A NOTE ON THE RITZ METHOD WITH AN APPLICATION TO OVERTONE STELLAR PULSATION THEORY

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1. Introduction

The Ritz method reduces eigenvalue problems involving linear operators on infinite dimensional spaces to finite matrix eigenvalue problems. This paper shows that for a certain class of linear operators it is possible to choose the coordinate functions so that numerical solution of the matrix equations is considerably simplified, especially when the matrices are large. The method is applied to the problem of overtone pulsations of stars.

This paper is mainly concerned with the special case of the eigenvalue problem

$$Au = \lambda Bu \tag{1}$$

with appropriate boundary conditions in which A and B have the form given by (3).

More generally if A and B are self-adjoint linear operators on some Hilbert space of functions, and u must satisfy given boundary conditions, numerical solutions of (1) may, if certain conditions are satisfied, be found very simply by applying the Ritz method to the corresponding variational problem. This gives rise to a matrix equation

$$\boldsymbol{P}\boldsymbol{x} = \Lambda \boldsymbol{Q}\boldsymbol{x} \tag{2}$$

where, if the space of allowable functions is reduced to one of dimension n, the matrices $\boldsymbol{P}$ and $\boldsymbol{Q}$ are $n \times n$. In this case denote the k^{th} eigenvalue of (2) by $\Lambda_k(n)$. Then, for all k, $\Lambda_k(n)$ provides an upper bound for the k^{th} eigenvalue, λ_k, of (1) and, provided the appropriate conditions are satisfied, $\lim_{n\to\infty} \Lambda_k(n) = \lambda_k$.

It is well known that the rate of convergence is generally very good for the first one or two eigenvalues, especially if the coordinate functions are chosen carefully. For larger k, much larger n must be used to obtain comparable accuracy. For very large n, (2) is more difficult to solve and, if $\boldsymbol{P}$ and $\boldsymbol{Q}$ had quite general form, considerable computer storage would

be required. Then one of the main advantages of the variational method, its speed even with limited computing facilities, would be lost.

It is shown here that if, when the Ritz method is applied to (4), the space of allowable functions is taken to be those continuous piecewise linear functions satisfying the required boundary conditions, then $\boldsymbol{P}$ and $\boldsymbol{Q}$ in (2) are symmetric and tri-diagonal and if the operator B_1 is positive then for most boundary conditions the matrix $\boldsymbol{Q}$ is positive definite. This result has been discovered independently for a slightly more restricted class of operators by Wendroff [1]. An extremely efficient method [2; p. 340] exists for solving equations of this form numerically. This method for solving (2) is considerably more efficient than that suggested in [1].

In section 4 this method is applied to the equation governing small radial adiabatic oscillations of stars. This equation is important in the theory of stellar stability [3].

The Ritz method has already been applied very successfully to this equation to determine the fundamental mode of oscillation — the mode for which the method converges most rapidly. Ledoux and Pekeris [4] used the Ritz method to determine the fundamental mode of the standard model (a classical stellar model). They also gave approximations to the first two overtones, but these were not accurate. Later Chandrasekhar and Lebovitz [5], in a paper which also deals with non-radial oscillations, used variational methods to determine the fundamental mode for a number of polytropic models.

For the very high modes, Ledoux has shown that excellent approximations for both eigenvalues [6] and eigenfunctions [7] may be obtained by an asymptotic method, provided (as is usually the case) that the outer layers of the star can be characterised by an effective polytropic index. But for modes in between these very high modes and the fundamental mode, the only method which has so far been used successfully is the tedious method of numerical integration of the differential equation. This method is slow and a completely new calculation is required if a slight change is made in the model. Also many methods of numerical integration require an initial guess for the eigenvalue. It would be useful if this could be provided rapidly and accurately by asymptotic or variational methods. Such methods also provide a useful check, and the analysis required for them is often useful in itself. Thus as well as providing an illustration of the methods described in section 3, the accurate determination of overtone pulsations (higher modes of oscillation) of stars by variational means is an important question in its own right.

Only one stellar model was considered in the present calculations but the same methods (and indeed the same computer programme, with only the data input changed) could be used for any other model. The model

used was one constructed by Van der Borght [8] for which molecular weight μ is constant, and $\mu^2 M = 10 M_\odot$ where M is the mass of the star and $M_\odot$ that of the sun. (These conditions are satisfied, for example, by a star of forty solar masses consisting entirely of ionized hydrogen.) In this model, the ratio β of gas pressure to total pressure varies from 0.794 at the centre to 0.904 at the surface.

Van der Borght [9] has used numerical integration to determine the first thirteen modes of oscillation of the model used here. He also modified Ledoux's asymptotic expression for the eigenvalues. Later (private communication) he determined the next seven modes by numerical integration.

Table 1 compares Van der Borght's eigenvalues with those obtained here and Table 2 compares the eigenfunctions. More detailed numerical results of the present calculations are given elsewhere [10].

2. The eigenvalue problem

Let A_1, B_1 be operators on the space of all real scalar-valued functions with domain $[a, b]$, defined by

$$A_1 u = -(f_1 u')' + f_2 u$$
$$B_1 u = -(f_3 u')' + f_4 u \tag{3}$$

where the f_i are given bounded integrable functions and f_1 and f_3 are differentiable. Then $A_1 u$, $B_1 u$ will be defined at those points where u is twice differentiable. Consider the eigenvalue problem

$$A_1 u = \lambda B_1 u \tag{4}$$

where at each boundary one of the following four conditions holds:

(i) Both f_1 and f_3 vanish and it is merely required that u be bounded. (This is the case at both boundaries in the problem considered in section 4.)

(ii) It is required that $u = 0$ at the boundary.

(iii) It is required that $u' = Ku$ at the boundary where K is some given constant, non-negative in the boundary condition at a and non-positive in that at b.

(iv) It is required that $u' = Ku$ at the boundary where K does not satisfy the sign requirement of condition (iii).

The designation "(4)" will be taken to mean the eigenvalue problem $A_1 u = \lambda B_1 u$ with one of the above four conditions imposed at each boundary. Clearly (4) is self-adjoint.

3. Choice of coordinate functions

Let $\{P_n\}$ be a sequence of partitions of the interval $[a, b]$ in which P_1 consists only of the points a and b and, for each n, P_{n+1} is a refinement of

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P_n containing exactly one extra point. Define a sequence $\{\phi_n\}$ of functions on $[a, b]$ as follows

$$
\begin{aligned}
\phi_0(x) &= 1 \\
\phi_1(x) &= (x-a)/(b-a) \\
\phi_n(x) &= \begin{cases} (x-a_i)/(a_n-a_i) & \text{for } a_i \leqq x \leqq a_n \\ (a_{i+1}-x)/(a_{i+1}-a_n) & \text{for } a_n \leqq x \leqq a_{i+1} \\ 0 & \text{for all other } x \text{ on } [a, b] \end{cases} \\
& \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad (n > 1)
\end{aligned} \tag{5}
$$

where, in the definition of ϕ_n, the points of P_{n-1} are labelled (in increasing order) $a_0, a_1, \cdots, a_{n-1}$, the extra point in P_n is a_n, and it is supposed that $a_i < a_n < a_{i+1}$.

The first $n+1$ functions of this sequence form a basis for the space of all continuous functions defined on $[a, b]$ whose derivatives are constant in each of the subintervals of P_n.

In solving (4) by the Ritz method when both boundary conditions are of type (i), $\{\phi_i\}$ defined by (5) may be taken as the sequence of coordinate functions. This was done in the example discussed in section 4. With other boundary conditions some minor modification of (5) is required. (For instance if both boundary conditions are of type (ii), ϕ_0 and ϕ_1 should be omitted from the sequence.)

The space spanned by the first $(n+1)$ members of the sequence defined by (5) is isomorphic with the space of all $(n+1)$tples $(x_0, \cdots, x_n)$ where x_i is the value of u at a_i and where now the points of P_n are labelled (in increasing order) $a_0, a_1, \cdots, a_n$. For any u in this space, and for $a_i < x < a_{i+1}$ where $0 \leqq i \leqq n-1$,

$$u(x) = x_i + (x-a_i)(x_{i+1}-x_i)/(a_{i+1}-a_i). \tag{6}$$

For each n define I_n and C_n on R^{n+1} by

$$
\begin{aligned}
I_n(x_0, \cdots, x_n) &= (A_1 u, u) \\
&= \int_a^b f_1(x)u'^2(x) + f_2(x)u^2(x)dx
\end{aligned} \tag{7a}
$$

$$
\begin{aligned}
C_n(x_0, \cdots, x_n) &= (B_1 u, u) \\
&= \int_a^b f_3(x)u'^2(x) + f_4(x)u^2(x)dx
\end{aligned} \tag{7b}
$$

where u is given by (6). The inner product (u_1, u_2) where u_1 and u_2 belong to the space is defined as usual as $\int_a^b u_1(x)u_2(x)dx$. The Ritz method may be carried out by extremising I_n/C_n for various n.

THEOREM. *If in* (4) *the boundary conditions are of types* (i), (ii), *or*

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(iii) *then when the Ritz method is applied to* (4) *by extremising* I_n/C_n *given by* (7), *the matrices* $\boldsymbol{P}$ *and* $\boldsymbol{Q}$ *in* (2) *are symmetric and tri-diagonal. If the operator* B_1 *is positive then* $\boldsymbol{Q}$ *is positive definite. If* A_1 *is positive then* $\boldsymbol{P}$ *is positive definite.*

PROOF. Consider first the case when both boundary conditions are of type (i). The equations to be solved in the Ritz method are then

$$\frac{\partial}{\partial x_i}(I_n - \Lambda C_n) = 0, \qquad i = 0, \cdots, n. \tag{8}$$

This system is of the form of (2) where $\boldsymbol{P} = (p_{ij})$, $\boldsymbol{Q} = (q_{ij})$ and p_{ij} and q_{ij} are the coefficients of x_{j-1} in $\partial I_n/\partial x_{i-1}$ and $\partial C_n/\partial x_{i-1}$ respectively. Substituting (6) in (7) shows that $I_n(x_0, \cdots, x_n)$ and $C_n(x_0, \cdots, x_n)$ are quadratic expressions in the x_i in which the coefficient of $x_i x_j$ is zero whenever $|i-j| > 1$. Thus $\boldsymbol{P}$ and $\boldsymbol{Q}$ are symmetric and tri-diagonal (that is $p_{ij} = q_{ij} = 0$ whenever $|i-j| > 1$).

If B_1 is positive then $(B_1 u, u) > 0$ for all non-zero u in the domain of B_1. In particular the inequality still holds when u is further restricted to be of the form given by (6). The quadratic form C_n and hence the matrix $\boldsymbol{Q}$ are then positive definite.

When the boundary condition at a is of type (ii) the equations to be solved in the Ritz method are the last n of the system (8). Since in this case $x_0 = 0$, $\boldsymbol{P}$ and $\boldsymbol{Q}$ will be $n \times n$ matrices. They are in fact precisely those obtained before with the first row and column deleted. Thus they are symmetric and tri-diagonal and when B_1 is positive, $\boldsymbol{Q}$ is positive definite.

Denote the point next after a in the partition P_n by $a+h$. Then if the boundary condition at a is of type (iii) it is required that $x_0 = x_1/(hK+1)$ so that x_0 will not occur explicitly in (8). Argument similar to that above shows that $\boldsymbol{P}$ and $\boldsymbol{Q}$ are symmetric tri-diagonal $n \times n$ matrices. Denote the value of $\boldsymbol{Q}$ in this case by $\boldsymbol{Q}_a$. Denote by $\boldsymbol{Q}_b$ the value of $\boldsymbol{Q}$ that would have been obtained had all the f_i been replaced by zero in the interval $[a, a+h)$ but left unchanged in $[a+h, b]$. Clearly $\boldsymbol{Q}_b$ is $n \times n$ and the argument used for boundary conditions of type (i) shows that if B_1 is positive then $\boldsymbol{Q}_b$ is positive definite unless $\int_{a+h}^{b} f_4(x)dx = 0$ in which case $\boldsymbol{Q}_b$ is singular but its principal minors are all positive. Substitution of (6) and (7) in (8) shows that $\boldsymbol{Q}_a$ may be obtained from $\boldsymbol{Q}_b$ by adding to its top left-hand element the non-negative number

$$\frac{2}{Kh^2+h}\int_a^{a+h} [(x-a)f_4(x) + K\big((x-a)^2 f_4(x) + f_3(x)\big)]\,dx.$$

In the case when $\int_{a+h}^{b} f_4(x)dx = 0$ this quantity must be positive for B_1 to be positive. Hence in all cases $\boldsymbol{Q}_a$ is positive definite.

Exactly the same arguments prove the result of the theorem when the boundary condition at b is of type (ii) or (iii).

Similarly it may be shown that if the operator A_1 is positive then $\boldsymbol{P}$ is positive definite whenever the boundary conditions are of types (i), (ii) or (iii).

If all functions in the domains of A_1 and B_1 are continuous then, provided at least one of the boundary conditions is of type (ii), (iii) (with $K \neq 0$) or (iv), a sufficient condition for B_1 to be positive is that $f_3 f_4 \geqq 0$ and $f_3 + f_4 > 0$ almost everywhere on (a, b). If both boundary conditions are of type (i) or type (iii) with $K = 0$, the above conditions are not sufficient, but if the additional condition $\int_a^b f_4(x)dx > 0$ is added then B_1 will be positive. Corresponding conditions for A_1 to be positive are obtained by replacing f_3 and f_4 in the above inequalities by f_1 and f_2 respectively.

If the boundary condition at a is of type (iv) then, provided P_n is chosen so that $Kh \neq -1$, $\boldsymbol{P}$ and $\boldsymbol{Q}$ will be symmetric and tri-diagonal. If B_1 is positive then sufficient conditions for $\boldsymbol{Q}$ to be positive definite in this case are that f_3 vanish and f_4 be positive in some neighbourhood of a (a condition always satisfied with the regular Sturm-Liouville system) and that h be sufficiently small.

If $\boldsymbol{P}$ and $\boldsymbol{Q}$ are symmetric and tri-diagonal and $\boldsymbol{Q}$ is positive definite then the eigenvalues of (2) may be found very rapidly by Wilkinson's [2; p. 340] simple generalisation of Givens' well-known method [2; p. 300] for the case where $\boldsymbol{Q}$ is the identity matrix. The method requires very little storage and, if $\boldsymbol{P}$ and $\boldsymbol{Q}$ are $n \times n$, the time required to find each eigenvalue increases linearly with n, so that very large n may be handled. Unless $\boldsymbol{Q}$ is very ill-conditioned the method is very accurate. If $\boldsymbol{P}$ is positive definite but not $\boldsymbol{Q}$ the eigenvalues Λ^{-1} may be obtained by the same method. A refinement of the method which gives an even better guarantee of accuracy for large n is given in [11]. Eigenvectors of (2), the sign-changes of which clearly locate the zeros of the approximate eigenfunctions u, may be found very accurately and simply using Wilkinson's recurrence relation $(\boldsymbol{P} - \Lambda \boldsymbol{Q})\boldsymbol{x}_r = \boldsymbol{Q}\boldsymbol{x}_{r-1}$.

The elements of $\boldsymbol{P}$ and $\boldsymbol{Q}$ are all integrals over subintervals of the partition. Each of the terms in the integrands is a quadratic expression multiplied by one of the f_i. Hence if the f_i do not vary greatly over the (small) subintervals of the partition, the integrals may be obtained very accurately by Simpson's rule.

It can be shown that if the maximum distance between a pair of consecutive points in P_n tends to zero as $n \to \infty$ then $\{\phi_i\}$ defined by (5) has the completeness properties required for the Ritz method to give convergence. In that case if u is any continuous real scalar-valued function, defined on $[a, b]$, which is differentiable except perhaps at a finite number

of points and whose derivative is bounded and Riemann integrable, then for all $\varepsilon > 0$ there is a $u_n = \sum_{i=0}^{n} c_i \phi_i$ (where the c_i are constants) such that $(A_1(u-u_n),\ u-u_n) < \varepsilon$. The value of n will generally depend on the fluctuation of u'. Sufficient conditions for convergence of eigenvalues and eigenfunctions are given in Mikhlin's [12] standard work. Further, Wendroff [13] has shown that with this choice of coordinate functions it is often possible to obtain actual lower bounds to λ_k from $\Lambda_k(n)$.

In the example considered in section 3, the higher eigenfunctions are known to have violently fluctuating first derivative. It might be expected that the method would give relatively slow convergence in that case. Yet useful results were obtained for quite high modes.

The use of continuous piecewise linear coordinate functions in the Ritz method can be generalised to many other cases of (1). For instance if (Au, u) and (Bu, u) have the form

$$\int_a^b \sum_{i=1}^{m} \sum_{j=1}^{m} (f_{ij1}(t)v_i(t)v_j(t) + f_{ij2}(t)v_i(t)v_j'(t) + f_{ij3}(t)v_i'(t)v_j'(t))\,dt,$$

the coordinate functions may be taken as $\{\phi_{ij}\}$ where ϕ_{ij} is the m-vector-valued function whose j^{th} component is ϕ_i defined by (5). Procedure similar to that described above leads to an equation of the form (2) where $\boldsymbol{P}$ and $\boldsymbol{Q}$ are symmetric and for suitable boundary conditions $\boldsymbol{Q}$ is positive definite if B is positive. This time $\boldsymbol{P}$ and $\boldsymbol{Q}$ are not tri-diagonal but of the form $(\boldsymbol{P}_{ij})$ where each of the $\boldsymbol{P}_{ij}$ $(i, j = 1, \cdots m)$ is itself a tri-diagonal matrix. The equation may no longer be solved by Wilkinson's simple method, but it is possible to utilize the large number of zero elements in the matrix to speed calculation and reduce storage requirements. A method is described elsewhere [10] for triangulating such matrices without introducing any extra non-zero elements. In this method m non-zero elements in the same row are eliminated simultaneously by subtracting an appropriate linear combination of m other rows.

Completeness of the coordinate functions and convergence of the method may be established for most generalisations. The special importance of (4) depends on the efficiency of Wilkinson's method for solving (2) in that case.

4. Radial oscillations of stars

For any spherically symmetric stellar model, define functions g_1, g_2, g_4 by

$$g_1(r) = \Gamma_1 p r^4, \quad g_2(r) = -r^3 \frac{d}{dr}\left((3\Gamma_1 - 4)p\right) \tag{9}$$
$$g_4(r) = \rho r^4$$

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where p and ρ are the equilibrium values of pressure and density respectively at a distance r from the centre and the adiabatic exponent Γ_1 is given by

$$\Gamma_1 = \beta+(4-3\beta)^2(\gamma-1)/(\beta+12(\gamma-1)(1-\beta))$$

where γ is the ratio of specific heats.

This section considers oscillations governed by the well-known equation

$$L\xi = \sigma^2 g_4 \xi \tag{10}$$

where the Lagrangian displacement at time t is $r\xi(r)e^{i\sigma t}$ and the operator L is defined by

$$L\xi = -(g_1\xi')'+g_2\xi.$$

Boundary conditions with which (10) must be solved are

$$r\xi(r) = 0 \qquad \text{at } r = 0 \tag{11a}$$

$$\Gamma_1 p(3\xi(r)+r\xi'(r)) = 0 \text{ at } r = R \text{ (the radius of the star).} \tag{11b}$$

Since $p(R) = 0$ in the model considered (and in most models) it is sufficient that ξ and ξ' be bounded. The problem is thus a special case of (4) with boundary conditions of type (i). When Van der Borght's model [7] is used the singularities at the boundaries are simple poles and present no serious difficulty.

Derivation of (10) and a discussion of the effects of relaxing some of the assumptions made in the derivation are given in an excellent review article by Ledoux and Walraven [14].

It can be deduced [10] from the differential equations on which Van der Borght's model is based that the operator L is positive in this case. It is not positive definite as ξ could be taken as zero except in an arbitrarily small neighbourhood of the boundaries. However the operator L^*, defined by $L^*\xi(r) = (\rho r^4)^{-1}L\xi(r)$, is positive definite and the Ritz method is applicable. Since L^* is positive definite, all eigenvalues σ^2 are positive. This rules out exponential instability for oscillations of the type considered here, but does not ensure stability.

In [4] and [5], ξ was assumed to be of the form

$$\xi(r) = \sum_{i=0}^{N} c_i r^{2i}.$$

Ledoux and Pekeris [4] considered the cases $N = 0, 1, 2$, while Chandrasekhar and Lebovitz [5] considered only the case $N = 1$. With that choice of coordinate functions, the matrices obtained for large N are much less tractable than those obtained by the method of section 3, which was used in the present calculations. As large N is necessary for accurate estimates of the higher modes, the method used here would seem more satisfactory for the calculation of these higher modes.

The P_n chosen for the present calculations divided the interval $[0, R]$ into n equal parts. Separate calculations of eigenvalues were made using $n = 10$, $n = 25$ and $n = 50$. Comparison of the results illustrates the rate of convergence. Eigenfunctions were calculated in the case $n = 50$ only. In Table 1 eigenvalues obtained with these three values of n are compared with those obtained by Van der Borght [9] using numerical integration and those given by his asymptotic formula. These asymptotic values appear to be slightly too small. Instead of σ^2, Table 1 shows the dimensionless ω^2 where $MG\omega^2 = R^3\sigma^2$ and G is the universal gravitational constant.

TABLE 1

Comparison of eigenvalues

	Variational method			Numerical integration	Asymptotic formula
k	$n = 50$	$n = 25$	$n = 10$		
1	2.050678	2.0509	2.052	2.050683	3.06
2	3.6522	3.663	3.73	3.6486	4.25
3	4.9939	5.043	5.29	4.9762	5.44
4	6.3084	6.437	6.65	6.2584	6.63
5	7.6239	7.864	7.91	7.5158	7.82
6	8.954	9.28	10.0	8.758	9.01
7	10.305	10.51	12.9	9.990	10.20
8	11.671	11.71	16.7	11.217	11.40
9	13.025	13.28	22.0	12.439	12.59
10	14.289	15.10	29.6	13.657	13.78
11	15.448	17.10	46.3	14.870	14.97
12	16.725	19.2		16.079	16.16
13	18.188	21.6		17.285	17.35
14	19.77	24.1		18.50	18.54
15	21.42	26.8		19.71	19.73
16	23.14	29.8		20.91	20.92
17	24.93	33.2		22.11	22.11
18	26.77	36.8		23.32	23.30
19	28.68	40.9		24.52	24.49
20	30.65	45.6		25.72	25.68

Even if much higher n had been used equation (2) would still have been easily handled. With $n = 50$, an IBM 1620 with a Fortran programme calculated the 20 eigenvalues shown in less than an hour. This computer would have taken far longer had the method of numerical integration of the differential equation been used. On faster modern computers, using the methods described here, calculation of the first few eigenvalues is practical even for $n = O(10^4)$.

The p_{ij} and q_{ij} in (2) were evaluated by Simpson's rule using three points. To test the accuracy of the integrations a separate calculation using five points was made in the case $n = 25$. The consequent difference in the

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eigenvalues was generally a fraction of 1 %. This is not surprising as the g_i vary slowly over most of the interval.

There will be errors in the numerical processes but consideration of the form of the eigenfunctions obtained by the variational method, and of the rapid decrease in accuracy as the mode increases or n decreases, suggests that in this case, except probably with the fundamental, the value of n, and not rounding errors, was the main factor limiting accuracy.

The eigenfunctions obtained by Van der Borght [9] and others have two important properties. (a) The zeros of ξ are not evenly distributed but become much denser near $r = R$. (b) The magnitude of ξ is much greater very near $r = R$ than in the rest of the star. These two effects increase as the mode increases.

There is some difficulty approximating a function whose derivative changes so rapidly by one of piecewise constant derivative. With the coordinate functions used here, separations of less than $1/n$ between zeros cannot be made clearly. Thus it is not surprising that zeros of the approximate eigenfunctions obtained by this method are slightly further from $r = R$ than those of the true eigenfunctions. Nevertheless all the approximate eigenfunctions obtained here had the correct number of zeros.

Let ξ_n be the approximation to ξ obtained using a partition containing n equal subintervals. If the last zero of ξ_n is to be very near $r = R$, $|\xi_n(R)/\xi_n(R-R/n)|$ must be small. This makes it difficult for ξ_n to satisfy property (b). Property (a) turns out to be much better satisfied. Even for quite high modes, the nodes and antinodes of ξ_{50} are in much the same position as those of ξ. But $|\xi_{50}(R)/\xi_{50}(0)|$ although it increases initially, levels off for the higher modes, ultimately decreasing almost exponentially. (It is less than one for modes higher than the twentieth.) A third property of ξ is rather better satisfied by ξ_{50}, but this property does not require rapid change of derivative. This is that, for higher modes, $\xi(0)$, although much less than $\xi(R)$, is still appreciably greater than the values of ξ at the intermediate maxima.

Because of the more rapid change of ξ' near $r = R$ it might seem that better results would be obtained from a P_n whose points were not evenly spaced but closer together near $r = R$. This may be so but experiments with this modification were inconclusive.

Table 2 compares the ratio $\xi(R)/\xi(0)$ and the value of r/R at the first two zeros of ξ for the first 15 eigenfunctions obtained by variational means ($\xi = \xi_{50}$) with those of Van der Borght. For each quantity the latter value is in the column immediately to the right of that showing the value for ξ_{50}. The first column shows the number of the mode. The agreement of the eigenfunctions is better than these figures might suggest, as the discrepancy in $\xi(R)/\xi(0)$ is mainly due to a very sharp increase in ξ very near

$r = R$ in Van der Borght's more accurate results. Until very near the surface the amplitudes agree quite well.

TABLE 2

Comparison of eigenfunctions

k	$\xi(R)/\xi(0)$		First zero		Scond zero	
1	2.54	2.57				
2	—27.5	—28.0	.603	.603		
3	61.3	69.6	.520	.520	.803	.803
4	—100	—126	.462	.463	.725	.726
5	136	194	.417	.420	.654	.659
6	—167	—277	.380	.384	.596	.604
7	198	362	.347	.354	.548	.558
8	—236	—460	.320	.328	.508	.520
9	295	566	.297	.306	.473	.487
10	—372	—678	.279	.286	.445	.458
11	327	792	.260	.265	.422	.432
12	—175	—908	.246	.253	.400	.409
13	85	1028	.230	.239	.375	.388
14	—44	—1150	.215	.226	.353	.370
15	24	1273	.200	.215	.332	.352

The use of variational methods to determine higher modes in the more difficult case of non-radial stellar oscillations will be considered in a forthcoming paper [15].

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