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Multi-Sided Braid Type Subfactors, II

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Abstract. We show that the multi-sided inclusion $R^{\otimes l} \subset R$ of braid-type subfactors of the hyperfinite II_1 factor R, introduced in *Multi-sided braid type subfactors* [E3], contains a sequence of intermediate subfactors: $R^{\otimes l} \subset R^{\otimes l-1} \subset \cdots \subset R^{\otimes 2} \subset R$. That is, every t-sided subfactor is an intermediate subfactor for the inclusion $R^{\otimes l} \subset R$, for $2 \leq t \leq l$. Moreover, we also show that if $t > m$ then $R^{\otimes t} \subset R^{\otimes m}$ is conjugate to $R^{\otimes t-m+1} \subset R$. Thus, if the braid representation considered is associated to one of the classical Lie algebras then the asymptotic inclusions for the Jones-Wenzl subfactors are intermediate subfactors.

1 Introduction

In this paper we show that the braid type subfactors of the hyperfinite II_1 factor R constructed in [E3]—called multi-sided or l-sided subfactors—contain a sequence of intermediate subfactors. This sequence of intermediate subfactors has the property that the inclusion of any two consecutive subfactors is conjugate to the two-sided pair, and that the inclusion of any t consecutive subfactors is conjugate to the t-sided pair.

The result says in particular that the two-sided inclusion has a special role among the multi-sided pairs, since any multi-sided inclusion can be obtained as a composition of inclusions all conjugate to the two-sided inclusion. In the case where the braid representations—used in the construction of the subfactors—come from Lie representation theory of classical type, the two-sided inclusion is also special, because it has been shown (see [E2], [G]) that they are conjugate to the asymptotic inclusions for the one-sided or Jones-Wenzl subfactors (see [W1], [W2]). This brings up the question of finding out what the asymptotic inclusion is for any multi-sided inclusion, and of whether the asymptotic inclusion of the asymptotic inclusion is again conjugate to a multi-sided subfactor.

The l-sided subfactors were defined in [E3] as inclusions $R^{\otimes l} \subset R$—with R the hyperfinite II_1 factor—where the embedding is defined from finite dimensional inclusions of the form $A_n^{\otimes l} \subset_{\hat{u}_n} A_{ln}$, where $\hat{u}_n$ is the conjugation by a special unitary in $u_n \in A_{ln}$, and the A_n's are finite dimensional braid group quotients. The l-sided subfactors generalize the construction of the already known two-sided subfactors [Ch], [E1]. As an interesting observation, in the examples associated to Lie type A, if β_1 is the index value for an l-sided subfactor (computed in [E3]) and if β_2 is that for the multiple interval subfactor corresponding to $2l$ intervals on S^1 for Wassermann's $\mathrm{SU}(n)$ construction, [Wa] (computed in [X]), then the relation $\beta_1 = \frac{1}{n^{l-1}}\beta_2$ holds.

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(In the previous version of this article and in [E3] the constant $\frac{1}{n^{l-1}}$ was mistakenly left out, as pointed out by the referee.)

The paper is organised as follows:

2. Preliminaries.
   2.1. Basic definitions, properties and assumptions of the braid groups and their representations.
   2.2. The construction of the multi-sided subfactors.
3. Intermediate inclusions.
4. References.

Acknowledgements I would like to thank Hans Wenzl for useful discussions.

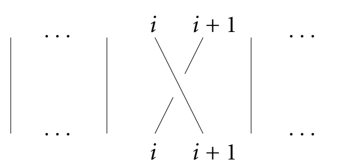
2 Preliminaries

2.1 Basic Definitions, Properties and Assumptions of the Braid Groups and Their Representations

Recall that the braid group $\mathbf{B}_n$ on n strands is defined by generators $\sigma_1, \dots, \sigma_{n-1}$ and the braid relations

(B1) $\sigma_{i+1}\sigma_i\sigma_{i+1} = \sigma_i\sigma_{i+1}\sigma_i$, for $i = 1, \dots, n-2$,
(B2) $\sigma_i\sigma_j = \sigma_j\sigma_i$, for $|i-j| \geq 2$.

A geometric picture of the standard generator σ_i is given by the following diagram:



and multiplication is given by concatenation of such diagrams (see [Bi] for more details). $\mathbf{B}_n$ is embedded into $\mathbf{B}_{n+1}$ by adding one vertical strand at the end of each generator of $\mathbf{B}_n$. Denote $\bigcup \mathbf{B}_n$ by $\mathbf{B}_\infty$.

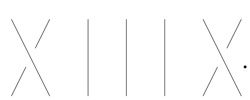
We state below some well known basic relations for elements in $\mathbf{B}_n$ that will be needed in the next sections. If $1 \leq i, j \leq n-1$, we denote by

$$(\sigma_i \cdots \sigma_j)$$

the element of $\mathbf{B}_n$ given by the increasing product $\sigma_i\sigma_{i+1}\cdots\sigma_j$ if $i < j$, or by the decreasing product $\sigma_i\sigma_{i-1}\cdots\sigma_j$ if $i > j$.

The algebraic tensor product $\mathbb{C}\mathbf{B}_n \otimes \mathbb{C}\mathbf{B}_m$ can be seen as a subalgebra of $\mathbb{C}\mathbf{B}_{n+m}$ via the embedding defined by juxtaposition. For example, if $n = 3$, $m = 4$

$$\sigma_1 \otimes \sigma_3 \longmapsto \overline{\sigma_1 \otimes \sigma_3} =$$



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In other words, the embedding $\mathbb{C}\mathbf{B}_n \otimes \mathbb{C}\mathbf{B}_m \hookrightarrow \mathbb{C}\mathbf{B}_{n+m}$ is defined by

$$\beta \otimes \gamma \longmapsto \overline{\beta \otimes \gamma} = \beta \,\mathrm{shift}_n \gamma,$$

where $\mathrm{shift}_n\, \sigma_i = \sigma_{i+n}$. Thus, the embedding $\mathbb{C}\mathbf{B}_n^{\otimes s} \hookrightarrow \mathbb{C}\mathbf{B}_{sn}$ is given by

$$\beta_1 \otimes \cdots \otimes \beta_s \longmapsto \overline{\beta_1 \otimes \cdots \otimes \beta_s} = \beta_1 \,\mathrm{shift}_n(\beta_2) \cdots \,\mathrm{shift}_{n(s-1)}(\beta_n).$$

We shall work with representations ρ of $\mathbb{C}\mathbf{B}_\infty$ that satisfy the following properties as in [E1]:

(i) ρ is locally finite dimensional: For every $n \in \mathbb{N}$, $\rho(\mathbb{C}\mathbf{B}_n)$ is a finite dimensional C*-algebra, so that we can write $\rho(\mathbb{C}\mathbf{B}_n) \simeq \bigoplus_{\lambda \in \Lambda_n} M_{a_\lambda}(\mathbb{C})$, for some index set Λ_n. Set $A_n = \rho(\mathbb{C}\mathbf{B}_n)$.

(ii) ρ is unitary: That is, $g_i = \rho(\sigma_i)$ is a unitary for all i.

(iii) The ascending sequence of finite dimensional C*-algebras $(A_n) = \big(\rho(\mathbb{C}\mathbf{B}_n)\big)$ is periodic, in the sense of Wenzl, [W1, Lemma 1.4].

(iv) Any element $x \in A_{n+1}$ can be written as a sum of elements $ag_n^{\pm 1}b + c$ with $a, b, c \in A_n$.

(v) The unique positive faithful normalised trace tr on $\bigcup A_n$ has the Markov property:

$$\mathrm{tr}(g_n^{\pm 1}x) = \eta^{(\pm)}\,\mathrm{tr}(x) \quad \text{for all } x \in A_n, \text{ for all } n,$$

where $\eta^{(+)}$, $\eta^{(-)}$ are fixed complex numbers. Given condition (iv), the Markov condition implies the multiplicativity property for the trace:

$$\mathrm{tr}(xy) = \mathrm{tr}(x)\,\mathrm{tr}(y),$$

if x and y are in subalgebras generated by disjoint subsets of generators $g_i^{\pm 1}$.

(vi) Existence of a projection p with the *contraction property*: $p \in A_k$ has the *contraction property* if for all $n \in \mathbb{N}$,

$$pA_{n+k}p \simeq pA_{k+1,n+k} \simeq A_{k+1,n+k},$$

where $A_{s,t}$ is the algebra generated by $\{1, g_s^{\pm 1}, \dots, g_{t-1}^{\pm 1}\}$. Note that since we already have the multiplicative property of the trace by (iv) and (v), the second isomorphism above is always true.

2.2 The Construction of the Multi-Sided Subfactors

The construction is done in detail in [E3, Section 3], where these subfactors are introduced. The inclusions are generated by pairs of ascending sequences of finite dimensional C* algebras $(A_n^l \subset A_{ln})_n$ where $l \geq 2$ is fixed and

(i) A_n is an n-braid quotient as in the preliminaries (i),

(ii) $A_n^l = u_n^l (A_n^{\otimes l}) {u_n^l}^*$ for a unitary $u_n^l \in A_{nl}$,

(iii) $(A_n^l)_n \subset (A_{ln})_n$ is periodic and has the commuting square property as in [W1].

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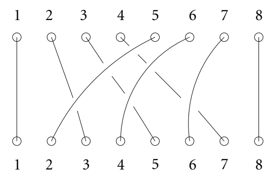
Having the conditions (i)–(iii) as above allows us to define an inclusion of subfactors of the hyperfinite II_1 factor R by taking $\overline{\bigcup_n A_n^l} \subset \overline{\bigcup_n A_{ln}}$, where we are taking the weak closure with respect to the GNS representation with respect to the unique positive trace on the unions.

These unitaries $u_n^{(l)}$ are images of braids $\gamma_n^{(l)} \in \mathbf{B}_{ln}$ defined by

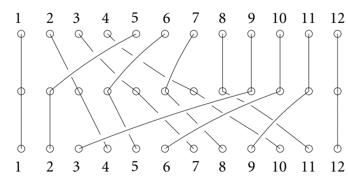
$$\gamma_n^{(l)} = \Phi_1^{(l)^{-1}} \Phi_2^{(l)^{-1}} \cdots \Phi_{n-1}^{(l)^{-1}},$$

where $\Phi_t^{(l)} = (\sigma_{(l-1)(t+1)} \cdots \sigma_{l(t+1)-2})(\sigma_{(l-2)(t+1)} \cdots \sigma_{l(t+1)-3}) \cdots (\sigma_{t+1} \cdots \sigma_{lt})$. (Notice that $\gamma_{n+1} = \gamma_n \Phi_n^{-1}$ for every n.) See below the geometric illustrations for some of these braids when $l = 4$.

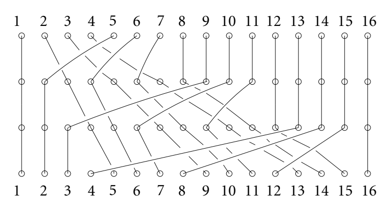
$$\gamma_2^{(4)} = \Phi_1^{(4)^{-1}} =$$



$$\gamma_3^{(4)} = \gamma_2^{(4)} \Phi_2^{(4)^{-1}} =$$

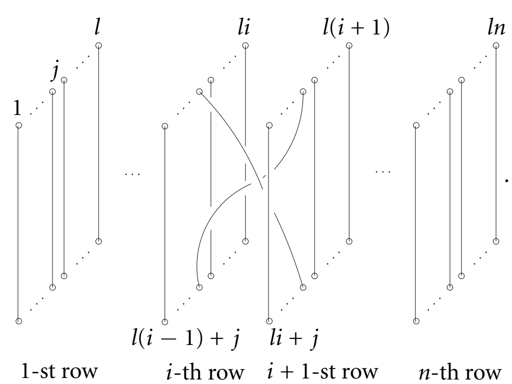


$$\gamma_4^{(4)} = \gamma_3^{(4)} \Phi_3^{(4)^{-1}} =$$



As a map from $\mathbf{B}_n^{\otimes l}$ into $\mathbf{B}_{ln}$, the conjugation Ψ_n by the braid $\gamma_n^{(l)}$ can be seen geometrically in a simple way, since conjugating by $(\Phi_t^{(l)})^{-1}$ pulls the $t+1$-st strand in the j-th tensor factor of $\mathbf{B}_n^{\otimes l}$ to the strand in $\mathbf{B}_{ln}$ labeled by $lt + j$. Below we include a picture of how conjugation by $\gamma_n^{(l)}$, Ψ_n, acts on each generator of $\mathbf{B}_n^{\otimes l}$: For $1 \leq i \leq n-1$ and $1 \leq j \leq l$,

$$(1 \otimes \cdots \otimes \underset{j\text{-th position}}{\sigma_i} \otimes \cdots \otimes 1) \mapsto$$



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Note that we are arranging the ln strands in n rows with l strands each, so that $\Psi_n(1 \otimes \cdots \otimes \underset{j\text{-th position}}{\sigma_i} \otimes \cdots \otimes 1)$ can be seen as a crossover between the j-th dots belonging to the i-th and $i+1$-st rows.

3 Intermediate Inclusions

Theorem 3 in this section is technical, and the main result of the article will follow as a corollary.

Proposition 1 *The two-sided inclusion $R^{\otimes 2} \subset R$ is stable, that is, $R^{\otimes 2} \otimes R \subset R \otimes R$ is conjugate to $R^{\otimes 2} \subset R$, with respect to the embedding $u^{(2)} \otimes \mathrm{id}_R$, where $u^{(2)} = \varinjlim \hat{u}_n^{(2)}$ is the two-sided embedding.*

Proof There is a necessary and sufficient condition for stableness due to Bisch and McDuff: An inclusion $N \subset M$ of II_1 factors is stable if and only if there exist two non-trivial non-commuting central sequences for $N \subset M$. In [E2, Section 3 Lemma 6] we have shown that the two-sided inclusions coming from representations of the braid group of type A have the Bisch-McDuff property (see [B]). This can be shown as well for the general two-sided inclusions defined from braid representations that just satisfy the properties listed in the preliminaries.

Let us recall that a sequence (x_n) in a II_1 factor M is central if for every $x \in M$, $\|[x, x_n]\|_2 \to 0$, where $\|.\|_2$ is the trace norm. A sequence (x_n) is trivial if there exists a complex sequence (λ_n) such that $\|x_n - \lambda_n\|_2 \to 0$. Finally, two sequences (x_n) and (y_n) are commuting if $\|[x_n, y_n]\|_2 \to 0$. A central sequence (x_n) for an inclusion of II_1 factors $N \subset M$ is central if (x_n) is contained in N and is central for M.

Let us define the following projections e_1 and e_2 in R. If the A_n's are the braid quotients as in the preliminaries (i)–(vi), there must be an $n_0 \in \mathbb{N}$ such that A_{n_0} has a subalgebra isomorphic to $M_2(\mathbb{C})$ (we assume that the A_n's have growing dimension since otherwise the weak closure of their union would be finite dimensional). In this subalgebra of A_{n_0}, we choose e_1 and e_2 to satisfy

$$(*) \qquad \begin{aligned} e_1 e_2 e_1 &= \alpha e_1, \\ e_2 e_1 e_2 &= \alpha e_2, \end{aligned}$$

where $\alpha \neq 0, 1$ is a complex number. (*E.g.*, take e_1 and e_2 to be the projections corresponding to the matrices $\left(\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}\right)$ and $\begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix}$.)

Define projections $e_n \in R$ for $n \in \mathbb{N}$ in the following way:

$$e_{2i-1} = \mathrm{shift}_{i-1}(e_1) = 1_{i-1} \otimes e_1 \in A_{i,i+n_0-1} \subset R,$$
$$e_{2i} = \mathrm{shift}_{i-1}(e_2) = 1_{i-1} \otimes e_2 \in A_{i,i+n_0-1} \subset R.$$

Now we are ready to produce two non-commuting, non-trival, central sequences for the two-sided inclusion $R^{\otimes 2} \subset R$: Take (x_n) and (y_n) in $R^{\otimes 2}$ with $x_n = e_n \otimes 1_R$ and $y_n = e_{n+1} \otimes 1_R$.

(a) (x_n) is non-trivial (and therefore so is (y_n)): Let (λ_n) be a sequence of complex numbers.

$$\begin{aligned}\|x_n - \lambda_n.1\|_2^2 &= \operatorname{tr}(x_n - x_n\overline{\lambda}_n - \lambda_n x_n + |\lambda_n|^2)\\ &= \operatorname{tr}(x_n) - 2\operatorname{Re}(\lambda_n)\operatorname{tr}(x_n) + |\lambda_n|^2\\ &= \lambda - 2\operatorname{Re}(\lambda_n)\lambda + |\lambda_n|^2\\ &= \lambda|1-\lambda_n|^2 + (1-\lambda)|\lambda_n|^2\\ &\geq \frac{1}{4}\min\{\lambda\,;\,1-\lambda\} > 0,\end{aligned}$$

where $\lambda := \operatorname{tr}(x_n) \in (0,1)$ (since the non-zero projections x_n are all conjugate). Thus, the sequence $\|x_n - \lambda_n.1\|_2$ cannot converge to zero.

(b) (x_n) is central for R (and therefore so is (y_n)): Let $x \in R$. Since the union of the algebras $A_j = \langle g_1, \dots, g_{j-1}\rangle$ is $\|.\|_2$-dense in R, there exists a sequence (w_j) with $w_j \in A_j$ such that $\|x - w_j\|_2 \to 0$. If $\varepsilon > 0$ take j such that $\|x - w_j\|_2 < \frac{\varepsilon}{2}$. By the definition of the projections x_n, $x_n \in A_{r(n),r(n)+n_0-1} \otimes 1_R$ (where $r(n) = [\frac{n+1}{2}]$). By the definition of the embedding $u^{(2)} = \varinjlim \hat{u}_n^{(2)}$, $u^{(2)}(x_n) \in A_{2(r(n)-1),2(r(n)+n_0)}$ (see the last figure in Section 2). Thus, if n is large enough then $u^{(2)}(x_n)$ and $w_j \in A_j$ will commute. Therefore

$$\begin{aligned}\|[u^{(2)}(x_n), x]\|_2 &= \|u^{(2)}(x_n)x - xu^{(2)}(x_n)\|_2\\ &\leq \|u^{(2)}(x_n)x - u^{(2)}(x_n)w_j\|_2 + \|u^{(2)}(x_n)w_j - w_ju^{(2)}(x_n)\|_2\\ &\qquad + \|w_ju^{(2)}(x_n) - xu^{(2)}(x_n)\|_2\\ &\leq 2\|u^{(2)}(x_n)\|\,\|x - w_j\|_2 < 2\frac{\varepsilon}{2} = \varepsilon.\end{aligned}$$

(c) The sequences (x_n) and (y_n) are non-commuting: We must show that the sequence (a_n) with $a_n = \|[x_n, y_n]\|_2 = \|[e_n, e_{n+1}]\|_2$ does not converge to zero. Consider the subsequence $(a_{2i-1})_i$. Using the relations $(*)$,

$$\begin{aligned}a_{2i-1}^2 &= \|[e_{2i-1}, e_{2i}]\|_2^2\\ &= \|[\operatorname{shift}_{i-1}(e_1), \operatorname{shift}_{i-1}(e_2)]\|_2^2\\ &= \operatorname{tr}\big(\operatorname{shift}_{i-1}(-e_1e_2e_1e_2 + e_1e_2e_1 + e_2e_1e_2 - e_2e_1e_2e_1)\big)\\ &= \operatorname{tr}(-e_1e_2e_1e_2 + e_1e_2e_1 + e_2e_1e_2 - e_2e_1e_2e_1)\\ &= 2\alpha(1-\alpha)\lambda \equiv \kappa > 0,\end{aligned}$$

where $\lambda = \operatorname{tr}(e_1) \neq 0$, and where the fourth equality above follows from the fact that $\operatorname{shift}_{2i-1}$ can be determined via conjugation by some unitary. The subsequence (a_{2i-1}) does not converge to zero, so neither does (a_n). ■

Proposition 2 *Let $l \geq 3$. The sequence of finite dimensional inclusions*

$$(A_n^{\otimes l} \underset{\hat{u}_n^{(2)}\otimes \mathrm{id}_n^{\otimes l-2}}{\subset} A_{2n} \otimes A_n^{\otimes l-2})_n$$

generates an inclusion of factors conjugate to the two-sided inclusion $R^{\otimes 2} \subset R$.

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Proof This is a corollary of Proposition 1. By definition of the maps, these finite dimensional inclusions generate the inclusion of factors

$$R^{\otimes 2} \otimes R^{\otimes l-2} \subset R \otimes R^{\otimes l-2},$$

via the embedding given by $u^{(2)} \otimes \mathrm{id}_{R^{\otimes l-2}}$, where $u^{(2)} = \varinjlim \hat{u}_n^{(2)}$ is the two-sided embedding. Since the hyperfinite II_1 factor R is isomorphic to $R^{\otimes m}$ for any $m \in \mathbb{N}$, it is easy to see that if the two-sided inclusion $R^{\otimes 2} \subset R$ is stable (which is the case by Proposition 1), then $R^{\otimes 2} \subset R$ is conjugate to $R^{\otimes 2} \otimes R^{\otimes m} \subset R \otimes R^{\otimes m}$ for any $m \in \mathbb{N}$ via the embedding $u^{(2)} \otimes \mathrm{id}_{R^{\otimes m}}$. ∎

Theorem 3 *Let $l \geq 3$ and $1 \leq t \leq l-2$. The sequence of finite dimensional inclusions*

$$(A_{(l-t)n} \otimes A_n^{\otimes t} \underset{\hat{u}_n^{(l)} \circ (\hat{u}_n^{(l-t)} \otimes \mathrm{id}_n^{\otimes t})^{-1}}{\subset} A_{ln})_n$$

generates an inclusion of factors conjugate to the $(t+1)$-sided inclusion $R^{\otimes t+1} \subset R$.

Proof In order to prove this statement, it will be enough to show the existence of a unitary $r_n^{(l,t)} \in A_{ln}$, for each $n \in \mathbb{N}$, such that the diagrams in items (a), (b), and (c) commute:

(a)

$$\begin{tikzcd}
A_{(l-t)n} \otimes A_n^{\otimes t} \arrow[r, "\hat{u}_n^{(l)} \circ (\hat{u}_n^{(l-t)} \otimes \mathrm{id}_n^{\otimes t})^{-1}"] & A_{ln} \\
A_n \otimes A_n^{\otimes t} \arrow[u, "\iota \otimes \iota^{\otimes t}"] \arrow[r, "\hat{u}_n^{(t+1)}"'] & A_{(t+1)n} \arrow[u, "\hat{r}_n^{(l,t)} \circ \iota"']
\end{tikzcd}$$

where the maps ι are the canonical embeddings (with $\iota \colon A_m \to A_{m+p}$, $m, p \in \mathbb{N}$; we shall denote all of these maps by ι, with no subindices), and where the map $\hat{r}_n^{(l,t)}$ denotes conjugation by $r_n^{(l,t)}$.

(b)

$$\begin{tikzcd}
A_{(t+1)(n+1)} \arrow[r, "\hat{r}_{n+1}^{(l,t)} \circ \iota"] & A_{l(n+1)} & A_{\left[\begin{smallmatrix} n+2 \\ l-t \end{smallmatrix}\right]t} \arrow[r, "{\hat{r}_{n+1}^{(l,t)}}^{-1} \circ \iota"] & A_{l(n+1)} \\
A_{(t+1)n} \arrow[u, "\iota"] \arrow[r, "\hat{r}_n^{(l,t)} \circ \iota"'] & A_{ln} \arrow[u, "\iota"] & A_{\left[\begin{smallmatrix} n+1 \\ l-t \end{smallmatrix}\right]t} \arrow[u, "\iota"] \arrow[r, "{\hat{r}_n^{(l,t)}}^{-1} \circ \iota"'] & A_{ln} \arrow[u, "\iota"']
\end{tikzcd}$$

(c)

$$\begin{tikzcd}
A_{(l-t)(n+1)} \otimes A_{n+1}^{\otimes t} \arrow[r, "\hat{u}_{n+1}^{(l)} \circ (\hat{u}_{n+1}^{(l-t)} \otimes \mathrm{id}_{n+1}^{\otimes t})^{-1}"] & A_{l(n+1)} \\
A_{(l-t)n} \otimes A_n^{\otimes t} \arrow[u, "\iota \otimes \iota^{\otimes t}"] \arrow[r, "\hat{u}_n^{(l)} \circ (\hat{u}_n^{(l-t)} \otimes \mathrm{id}_n^{\otimes t})^{-1}"'] & A_{ln} \arrow[u, "\iota"']
\end{tikzcd}$$

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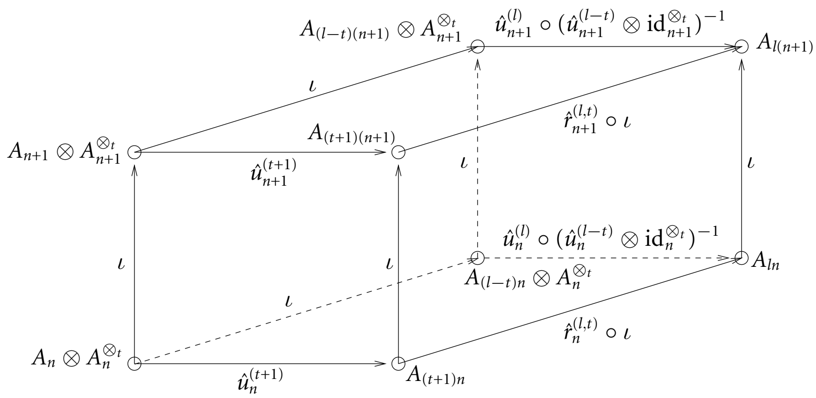
The reason that having (a), (b), and (c) will be enough for proving this proposition is the following: Firstly, the faces of the cube-diagram shown below with the embeddings defined as in (a), (b), and (c), will all be commuting:



Secondly, because the canonical maps will generate the trivial inclusion, and because the map $\varinjlim(\hat{r}_n^{(l,t)} \circ \iota)$ will be an automorphism, then the inclusions

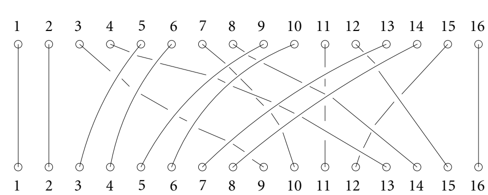
$$\hat{u}_n^{(l)} \circ (\hat{u}_n^{(l-t)} \otimes \mathrm{id}_n^{\otimes t})^{-1} \colon A_{(l-t)n} \otimes A_n^{\otimes t} \hookrightarrow A_{ln}$$

will generate an inclusion of factors conjugate to that generated by the embeddings $\hat{u}_n^{(t+1)} \colon A_n^{\otimes t+1} \hookrightarrow A_{(t+1)n}$, namely, the $(t+1)$-sided inclusion, which is what we wanted to show.

Let us remark that having the commuting diagrams in item (b) is the reason why the morphism $\varinjlim(\hat{r}_n^{(l,t)} \circ \iota)$ is an automorphism: The first commuting diagram in (b) will force $\varinjlim(\hat{r}_{n+1}^{(l,t)^{-1}} \circ \iota) \circ (\hat{r}_n^{(l,t)} \circ \iota)$ to be the trivial inclusion, so that $\varinjlim(\hat{r}_{n+1}^{(l,t)^{-1}} \circ \iota)$ is the right inverse for $\varinjlim(\hat{r}_n^{(l,t)} \circ \iota)$, and the second commuting diagram in (b) will force $\varinjlim(\hat{r}_{n+1}^{(l,t)^{-1}} \circ \iota) \circ (\hat{r}_n^{(l,t)} \circ \iota)$ to be the trivial inclusion, so that $\varinjlim(\hat{r}_{n+1}^{(l,t)^{-1}} \circ \iota)$ is the left inverse for $\varinjlim(\hat{r}_n^{(l,t)} \circ \iota)$.

Now we shall proceed by describing the embeddings $\hat{u}_n^{(l)} \circ (\hat{u}_n^{(l-t)} \otimes \mathrm{id}_n^{\otimes t})^{-1}$ at the braid level, so that we can find the right unitaries r_n. By definition (Section 2.2), $u_n^{(l)} = \rho(\gamma_n^{(l)})$, where ρ is the braid representation considered, and $\gamma_n^{(l)} \in \mathbf{B}_{ln}$ (see also the geometric description). Thus, $u_n^{(l)} \circ (u_n^{(l-t)} \otimes 1_n^{\otimes t})^{-1} = \rho\big(\gamma_n^{(l)} \circ (\gamma_n^{(l-t)} \otimes 1_n^{\otimes t})^{-1}\big)$. See below the diagrams for $\gamma_n^{(l)} \circ (\gamma_n^{(l-t)} \otimes 1_n^{\otimes t})^{-1}$, for the cases $n = 4$, $n = 5$, $l = 4$ and $t = 2$:

$\gamma_4^{(4)} \circ (\gamma_4^{(2)} \otimes 1_4^{\otimes 2})^{-1} =$



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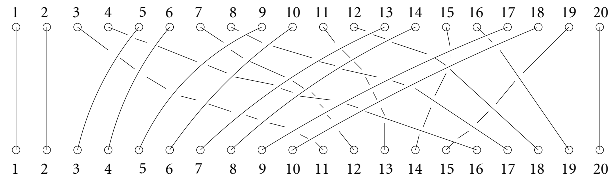
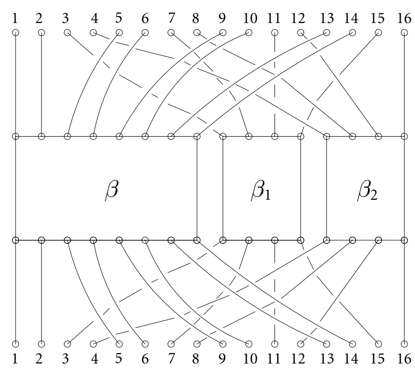
$$\gamma_5^{(4)} \circ (\gamma_5^{(2)} \otimes 1_5^{\otimes 2})^{-1} =$$



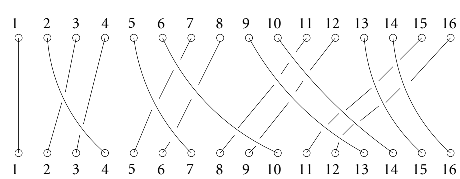
One can describe geometrically $\hat{\gamma}_n^{(l)} \circ (\hat{\gamma}_n^{(l-t)} \otimes \mathrm{id}_n^{\otimes t})^{-1} \colon \mathbf{B}_{(l-t)n} \otimes \mathbf{B}_n^{\otimes t} \to \mathbf{B}_{ln}$ in the following way: This map acts by pulling the i-th strand ($1 \leq i \leq n$) from the p-th non-trivial tensor factor in $1_{(l-t)n} \otimes \mathbf{B}_n^{\otimes t} \subset \mathbf{B}_{(l-t)n} \otimes \mathbf{B}_n^{\otimes t}$, and by placing it right after the $(l-t+p-1)i$-th strand in $\mathbf{B}_{ln}$, for $p = 1, \dots, t$. This becomes clear if one looks at the diagrams for the examples below, for $n = 4$, $l = 4$, and $t = 2$: Take $\beta \otimes \beta_1 \otimes \beta_2 \in \mathbf{B}_{(l-t)n} \otimes \mathbf{B}_n^{\otimes t} = \mathbf{B}_8 \otimes \mathbf{B}_4 \otimes \mathbf{B}_4$.

$$\big(\hat{\gamma}_4^{(4)} \circ (\hat{\gamma}_4^{(2)} \otimes \mathrm{id}_4^{\otimes 2})^{-1}\big)(\beta \otimes \beta_1 \otimes \beta_2) =$$



Now that one can see what happens with this conjugation geometrically, one can compare this with what happens with the conjugation $\hat{u}_n^{(t+1)} \colon A_n^{\otimes t+1} \to A_{(t+1)n}$, followed by the embedding $\iota \colon A_{(t+1)n} \to A_{ln}$. From this comparison, one can define a unitary $r_n^{(l,t)}$ that satisfies what is required in (a), (b), (c), as follows: At the braid level, we want a braid $\Omega = \Omega(n,l,t) \in \mathbf{B}_{ln}$ such that its conjugation $\hat{\Omega}$ acts by pulling strands that are labeled by $i(t+1) - j$, with $i = 1, \dots, n$ and $j = 0, \dots, t-1$, to the strands labeled by $il - j$. The braid Ω can be given geometrically by the diagram of the permutation in $\mathcal{S}_{ln}$ defined by $il - j \mapsto i(t+1) - j$, for $i = 1, \dots, n$ and $j = 0, \dots, t-1$, where all the crossings "from left to right" are over-crossings. See the examples below.

$$\Omega(4,4,2) =$$



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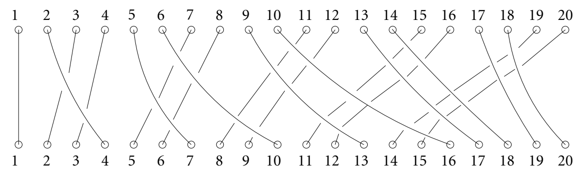
$\Omega(5,4,2) =$ 

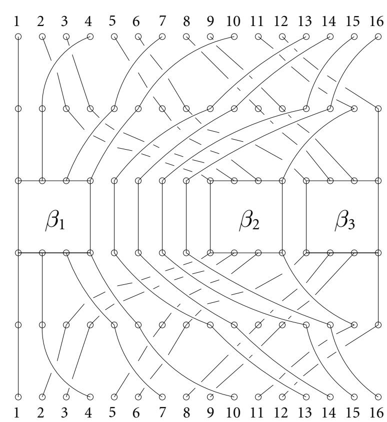
Now one can define the desired unitary by $r_n^{(l,t)} := \rho\big(\Omega(n,l,t)\big) \in A_{ln}$, written as $r_n^{(l,t)} = (g_{l-t}\cdots g_2)\cdots(g_{l-2}\cdots g_t)\,(g_{l-1}\cdots g_{t+1})\,\big(g_{(n-1)l-(t-1)-1}\cdots g_{(n-1)(t+1)-(t-1)}\big)$ $\cdots\big(g_{(n-1)l-2}\cdots g_{(n-1)(t+1)-1}\big)\ \big(g_{(n-1)l-1}\cdots g_{(n-1)(t+1)}\big)\ \big(g_{nl-(t-1)-1}\cdots g_{n(t+1)-(t-1)}\big)$ $\cdots\big(g_{nl-2}\cdots g_{n(t+1)-1}\big)\big(g_{nl-1}\cdots g_{n(t+1)}\big)$.

The algebraic formal proof that the diagrams in (a), (b), (c) are commuting is very tedious, so we shall omit this formalism, but we shall show that these diagrams are commuting at the braid level (and therefore in the braid quotients) by using braid diagrams. To simplify the diagrams, we shall do it in the cases $n = 4$, $l = 4$, and $t = 2$, using the last figures. We shall show first that the diagram in (a) is commuting at the braid level:

$$\begin{tikzcd}
\mathbf{B}_8 \otimes \mathbf{B}_4^{\otimes 2} \arrow[r, "\hat\gamma_4^{(4)}\circ(\hat\gamma_4^{(2)}\otimes \mathrm{id}_4^{\otimes 2})^{-1}"] & \mathbf{B}_{16} \\
\mathbf{B}_4 \otimes \mathbf{B}_4^{\otimes 2} \arrow[u, "\iota\otimes \mathrm{id}_4^{\otimes 2}"] \arrow[r, "\hat\gamma_4^{(3)}"'] & \mathbf{B}_{12} \arrow[u, "\hat\Omega(4,4,2)\circ\iota"']
\end{tikzcd}$$

If $\beta_1 \otimes \beta_2 \otimes \beta_3 \in \mathbf{B}_4^{\otimes 3}$,

$$\big(\hat\Omega(4,4,2)^{-1} \circ \hat\gamma_4^{(4)} \circ (\hat\gamma_4^{(2)} \otimes \mathrm{id}_4^{\otimes 2})^{-1} \circ (\iota \otimes \mathrm{id}_4^{\otimes 2})\big)\,(\beta_1 \otimes \beta_2 \otimes \beta_3)$$

$=$ 

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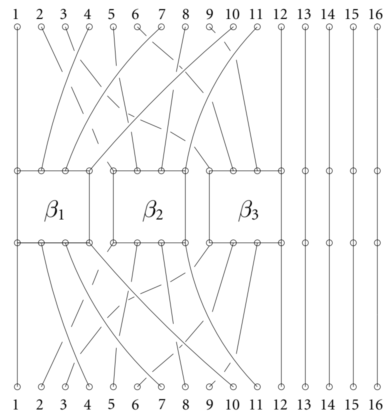
$=$



$$= (\iota \circ \hat{\gamma}_4^{(3)})(\beta_1 \otimes \beta_2 \otimes \beta_3).$$

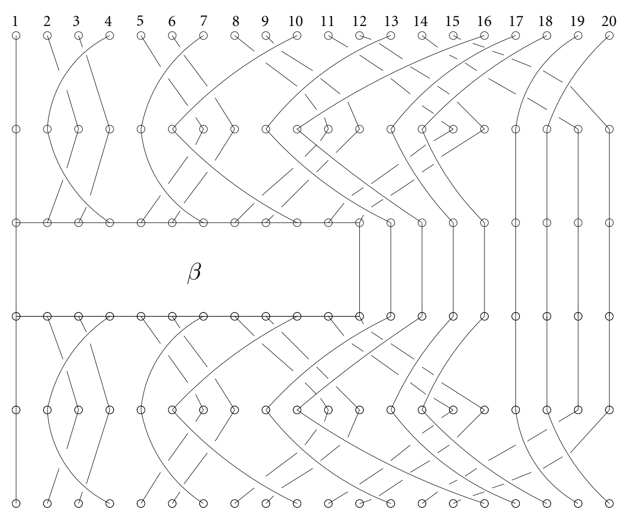
Now we shall show that the diagrams in (b) are commuting at the braid level:

$$\begin{tikzcd}
\mathbf{B}_{15} \arrow[r, "\hat{\Omega}(5,4,2)\circ\iota"] & \mathbf{B}_{20} & \mathbf{B}_{6} \arrow[r, "\hat{\Omega}(5,4,2)^{-1}\circ\iota"] & \mathbf{B}_{20} \\
\mathbf{B}_{12} \arrow[u, "\iota"] \arrow[r, "\hat{\Omega}(4,4,2)\circ\iota"'] & \mathbf{B}_{16} \arrow[u, "\iota"'] & \mathbf{B}_{4} \arrow[u, "\iota"] \arrow[r, "\hat{\Omega}(4,4,2)^{-1}\circ\iota"'] & \mathbf{B}_{16} \arrow[u, "\iota"']
\end{tikzcd}$$

Take $\beta \in \mathbf{B}_{12}$. Then

$$\left(\hat{\Omega}(5,4,2)^{-1} \circ \iota \circ \Omega(4,4,2) \circ \iota\right)(\beta)$$

$=$



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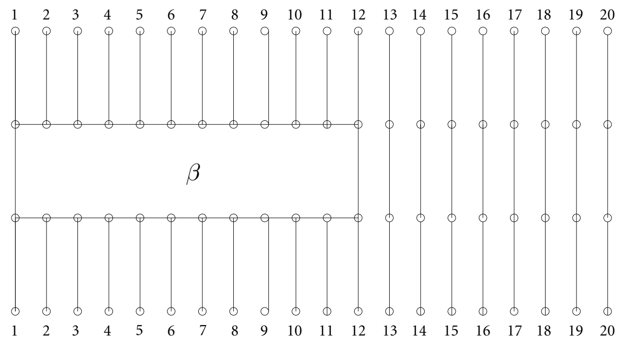
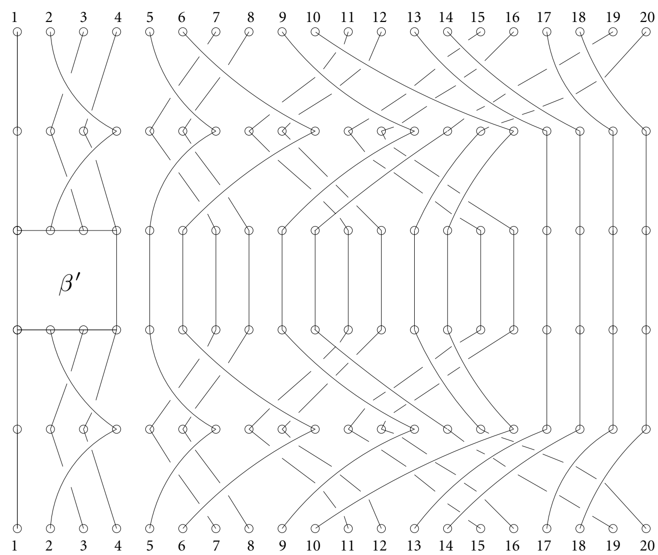


$= \iota(\beta).$

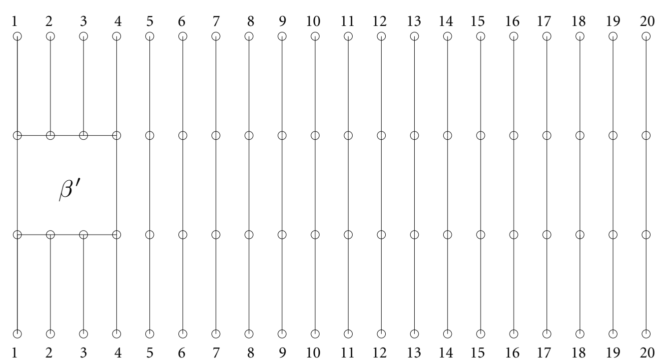
Take $\beta' \in \mathbf{B}_4$. Then

$\left(\hat{\Omega}(5,4,2) \circ \iota \circ \Omega(4,4,2)^{-1} \circ \iota\right)(\beta')$

$=$



$=$



$= \iota(\beta').$

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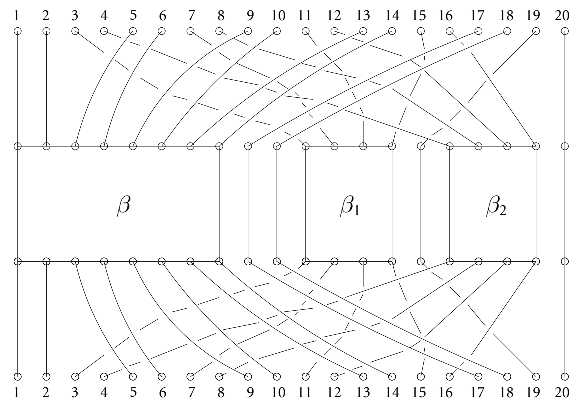
And finally, we show that the diagram (c) is commuting at the braid level:

$$\begin{tikzcd} \mathbf{B}_{10} \otimes \mathbf{B}_5^{\otimes 2} \arrow[r, "\hat{\gamma}_5^{(4)} \circ (\hat{\gamma}_5^{(2)} \otimes \mathrm{id}_5^{\otimes 2})^{-1}"] & \mathbf{B}_{20} \\ \mathbf{B}_8 \otimes \mathbf{B}_4^{\otimes 2} \arrow[u, "\iota \otimes \mathrm{id}_4^{\otimes 2}"] \arrow[r, "\hat{\gamma}_4^{(4)} \circ (\hat{\gamma}_4^{(2)} \otimes \mathrm{id}_4^{\otimes 2})^{-1}"'] & \mathbf{B}_{16} \arrow[u, "\iota"'] \end{tikzcd}$$

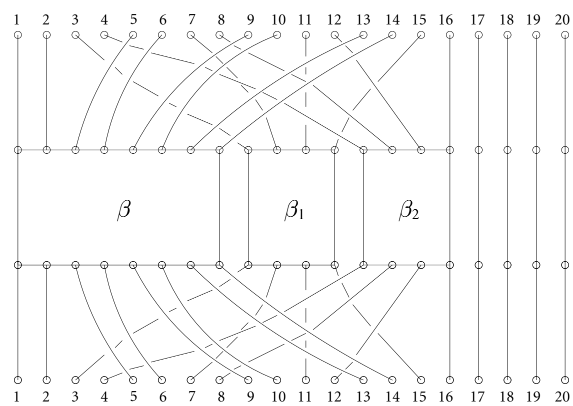
If $\beta \otimes \beta_1 \otimes \beta_2 \in \mathbf{B}_8 \otimes \mathbf{B}_4 \otimes \mathbf{B}_4$, then

$$\left(\hat{\gamma}_5^{(4)} \circ (\hat{\gamma}_5^{(2)} \otimes \mathrm{id}_5^{\otimes 2})^{-1} \circ (\iota \otimes \mathrm{id}_4^{\otimes 2})\right)(\beta \otimes \beta_1 \otimes \beta_2)$$

$=$



$=$



$$= \left(\iota \circ \hat{\gamma}_4^{(4)} \circ (\hat{\gamma}_4^{(2)} \otimes \mathrm{id}_4^{\otimes 2})^{-1}\right)(\beta \otimes \beta_1 \otimes \beta_2). \qquad \blacksquare$$

Corollary 4 *The l-sided inclusion $R^{\otimes l} \subset R$ contains the s-sided subfactors $R^{\otimes s} \subset R$ (for $s = 2, \ldots, l$) as a sequence of intermediate subfactors*

$$R^{\otimes l} \subset R^{\otimes l-1} \subset R^{\otimes l-2} \subset \cdots \subset R^{\otimes 2} \subset R.$$

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Furthermore, for $j = 0, \dots, l-2$ and $t = 1, \dots, l-j-1$ the inclusion

$$R^{\otimes l-j} \subset R^{\otimes l-j-t}$$

is conjugate to the $(t+1)$-sided inclusion $R^{\otimes t+1} \subset R$. In particular, the two-sided inclusion is an intermediate subfactor for all the multi-sided inclusions, and any multi-sided inclusion can be written as a composition of inclusions all conjugate to the two-sided inclusion.

Proof For any $n \in \mathbb{N}$, one can write

$$\begin{aligned} u_n^{(l)} &= \left(u_n^{(l)} \circ (u_n^{(l-1)} \otimes 1_n)^{-1} \right) \circ \left(u_n^{(l-1)} \circ (u_n^{(l-2)} \otimes 1_n)^{-1} \otimes 1_n \right) \\ &\quad \circ \cdots \circ \left(u_n^{(3)} \circ (u_n^{(2)} \otimes 1_n)^{-1} \otimes 1_n^{\otimes l-3} \right) \circ \left(u_n^{(2)} \otimes 1_n^{\otimes l-2} \right). \end{aligned}$$

That is, the finite dimensional inclusion $A_n^{\otimes l} \underset{\hat{u}_n^{(l)}}{\subset} A_{ln}$ can be written as the composition

$$\begin{aligned} &A_n^{\otimes 2} \otimes A_n^{\otimes l-2} \xrightarrow[\hat{u}_n^{(2)} \otimes \mathrm{id}_n^{\otimes l-2}]{} A_{2n} \otimes A_n^{\otimes l-2} \xrightarrow[\hat{u}_n^{(3)} \circ (\hat{u}_n^{(2)} \otimes \mathrm{id}_n)^{-1} \otimes \mathrm{id}_n^{\otimes l-3}]{} A_{3n} \otimes A_n^{\otimes l-3} \\ &\qquad \cdots \longrightarrow \cdots \xrightarrow[\hat{u}_n^{(l-1)} \circ (\hat{u}_n^{(l-2)} \otimes \mathrm{id}_n)^{-1} \otimes \mathrm{id}_n]{} A_{(l-1)n} \otimes A_n \xrightarrow[\hat{u}_n^{(l)} \circ (\hat{u}_n^{(l-1)} \otimes \mathrm{id}_n)^{-1}]{} A_{ln}. \end{aligned}$$

By the Proposition 2 and Theorem 3, the composition of any t consecutive embeddings above, for $j = 1, \dots, l-1$, and $t = 1, \dots, l-j$, gives us a pair of ascending sequences

$$(A_{(j+1-t)n} \otimes A_n^{\otimes l-(j+1-t)} \underset{\hat{u}_n^{(j+1)} \circ (\hat{u}_n^{(j+1-t)} \otimes \mathrm{id}_n^{\otimes t})^{-1} \otimes \mathrm{id}_n^{\otimes l-j-1}}{\subset} A_{(j+1)n} \otimes A_n^{\otimes l-j-1})_n$$

which generates an inclusion $R^{\otimes l-j+1} \subset R^{\otimes l-j+1-t}$ which is conjugate to the $(t+1)$-sided inclusion. ∎

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