. Such solutions only exist for certain particular values of the constant a; there are $2n+1$ such values and $2n+1$ corresponding Lamé functions.

Consider now the integral equation

$$u(x) = \lambda \int_{-2K}^{2K} N(x, s)\, u(s)\, ds$$

where $4K$ is the real period of the elliptic function $\operatorname{sn} x$ and where $N(x, s)$ is a polynomial in the elliptic functions of argument x and of argument s, and is a solution of the equation

$$\frac{\partial^2 N}{\partial x^2} - \frac{\partial^2 N}{\partial s^2} - n(n+1) k^2 (\operatorname{sn}^2 x - \operatorname{sn}^2 s) N = 0.$$

Professor Whittaker[1] has shown that the *Eigenfunktionen* of such an integral equation are either the whole set or some subclass of the $2n+1$ Lamé functions of degree n. He has given various particular forms which the nucleus may have. In this note, the general form of the nucleus is discussed and also the connection between the various particular forms.

§ 2. The equation of Laplace in three dimensions

$$\frac{\partial^2 V}{\partial X^2} + \frac{\partial^2 V}{\partial Y^2} + \frac{\partial^2 V}{\partial Z^2} = 0$$

[1] *Proc. Lond. Math. Soc.* (2) **14** (1915) 260.
Proc. R. S. Edin. **35** (1914–15) 70.
See also Whittaker and Watson, *Modern Analysis* (3rd Edition, 1920), Ch. XXIII.

has the form

$$\frac{\partial^2 V}{\partial x^2} - \frac{\partial^2 V}{\partial s^2} - k^2(\text{sn}^2 x - \text{sn}^2 s)\frac{\partial}{\partial r}\left(r^2 \frac{\partial V}{\partial r}\right) = 0$$

when expressed in terms of the curvilinear coordinates r, x, s given by

$$\begin{aligned} X &= kr \,\text{sn}\, x \,\text{sn}\, s \\ Y &= r \,\text{dn}\, x \,\text{dn}\, s/k' \\ Z &= ikr \,\text{cn}\, x \,\text{cn}\, s/k', \end{aligned}$$

where $k' = \sqrt{1-k^2}$. We obviously have

$$X^2 + Y^2 + Z^2 = r^2.$$

It follows that, if $N(x, s)$ is the required nucleus, then $r^n N(x, s)$ is a rational integral solid harmonic, and thence that $N(x, s)$ is a rational integral surface harmonic $S_n(\theta, \phi)$[1] where

$$\begin{aligned} \cos\theta &= k \,\text{sn}\, x \,\text{sn}\, s \\ \sin\theta\cos\phi &= \text{dn}\, x \,\text{dn}\, s/k' \\ \sin\theta\sin\phi &= ik \,\text{cn}\, x \,\text{cn}\, s/k'. \end{aligned}$$

From this result, all the known integral equations can be obtained.

A particular surface harmonic is given by

$$S_n(\theta, \phi) = P_n[\cos\theta\cos\theta_0 + \sin\theta\sin\theta_0\cos(\phi - \phi_0)]$$

where θ_0, ϕ_0 are arbitrary constants. From this, we derive the nucleus

$$P_n[k^2 \,\text{sn}\, x \,\text{sn}\, s \,\text{sn}\, x_0 \,\text{sn}\, s_0 + \frac{1}{k'^2}\text{dn}\, x \,\text{dn}\, s \,\text{dn}\, x_0 \,\text{dn}\, s_0 - \frac{k^2}{k'^2}\text{cn}\, x \,\text{cn}\, s \,\text{cn}\, x_0 \,\text{cn}\, s_0]$$

where x_0, s_0, are arbitrary constants; this nucleus will give us all the $(2n+1)$ Lamé functions of degree n, whereas those obtained by assigning particular values to x_0 and s_0, in general, do not. By writing $x_0 = K$, $s_0 = K + iK'$, or $x_0 = 0$, $s_0 = K + iK'$, or $x_0 = 0$, $s_0 = K$ respectively, we obtain the three following nuclei, due to Professor Whittaker:—

$$P_n(k \,\text{sn}\, x \,\text{sn}\, s)$$

$$P_n\left(\frac{ik}{k'}\text{cn}\, x \,\text{cn}\, s\right)$$

$$P_n\left(\frac{1}{k'}\text{dn}\, x \,\text{dn}\, s\right).$$

Professor Whittaker's fourth nucleus is

$$(\text{dn}\, x \,\text{dn}\, s + k\cosh\eta \,\text{cn}\, x \,\text{cn}\, s + kk'\sinh\eta \,\text{sn}\, x \,\text{sn}\, s)^n;$$

[1] *Cf.* Heine, *Theorie der Kugelfunctionen*, (1878), 355.

this is a constant multiple of the nucleus

$$\lim_{\epsilon \to 0} \epsilon^n P_n[k^2 \operatorname{sn} x \operatorname{sn} s \operatorname{sn} x_0 \operatorname{sn}(iK' + \epsilon) + \ldots]$$

where $\cosh \eta = \operatorname{cd} x_0$.

Lastly, from the surface harmonics

$$P_n''(\cos\theta) \sin\phi \cos\phi \sin^2\theta$$
$$P_n''(\sin\theta \sin\phi) \sin\theta \cos\phi \cos\theta$$
$$P_n''(\sin\theta \cos\phi) \sin\theta \sin\phi \cos\theta$$

where $P_n''(t) = d^2 P_n(t)/dt^2$, we obtain the three forms of nucleus

$$P_n''(k \operatorname{sn} x \operatorname{sn} s) \operatorname{cn} x \operatorname{dn} x \operatorname{cn} s \operatorname{dn} s$$
$$P_n''(ik \operatorname{cn} x \operatorname{cn} s/k') \operatorname{sn} x \operatorname{dn} x \operatorname{sn} s \operatorname{dn} s$$
$$P_n''(\operatorname{dn} x \operatorname{dn} s/k') \operatorname{sn} x \operatorname{cn} x \operatorname{sn} s \operatorname{cn} s$$

given by Whittaker and Watson (*loc. cit.* § 23. 61).

We see then that all the known forms of nucleus for the Lamé functions are really particular cases of the nucleus $S_n(\theta, \phi)$ given at the beginning of this section.

In a similar way, Poole[1] and S. C. Dhar[2] have obtained, from the solutions of the equation of wave motions in two dimensions, the various forms of the nucleus of the homogeneous integral equation satisfied by the Mathieu functions.

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[1] *Proc. Lond. Math. Soc.* (2) **20** (1921), 374.

[2] *Journ. of Dept. of Sc., Calcutta University* 111 (1922). (Unfortunately I have been unable to verify this reference, as the journal is not in any of the Edinburgh libraries.)