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ALMOST IDENTICAL IMITATIONS OF (3,1)-DIMENSIONAL MANIFOLD PAIRS AND THE MANIFOLD MUTATION

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Abstract

In this paper, we construct from any given good (3,1)-dimensional manifold pair finitely many almost identical imitations of it whose exteriors are mutative hyperbolic 3-manifolds. The equivariant versions with the mutative reduction property on the isometry group are also established. As a corollary, we have finitely many hyperbolic 3-manifolds with the same volume and the same isometry group.

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0. Introduction

By a *manifold pair*, we mean a pair (M, L) such that M is a compact connected oriented 3-manifold with or without boundary and L is $\emptyset$ or a proper (possibly disconnected) 1-submanifold. By the *exterior* $E(L, M)$ of a manifold pair (M, L), we mean the 3-manifold $\mathrm{cl}(M - N(L))$ for a tubular neighbourhood $N(L)$ of L in M. (Throughout this paper, we understand that $N(L) = \emptyset$ if $L = \emptyset$.) A manifold pair (M, L) is identified with the 3-manifold M if $L = \emptyset$ or called a (3,1)-*manifold pair* if $L \neq \emptyset$. A manifold pair (M, L) is said to be *good* if any 2-sphere component of ∂M intersects L in three or more points. We consider the concept of mutation on a manifold pair, associated with a symmetry of a closed separating surface of genus 2, due to Ruberman [4]. For our purpose, it is convenient to call a finite sequence of such mutations a

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mutation. To state this definition precisely, let F be a closed connected oriented surface. An involution ρ on F is called a *symmetry* of F if the orbit space F/ρ is homeomorphic to S^2. A manifold pair (M', L') is an *e-mutation* of a manifold pair (M, L) if there is a closed separating surface F of genus 2 in $\operatorname{int} E(L, M)$ such that (M', L') is a manifold pair obtained from (M, L) by cutting it along F and re-gluing by a symmetry ρ of F and the orientation of M' is inherited from M by the cut and re-gluing operation. In this case (M, L) is also an e-mutation of (M', L') and we can say without ambiguity that the manifold pairs (M, L), (M', L') are *e-mutative*.

DEFINITION. Two manifold pairs (M, L), (M', L') are *mutative* if there is a finite sequence of manifold pairs $(M^{(n)}, L^{(n)})$, $n = 0, 1, \ldots, s$, with $(M, L) = (M^{(0)}, L^{(0)})$ and $(M', L') = (M^{(s)}, L^{(s)})$ such that $(M^{(n)}, L^{(n)})$ and $(M^{(n+1)}, L^{(n+1)})$ are e-mutative for all n.

The mutative relation gives an equivalence relation among manifold pairs which we call the *mutation*. The mutation on (3,1)-manifold pairs generalizes a certain kind of Conway mutation on links, containing any Conway mutation on knots. (See Lemma 2.1.) When a distinction from the Conway mutation is emphasized, this mutation will be called a *manifold mutation*. Noting that a hyperbolic 3-manifold is unchanged under e-mutation associated with any symmetry of a closed *compressible* surface of genus 2, (see Lemma 2.2), we see from a result of Ruberman [4] that if (M, L) and (M', L') are mutative manifold pairs and $E(L, M)$ is hyperbolic, then $E(L', M')$ is also hyperbolic and we have the same hyperbolic volume $\operatorname{Vol} E(L, M) = \operatorname{Vol} E(L', M')$. The purpose of this paper is to construct from any given good (3,1)-manifold pair (M, L) finitely many mutative, non-diffeomorphic (3,1)-manifold pairs (M, L_*) with the hyperbolic covering property which are almost identical imitations of (M, L). This also enables us to construct from any given good 3-manifold M, finitely many mutative, non-diffeomorphic, hyperbolic 3-manifolds M_* which are normal imitations of M. These results are given in Theorems 2.4 and 2.5 of Section 2 together with stronger conditions on the hyperbolic volume and the covering isometry group. Equivariant versions of these results are not far from these results by advantages of the topological imitation theory (specially based on the main result of [3]) and we establish an equivariant version together with a mutative reduction property on the isometry group in Theorem 3.2 of Section 3. In Section 1, a slightly stronger version of the main result of [3] is established. Throughout this paper, terminologies of [3] will be used unless otherwise stated. In particular, a 3-manifold will be said to be *hyperbolic*, if the

3-manifold obtained from it by removing all of the torus boundary components has a complete hyperbolic structure with the non-torus boundary components totally geodesic.

1. A basic result in the topological imitation theory

For a manifold pair (M, L) and some sphere components S_i, $i = 1,2,\dots,s$, of ∂M, we call the operation of adding the cones over $(S_i, S_i \cap L)$, $i = 1,2,\dots,s$, to (M, L) a *spherical completion* of (M, L). For convenience, we allow the unoperated case $s = 0$ as a spherical completion. By a spherical completion, we can obtain from any manifold pair (M, L) a unique good manifold pair, which we denote by $(M, L)_\wedge$. By a *normal covering* of a good manifold pair (M, L), we mean a finite regular covering $p : (\tilde{M}, \tilde{L}) \to (M, L)$ with branch set a component union of L (possibly $\emptyset$), denoted by F_L, such that $\tilde{M}$ is connected. We denote by $G(\tilde{M} \to M)$ the covering transformation group of the covering $p|\tilde{M} \to M$. For a normal covering $p : (\tilde{M}, \tilde{L}) \to (M, L)$ of a good manifold pair (M, L), let L_0 be a component union (possibly $\emptyset$) of L such that $L_0 \supset L - F_L$. The good manifold pair $(\tilde{M}, \tilde{L}_0)_\wedge$ obtained uniquely from the manifold pair $(\tilde{M}, \tilde{L}_0)$ with $\tilde{L}_0 = p^{-1}L_0$ is simply denoted by $(\check{M}, \check{L})$ and called a *branch-missing good manifold pair* of the normal covering $p : (\tilde{M}, \tilde{L}) \to (M, L)$. The action of $G(\tilde{M} \to M)$ on $(\tilde{M}, \tilde{L}_0)$ naturally extends to an action on $(\check{M}, \check{L})$.

DEFINITION. A good manifold pair (M, L) has the *hyperbolic covering property* if $E(\check{L}, \check{M})$ is hyperbolic for all branch-missing good manifold pairs $(\check{M}, \check{L})$ of any normal covering $p : (\tilde{M}, \tilde{L}) \to (M, L)$.

When $L \neq \emptyset$, this definition coincides with the definition given in [3]. We see that M (that is, $(M, \emptyset)$) has the hyperbolic covering property if and only if M is a hyperbolic 3-manifold. Let $I = [-1,1]$. For a manifold pair (M, L), a smooth involution α on $(M, L) \times I$ is called a *reflection* in $(M, L) \times I$ if $\alpha((M, L) \times 1) = (M, L) \times (-1)$ and $\mathrm{Fix}(\alpha, (M, L) \times I)$ is a manifold pair.

DEFINITION. ([3]) Let α be a reflection in $(M, L) \times I$.

(1) α is *standard* if $\alpha(x, t) = (x, -t)$ for all $(x, t) \in M \times I$,
(2) α is *normal* if $\alpha(x, t) = (x, -t)$ for all $(x, t) \in \partial(M \times I) \cup N(L) \times I$, where $N(L)$ denotes a tubular neighbourhood of L in M,

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(3) α is *isotopically standard* if $f^{-1}\alpha f$ is standard for a diffeomorphism f of $M \times I$ isotopic to the identity by an isotopy relative to $\partial(M \times I) \cup N(L) \times I$,
(4) α is *isotopically almost standard* if $L \neq \emptyset$ and $\alpha|(M, L - a) \times I$ is isotopically standard for each component a of L.

An embedding ϕ from a manifold pair (M_*, L_*) to $(M, L) \times I$ is called a *reflector* of a reflection α in $(M, L) \times I$ if $\phi(M_*, L_*) = \mathrm{Fix}(\alpha, (M, L) \times I)$. The composite $q = p_1\phi : (M_*, L_*) \to (M, L)$ of a reflector ϕ of a reflection α in $(M, L) \times I$ and the projection $p_1 : (M, L) \times I \to (M, L)$ is called an *imitation*. (M_*, L_*) is also called an *imitation* of (M, L) with *imitation map* q. In particular, if the reflection α is normal or isotopically almost standard, then q is called a *normal imitation* or an *almost identical imitation*, respectively. When $q : (M_*, L_*) \to (M, L)$ is an almost identical imitation, we can write (M_*, L_*) as (M, L_*). Let $q : (M_*, L_*) \to (M, L)$ be a normal imitation . For any normal covering $p : (\tilde{M}, \tilde{L}) \to (M, L)$, we have the following commutative diagram:

$$\begin{array}{ccc} (\tilde{M}_*, \tilde{L}_*) & \xrightarrow{\tilde{q}} & (\tilde{M}, \tilde{L}) \\ p_* \downarrow & & p \downarrow \\ (M_*, L_*) & \xrightarrow{q} & (M, L) \end{array}$$

where $p_* : (\tilde{M}_*, \tilde{L}_*) \to (M_*, L_*)$ is a normal covering, called the *lift* of p by q and $\tilde{q} : (\tilde{M}_*, \tilde{L}_*) \to (\tilde{M}, \tilde{L})$ is a normal imitation called the *lift* of q by p.

DEFINITION. A normal imitation $q : (M_*, L_*) \to (M, L)$ of a good manifold pair (M, L) with $E(L_*, M_*)$ hyperbolic is *rigid* if

$$\mathrm{Isom}\, E(\tilde{L}_*, \tilde{M}_*) \cong G(\tilde{M} \to M)$$

for the lift $p_* : (\tilde{M}_*, \tilde{L}_*) \to (M_*, L_*)$ of any normal covering $p : (\tilde{M}, \tilde{L}) \to (M, L)$ by q.

DEFINITION. A normal imitation $q : (M_*, L_*) \to (M, L)$ of a good manifold pair (M, L) such that (M_*, L_*) has the hyperbolic covering property is *J-rigid* for a positive integer J if we have the following (1) – (3):

(1) q is rigid,
(2) $$\mathrm{Isom}\, E(\check{L}_*, \check{M}_*) \cong G(\tilde{M} \to M)$$
for any branch-missing good manifold pair $(\check{M}_*, \check{L}_*)$ of the lift $p_* : (\tilde{M}_*, \tilde{L}_*) \to (M_*, L_*)$ of any normal covering $p : (\tilde{M}, \tilde{L}) \to (M, L)$ of degree $\leq J$ by q,

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(3) Every normal covering $p_* : (\tilde{M}_*, \tilde{L}_*) \to (M_*, L_*)$ of degree $\leq J$ is the lift of a normal covering $p : (\tilde{M}, \tilde{L}) \to (M, L)$ by q.

In this definition, note that when $L = \emptyset$, (2) is implied by (1). By a *good family*, we mean an infinite family $\mathfrak{I}$ of good manifold pairs (M_m, L_m) with m ranging over an infinite subset $I_{\mathbf{Z}}$ of Z. Such a good family $\mathfrak{I}$ is also called a 3-*family* if $L_m = \emptyset$ for all m, or a (3,1)-*family* if $L_m \neq \emptyset$ for all m. For a positive number C, such a good family $\mathfrak{I}$ is said to be C-*hyperbolic*, if the exterior $E(L_m, M_m)$ of each member $(M_m, L_m) \in \mathfrak{I}$ is a hyperbolic 3-manifold with

$$C < \operatorname{Vol} E(L_m, M_m) < \sup_{m \in I_{\mathbf{Z}}} \operatorname{Vol} E(L_m, M_m) < +\infty.$$

The following theorem is a slightly stronger version of the main theorem of [3].

THEOREM 1.1. *For any good* (3,1)-*manifold pair* (M, L) *and any positive integer* J *and any positive number* C, *there is a* C-*hyperbolic* (3,1)-*family* $\mathfrak{I}$ *whose member* (M, L_m), $m \in I_{\mathbf{Z}}$, *has the hyperbolic covering property and a* J-*rigid almost identical imitation map* $q_m : (M, L_m) \to (M, L)$.

For this proof, we use a notion of co-identical imitation. Let (X, L_X) be a manifold pair and F be a proper surface in X meeting L_X transversely such that F splits (X, L_X) into a good manifold pair (M, L) and a manifold pair (A, L_A), where L_X, L or L_A may be $\emptyset$.

DEFINITION. A normal imitation $q : (M_*, L_*) \to (M, L)$ is (A, L_A)-*co-identical* if the reflection α_X in $(X, L_X) \times I$ defined by the normal reflection α in $(M, L) \times I$ used for q and the standard reflection in $(A, L_A) \times I$ is isotopically standard.

LEMMA 1.2. *Let* (M, L) *be a good* (3,1)-*manifold pair. Assume that there is an incompressible, non-*∂*-parallel loop in* $(\partial M) \cap E(L, M)$ *which bounds a disk* D *in* $E(L_A, A)$. *Then for any positive number* C, *there is a* C-*hyperbolic* (3,1)-*family* $\mathfrak{I}$ *whose member* (M, L_m) *has the hyperbolic covering property and a rigid* (A, L_A)-*co-identical almost identical imitation map* $q_m : (M, L_m) \to (M, L)$.

PROOF. Take a bi-collar $D \times I$ of D in $E(L_A, A)$ and regard it as a 2-handle attaching to M. Let $a = p \times I$ for a point $p \in \operatorname{int} D$ and $M^+ = M \cup D \times I$.

Then $(M^+, L \cup a)_\wedge$ is a good (3,1)-manifold pair and we obtain from [3, Main Theorem] a C-hyperbolic (3,1)-family $\mathfrak{I}^+$ whose member $(M^+, L_m \cup a)_\wedge, m \in I_{\mathbf{Z}}$, has the hyperbolic covering property and a rigid almost identical imitation $q_m^+ : (M^+, L_m \cup a)_\wedge \to (M^+, L \cup a)_\wedge$. Identifying $E(a, M^+)$ with M, the rigid almost identical imitation map q_m^+ defines a rigid (A, L_A)-co-identical almost identical imitation map $q_m : (M, L_m) \to (M, L)$ such that (M, L_m) has the hyperbolic covering property. Hence $\mathfrak{I} = \{(M, L_m) | m \in I_{\mathbf{Z}}\}$ is a desired C-hyperbolic (3,1)-family. This completes the proof.

LEMMA 1.3. *Let (M, L) be a good (3,1)-manifold pair and $O(s) = \bigcup_{i=1}^{s} O_i$ be an s-component trivial link in $E(L, M)$ and J be a positive integer. Let $q : (M, L_*) \to (M, L)$ be an almost identical imitation obtained from an almost identical imitation $q^O : (M, L_\# \cup O(s)) \to (M, L \cup O(s))$ by the $(1/m_i)$-Dehn surgery along O_i such that $m_i \equiv 0 \pmod{J!}$ for all i. Then every normal covering $(\tilde{M}_*, \tilde{L}_*) \to (M, L_*)$ of degree $\leq J$ is the lift of a normal covering $(\tilde{M}, \tilde{L}) \to (M, L)$ by q.*

PROOF. Every normal covering $p_* : (\tilde{M}_*, \tilde{L}_*) \to (M, L_*)$ of degree $\leq J$ induces a normal unbranched covering $p_E^O : \tilde{E}(L_\# \cup O(s), M) \to E(L_\# \cup O(s), M)$ by restriction, which lifts each meridian of $O(s)$ in $E(L_\# \cup O(s), M)$ trivially by our assumption on the Dehn surgery coefficients. Hence this covering p_E^O extends to a normal unbranched covering $p_E : \tilde{E}(L_\#, M) \to E(L_\#, M)$ which defines a normal covering $p : (\tilde{M}, \tilde{L}_\#) \to (M, L_\#)$. Since $(M, L_\# \cup O(s))$ is an almost identical imitation of $(M, L \cup O(s))$, we see that each component of O(s) is null-homotopic in $E(L_\#, M)$, so that the normal covering $p_E : \tilde{E}(L_\#, M) \to E(L_\#, M)$ lifts $O(s)$ trivially. Since the covering $p : (\tilde{M}, \tilde{L}_\#) \to (M, L_\#)$ is the lift of a normal covering $(\tilde{M}, \tilde{L}) \to (M, L)$ by the imitation map q^O, it follows that every normal covering $p_* : (\tilde{M}_*, \tilde{L}_*) \to (M, L_*)$ of degree $\leq J$ is the lift of a normal covering $(\tilde{M}, \tilde{L}) \to (M, L)$ by the imitation map q. This completes the proof.

PROOF OF THEOREM 1.1. By [3, Lemma 5.1] and Lemma 1.3, we may consider that (M, L) has the hyperbolic covering property and $\operatorname{Vol} E(L, M) \geq C$. Let (B, t) be a trivial basic tangle for (M, L) with complement (M', L'). Let O be a trivial knot in $B - t$ and $(B^{(0)}, t^{(0)}), t^{(0)} = O \cap B^{(0)}$, a piece tangle of O in $(B, t \cup O)$ (see Figure 1). Let $B' = B - \operatorname{int} B^{(0)}$ and $(t \cup O)' = B' \cap (t \cup O)$. By Lemma 1.2, we have a $(B', (t \cup O)')$-co-identical almost identical imitation

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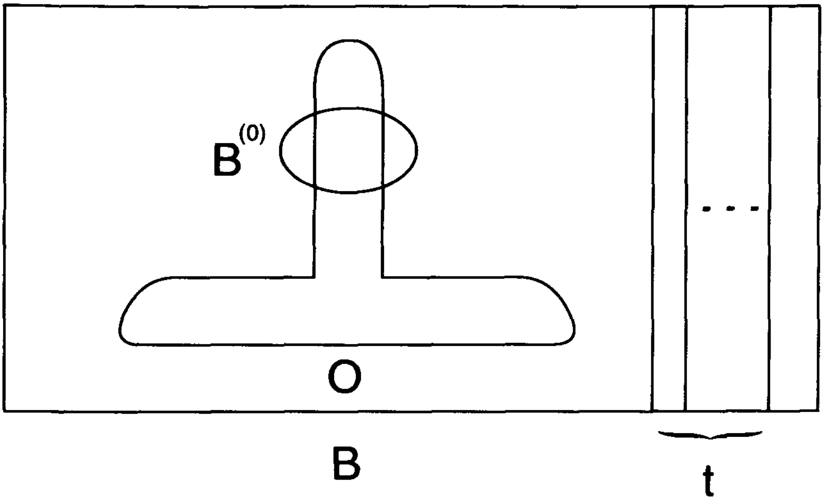

FIGURE 1

$(B^{(0)}, t_*^{(0)}) \to (B^{(0)}, t^{(0)})$ such that $(B^{(0)}, t_*^{(0)})$ has the hyperbolic covering property and no periodic map. Replacing $(B^{(0)}, t^{(0)})$ by $(B^{(0)}, t_*^{(0)})$ and $(B', (t \cup O)')$ by its almost identical imitation $(B', (t \cup O)'_*)$ with hyperbolic covering property, we obtain an almost identical imitation $(B, t_\# \cup O) \to (B, t \cup O)$. Next, replacing (M', L') by its $(B, t \cup O)$-co-identical almost identical imitation (M', L'_*) with hyperbolic covering property, and $(B, t \cup O)$ by $(B, t_\# \cup O)$, we obtain an almost identical imitation $q : (M, L_\# \cup O) \to (M, L \cup O)$. By Myers' gluing lemma, $E(L_\# \cup O, M)$ is hyperbolic. Let $q_m : (M, L_m) \to (M, L)$ be the almost identical imitation obtained from $q : (M, L_\# \cup O) \to (M, L \cup O)$ by taking the $1/m$-Dehn surgery along O. By an argument of [3, Proof of Main Theorem], we have an integer $m_0 > 0$ such that:

(0) (M, L_m) has the hyperbolic covering property for all $m \geq m_0$ and $\Im_0 = \{(M, L_m) | m \geq m_0\}$ forms a C-hyperbolic (3,1)-family,

(1) $q_m : (M, L_m) \to (M, L)$ is rigid for all $m \geq m_0$,

(2) Isom $E(\check{L}_m, \check{M}_m) \cong G(\tilde{M} \to M)$ for any branch-missing good manifold pair $(\check{M}_m, \check{L}_m)$ of the lift $(\tilde{M}_m, \tilde{L}_m) \to (M, L_m)$ of any normal covering $(\tilde{M}, \tilde{L}) \to (M, L)$ of degree $\leq J$ by the imitation map q_m.

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Let $I_{\mathbf{Z}} = \{m \in \mathbf{Z} | m \geq m_0, \ m \equiv 0 \pmod{J!}\}$. Then by Lemma 1.3 the subfamily $\mathfrak{I} = \{(M, L_m) | m \in I_{\mathbf{Z}}\}$ of $\mathfrak{I}_0$ is a desired family. This completes the proof.

2. Mutative manifold pairs

We first observe a relation between the manifold mutation and the Conway mutation. A (3,1)-manifold pair (M', L') is called a *Conway mutation* of a (3,1)-manifold pair (M, L) (or (M, L) and (M', L') are said to be *Conway mutative*) if there are a separating 4-pointed sphere S for (M, L), called a *Conway sphere*, and a symmetry ρ of S with $\rho(S \cap L) = S \cap L$ and $\mathrm{Fix}(\rho, S) \cap L = \emptyset$, called a *symmetry of* the *Conway sphere* S, such that (M', L') is obtained from (M, L) by cutting along S and re-gluing by ρ. For example, if L is the split union of two copies of a non-trivial knot K in S^3 and L' is the split union of the knot sum $K \# K$ and a trivial knot O in S^3, then (S^3, L) and (S^3, L') are Conway mutative, but not manifold mutative (cf. Lemma 2.2). However, we have the following:

LEMMA 2.1. *Two* (3,1)*-manifold pairs* (M, L) *and* (M', L') *are mutative if they are Conway mutative by a symmetry* ρ *of a Conway sphere* S *for* (M, L) *such that* p, $\rho(p)$ *belong to the same component of* L *for each point* $p \in S \cap L$.

PROOF. Let $S \cap L = \{p, \rho(p), p', \rho(p')\}$. Let ℓ and ℓ' be the components of L containing p, $\rho(p)$ and p', $\rho(p')$, respectively. When $\ell \neq \ell'$, we have a closed separating surface F of genus 2 in $E(L, M)$ obtained from S by the 1-handle surgeries along any two arcs $a \subset \ell$ and $a' \subset \ell'$ with $\partial a = p \cup \rho(p)$ and $\partial a' = p' \cup \rho(p')$. Since the symmetry ρ on S induces a symmetry of F, we see that (M, L) and (M', L') are mutative. Assume that $\ell = \ell'$. Let $a, a' \subset \ell$ be arcs with $\partial a = p \cup \rho(p)$ and $\partial a' = p' \cup \rho(p')$. If $a \cap a' = \emptyset$, then we see from the same reason that (M, L) and (M', L') are mutative. If $a \supset a'$, then we have also a closed separating surface F of genus 2 (with a symmetry induced from ρ) by the 1-handle surgeries on S along first a' and then a. Hence (M, L) and (M', L') are mutative. It is similar for $a \subset a'$. Assume that a contains only one point of p', $\rho(p')$. Then note that ρ is the composite of symmetries ρ', ρ'' of S such that $\rho'(p) = p'$ and $\rho'(\rho(p)) = \rho(p')$ and $\rho''(p) = \rho(p')$ and $\rho''(\rho(p)) = p'$. Applying the above argument to ρ' and ρ'', we see that (M, L) and (M', L') are mutative. This completes the proof.

The following lemma is suggested by Ruberman [4, Lemma 5.3].

LEMMA 2.2. *Let two manifold pairs* (M, L) *and* (M', L') *be e-mutative by a symmetry of a closed separating surface* F *of genus 2. If* $E(L, M)$ *is simple and* F *is compressible in* $E(L, M)$, *then we have a diffeomorphism* $(M, L) \cong (M', L')$ *sending* M *to* M' *orientation-preservingly.*

PROOF. Let ℓ be a compressible simple loop in F. Since F is of genus 2, we see that ℓ is isotopic to a ρ-invariant simple loop ℓ_* with $\mathrm{Fix}(\rho, \ell_*) = \emptyset$ or S^0 according to whether ℓ is null-homologous in F or not. In fact, we have an orientation-preserving diffeomorphism f of F such that $f(\ell)$ is ρ-invariant with $\mathrm{Fix}(\rho, f(\ell)) = \emptyset$ or S^0 according to whether ℓ is null-homologous in F or not. By [1, p.188] , f is isotopic to a ρ-equivariant diffeomorphism f_*. Then $\ell_* = (f_*)^{-1} f(\ell)$ is ρ-invariant and isotopic to ℓ, as desired. Thus, (M', L') is obtained from (M, L) by cutting along one or two disjoint tori and re-gluing by symmetries induced from ρ. Since $E(L, M)$ is simple, any torus is compressible or ∂-parallel in $E(L, M)$. For a compressible torus T with a symmetry ρ', we see (from an argument similar to the above argument on F) that a compressible simple loop ℓ' in T is isotopic to a ρ'-invariant simple loop ℓ'_* with $\mathrm{Fix}(\rho', \ell'_*) = S^0$. This implies that if (M'', L'') is obtained from (M, L) by cutting along a compressible torus in $E(L, M)$ and re-gluing by a symmetry, then (M'', L'') is obtained from (M, L) by cutting along a sphere and re-gluing by a symmetry, so that $(M'', L'') \cong (M, L)$. If (M'', L'') is obtained from (M, L) by cutting along a ∂-parallel torus in $E(L, M)$ and re-gluing by a symmetry, we have $(M'', L'') \cong (M, L)$ directly. This completes the proof.

The following corollary is direct from Lemma 2.2 and the main result of Ruberman [4].

COROLLARY 2.3. *If two* (3,1)*-manifold pairs* (M, L) *and* (M', L') *are mutative and* $E(L, M)$ *is hyperbolic, then* $E(L', M')$ *is hyperbolic and we have* $\mathrm{Vol}\, E(L, M) = \mathrm{Vol}\, E(L', M')$.

For two normal imitations $q : (M_*, L_*) \to (M, L)$ and $q : (M'_*, L'_*) \to (M, L)$ of a good manifold pair (M, L) and a positive integer J, we make the following two definitions:

DEFINITION. q, q' are *properly mutative* if $(\tilde{M}_*, \tilde{L}_*)$ and $(\tilde{M}'_*, \tilde{L}'_*)$ are mutative and $E(\tilde{L}_*, \tilde{M}_*)$ and $E(\tilde{L}'_*, \tilde{M}'_*)$ are non-diffeomorphic for the lifts $(\tilde{M}_*, \tilde{L}_*) \to (M_*, L_*)$ and $(\tilde{M}'_*, \tilde{L}'_*) \to (M'_*, L'_*)$ of any normal covering $p : (\tilde{M}, \tilde{L}) \to (M, L)$ by q, q'.

DEFINITION. q, q' are *J-properly mutative* if q, q' are properly mutative and $E(\breve{L}_*, \breve{M}_*)$ and $E(\breve{L}'_*, \breve{M}'_*)$ are non-diffeomorphic for any branch-missing good manifold pairs $(\breve{M}_*, \breve{L}_*)$ and $(\breve{M}'_*, \breve{L}'_*)$ of the lifts $(\tilde{M}_*, \tilde{L}_*) \to (M_*, L_*)$ and $(\tilde{M}'_*, \tilde{L}'_*) \to (M'_*, L'_*)$ of any normal covering $p : (\tilde{M}, \tilde{L}) \to (M, L)$ of degree $\leq J$ by q, q'.

We shall obtain the following mutative version of Theorem 1.1:

THEOREM 2.4. *For any good* (3,1)*-manifold pair* (M, L) *and any positive integers J, N and any positive number C, there are C-hyperbolic* (3,1)*-families* $\mathfrak{I}^{(n)}$, $n = 1, 2, \ldots, 2^N$, *with the same index set* $I_{\mathbf{Z}}$ *whose member* $(M, L_m^{(n)}) \in \mathfrak{I}^{(n)}$, $m \in I_{\mathbf{Z}}$, *has the hyperbolic covering property and a J-rigid almost identical imitation map* $q_m^{(n)} : (M, L_m^{(n)}) \to (M, L)$ *such that* $q_m^{(n)}$ *and* $q_m^{(n')}$ *are J-properly mutative for all* n, n' *with* $n \neq n'$ *and all* $m \in I_{\mathbf{Z}}$.

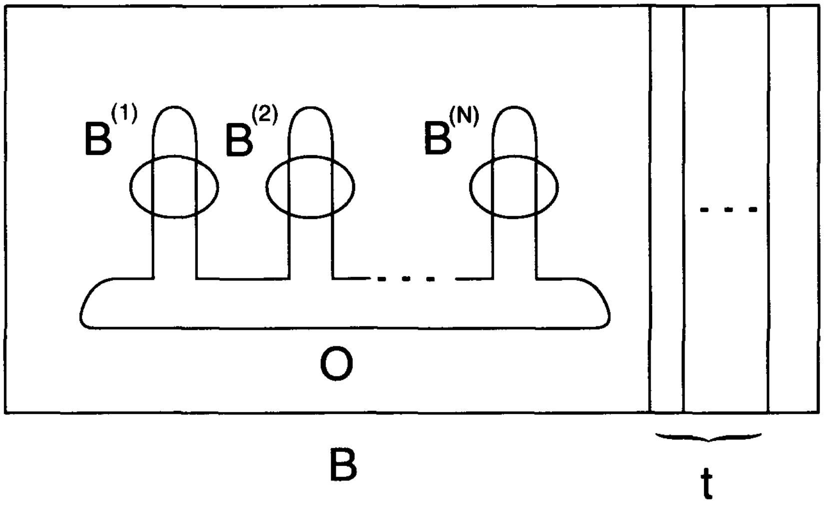

FIGURE 2

PROOF. The proof will be analogous to the proof of Theorem 1.1 except that we consider N trivial 2-string tangles instead of one such tangle as the starting point. By [3, Lemma 5.1] and Lemma 1.3, we may consider that (M, L) has the hyperbolic covering property and $\operatorname{Vol} E(L, M) \geq C$. Let (B, t) be a trivial basic tangle for (M, L) with complement (M', L'). Let O be an oriented trivial knot in $B - t$ and $(B^{(k)}, t^{(k)})$, $t^{(k)} = O \cap B^{(k)}$,

$k = 1,2,\ldots,N$, be piece tangles of O in $(B, t \cup O)$, illustrated in Figure 2. Let $B' = B - \cup_{k=1}^{N} \operatorname{int} B^{(k)}$ and $(t \cup O)' = B' \cap (t \cup O)$. By Lemma 1.2, we have a $(B', (t \cup O)')$-co-identical almost identical imitation $(B^{(k)}, t_*^{(k)}) \to (B^{(k)}, t^{(k)})$ such that $(B^{(k)}, t_*^{(k)})$ has the hyperbolic covering property and no periodic map. We assume that $\operatorname{Vol} E(t_*^{(k)}, B^{(k)})$, $k = 1,2, \ldots, N$, are mutually different. Replacing $(B^{(k)}, t^{(k)})$ by $(B^{(k)}, t_*^{(k)})$, $k = 1,2, \ldots, N$, and $(B', (t \cup O)')$ by its almost identical imitation $(B', (t \cup O)'_*)$ with hyperbolic covering property, we obtain an almost identical imitation $(B, t_\# \cup O) \to (B, t \cup O)$. Next, replacing (M', L') by its $(B, t \cup O)$-co-identical almost identical imitation (M', L'_*) with hyperbolic covering property, and $(B, t \cup O)$ by $(B, t_\# \cup O)$, we obtain an almost identical imitation $q : (M, L_\# \cup O) \to (M, L \cup O)$. By Myers' gluing lemma, $E(L_\# \cup O, M)$ is hyperbolic. Let $q_m : (M, L_m) \to (M, L)$ be the almost identical imitation map obtained from the imitation map $(M, L_\# \cup O) \to (M, L \cup O)$ by taking the $1/m$-Dehn surgery along O. The argument of [3, Proof of Main Theorem] and Lemma 1.3 show that there is an integer $m_0 > 0$ such that the family $\Im = \{(M, L_m) | m \in I_{\mathbf{Z}}\}$ with $I_{\mathbf{Z}} = \{m | m \geq m_0,\ m \equiv 0 \pmod{J!}\}$ forms a C-hyperbolic (3,1)-family whose member (M, L_m) has the hyperbolic covering property and whose member's almost identical imitation map $q_m : (M, L_m) \to (M, L)$ is J-rigid, as it is observed in the proof of Theorem 1.1. Let ρ_k be a symmetry of the Conway sphere $\partial B^{(k)}$ for (M, L). Using $\rho_k, k = 1,2,\ldots,N$, we obtain from $(M, L_\# \cup O)$ Conway mutative (3,1)-manifold pairs $(M, L_\# \cup O^{(n)}), n = 1, 2, \ldots, 2^N$ admitting almost identical imitation maps $(M, L_\# \cup O^{(n)}) \to (M, L \cup O)$. Since m_0 is taken sufficiently large, we may consider that the family $\Im^{(n)} = \{(M, L_m^{(n)}) | m \in I_{\mathbf{Z}}\}$, obtained by starting from $(M, L_\# \cup O^{(n)})$ instead of $(M, L_\# \cup O)$ is a C-hyperbolic (3,1)-family whose member $(M, L_m^{(n)})$ has the hyperbolic covering property and a J-rigid almost identical imitation $q_m^{(n)} : (M, L_m^{(n)}) \to (M, L)$. Let $E_m^{(n)} = E(L_m^{(n)}, M)$ and $E_\# = E(L_\#, M)$. Let $O_m^{(n)} \subset E_m^{(n)}$ be the core of the solid torus used for the $1/m$-Dehn surgery along $O^{(n)}$. Let $(\breve{M}_m^{(n)}, \breve{L}_m^{(n)})$ and $(\breve{M}_\#, \breve{L}_\#)$ be any branch-missing good manifold pairs of the lifts $(\tilde{M}_m^{(n)}, \tilde{L}_m^{(n)}) \to (M_m^{(n)}, L_m^{(n)})$ and $(\tilde{M}_\#, \tilde{L}_\#) \to (M_\#, L_\#)$ of any normal covering $p : (\tilde{M}, \tilde{L}) \to (M, L)$ of degree $\leq J$ by $q_m^{(n)}$ and $q_\#$. Since m_0 is taken sufficiently large, we can assume that $\breve{E}_m^{(n)} = E(\breve{L}_m^{(n)}, \breve{M}_m^{(n)})$ and $\breve{E}_\# = E(\breve{L}_\#, \breve{M}_\#)$ do not have the same volume if $\breve{E}_\#$ is hyperbolic, and that the (trivial) lift $\breve{O}_m^{(n)}$ of $O_m^{(n)}$ to $\breve{E}_m^{(n)}$ consists of the shortest geodesics for all $m \in I_{\mathbf{Z}}$ and all n. We show that $\breve{E}_m^{(n)}$ and $\breve{E}_m^{(n')}$ are not diffeomorphic for any n, n' with $n \neq n'$ and any $m \in I_{\mathbf{Z}}$. Suppose there is a diffeomorphism $f : \breve{E}_m^{(n)} \cong \breve{E}_m^{(n')}$, which is isotopic to an isometry f'. Then $f'(\breve{O}_m^{(n)}) = \breve{O}_m^{(n')}$. Note that f' preserves the meridian-longitude systems of $\breve{O}_m^{(n)}$

and $\breve{O}_m^{(n')}$ (up to orders and signs). If f' is orientation-reversing with respect to the orientations $\breve{E}_m^{(n)}$ and $\breve{E}_m^{(n')}$ induced from the orientation of M, then f' sends the slope $-1/m$ around $\breve{O}_m^{(n)}$ to the slope 1/m around $\breve{O}_m^{(n')}$ and hence $\breve{E}_{\#}$ must be diffeomorphic to $\breve{E}_{-2m}^{(n')}$, a contradiction. Thus, f' is orientation-preserving and slope-preserving. Then we have an orientation-preserving diffeomorphism $f'' : (\breve{E}_{\#}, \breve{O}^{(n)}) \cong (\breve{E}_{\#}, \breve{O}^{(n')})$ for the lifts $\breve{O}^{(n)}$, $\breve{O}^{(n')}$ of $O^{(n)}$, $O^{(n')}$ to $\breve{E}_{\#}$. Let $(\breve{B}^{(k)}, \breve{t}_*^{(k)})$ be the trivial lift of $(B^{(k)}, t_*^{(k)})$ to $\breve{E}_{\#}$. By [2], f'' is isotopic to a diffeomorphism f_* with $f_*(\breve{B}^{(k)}, \breve{t}_*^{(k)}) = (\breve{B}^{(k)}, \breve{t}_*^{(k)})$, $k = 1,2, \ldots, N$. Taking N so that $N > 1$, we see that f_* preserves the orientations of $\breve{O}^{(n)}$ and $\breve{O}^{(n')}$ inherited from the orientation of $O \cap B'$. For $n \neq n'$, there is a symmetry ρ_k used only for either construction of $O^{(n)}$ or $O^{(n')}$. Then f_* induces an orientation preserving self-diffeomorphism g of $(B^{(k)}, t_*^{(k)})$ inducing a non-identity map on $H_1(B^{(k)} - t_*^{(k)}; \mathbf{Z})$. By Mostow rigidity on the complete hyperbolic 3-manifold $B^{(k)} - t^{(k)}$ with totally geodesic boundary $\partial(B^{(k)} - t^{(k)})$, g is isotopic to a periodic map, which must be non-trivial, a contradiction. Thus, $\breve{E}_m^{(n)}$ and $\breve{E}_m^{(n')}$ are non-diffeomorphic for any n, n' with $n \neq n'$ and any $m \in I_{\mathbf{Z}}$. When we consider $\tilde{E}_m^{(n)} = E(\tilde{L}_m^{(n)}, \tilde{M}_m^{(n)})$ and $\tilde{E}_{\#} = E(\tilde{L}_{\#}, \tilde{M}_{\#})$ for any normal covering $p : (\tilde{M}, \tilde{L}) \to (M, L)$ instead of $\breve{E}_m^{(n)}$ and $\breve{E}_{\#}$, we can show that $q_m^{(n)}$ and $q_m^{(n')}$ $(n \neq n',\ m \in I_{\mathbf{Z}})$ are properly mutative for a similar reason, for the (trivial) lift $\tilde{O}_m^{(n)}$ of $O_m^{(n)}$ to $\tilde{E}_m^{(n)}$ still consists of the shortest geodesics. Hence $q_m^{(n)}$ and $q_m^{(n')}$ $(n \neq n',\ m \in I_{\mathbf{Z}})$ are J-properly mutative. This completes the proof.

The following is essentially a corollary to Theorem 2.4:

THEOREM 2.5. *For any good 3-manifold and any positive integers J, N and any positive number C, there are C-hyperbolic 3-families $\mathfrak{I}^{(n)}$, $n = 1, 2, \ldots, 2^N$, with the same index set $I_{\mathbf{Z}}$ whose member $M_m^{(n)} \in \mathfrak{I}^{(n)}$ has a J-rigid normal imitation map $q_m^{(n)} : M_m^{(n)} \to M$ such that $q_m^{(n)}$ and $q_m^{(n')}$ are properly mutative for all n and n' with $n \neq n'$ and all $m \in I_{\mathbf{Z}}$.*

PROOF. Let O be a trivial knot in $\operatorname{int} M$. By Theorem 2.4, we have (3,1)-manifold pairs $(M, O_*^{(n)})$, $n = 1, 2, \ldots, 2^N$, which have the hyperbolic covering property with $\operatorname{Vol} E(O_*^{(n)}, M) > C$ and J-rigid, properly mutative, almost identical imitation maps $q^{(n)} : (M, O_*^{(n)}) \to (M, O)$. By the $1/m$-Dehn surgery along $O_*^{(n)}$ and O, we obtain from $q^{(n)}$ a normal imitation map $q_m^{(n)} : M_m^{(n)} \to M$. By [3, Proof of Theorem 6.1] and Lemma 1.3, there is an integer $m_0 > 0$ such that if we take $I_{\mathbf{Z}} = \{m \in \mathbf{Z} | m \geq m_0, m \equiv 0 \pmod{J!}\}$, then $\mathfrak{I}^{(n)} = \{M_m^{(n)} | m \in I_{\mathbf{Z}}\}$ is a C-hyperbolic 3-family whose member's normal imitation map $q_m^{(n)}: M_m^{(n)} \to M$ is J-rigid. Let $O_m^{(n)} \subset M_m^{(n)}$ be the core of the solid torus used for the $1/m$-Dehn surgery along $O_*^{(n)}$. Since m_0 is taken sufficiently large, we can assume that $O_m^{(n)}$ is the shortest geodesic in $M_m^{(n)}$ for all n and all $m \in I_{\mathbf{Z}}$. Let $\tilde{M}_m^{(n)} \to M_m^{(n)}$ be the lift of any good covering $\tilde{M} \to M$ by $q_m^{(n)}$. Note that the lift $\tilde{O}_m^{(n)}$ of $O_m^{(n)}$ to $\tilde{M}_m^{(n)}$ consists of the shortest geodesics for all n and all $m \in I_{\mathbf{Z}}$. Let $\tilde{O}_*^{(n)}$ be the lift of $O_*^{(n)}$ to $\tilde{M}$. Suppose that there is a diffeomorphism $f: \tilde{M}_m^{(n)} \cong \tilde{M}_m^{(n')}$ for some n, n' with $n \neq n'$, which is homotopic to an isometry f'. Then $f'(\tilde{O}_m^{(n)}) = \tilde{O}_m^{(n')}$ and we have a diffeomorphism

$$E(\tilde{O}_*^{(n)}, \tilde{M}) \cong E(\tilde{O}_m^{(n)}, \tilde{M}_m^{(n)}) \cong E(\tilde{O}_m^{(n')}, \tilde{M}_m^{(n')}) \cong E(\tilde{O}_*^{(n')}, \tilde{M}).$$

This contradicts that $q^{(n)}$ and $q^{(n')}$ are properly mutative. Since $\tilde{M}_m^{(n)}$ and $\tilde{M}_m^{(n')}$ are mutative, we showed that $q_m^{(n)}$ and $q_m^{(n')}$ are properly mutative for all n, n' with $n \neq n'$ and all $m \in I_{\mathbf{Z}}$. This completes the proof.

3. Mutative manifold pairs with group action

In this section, we shall obtain an equivariant version of a combined result of Theorems 2.4 and 2.5. A *good G-manifold pair* is a good manifold pair (M, L) on which a finite group G acts faithfully so that the G-orbit pair $(M, L \cup \bar{F}(G, M))/G$ is a spherical completion of a good manifold pair, where $\bar{F}(G, M)$ denotes the union of the fixed point set $\mathrm{Fix}(g, M)$ for all non-trivial elements $g \in G$. A good G-manifold pair (M_*, L_*) is a *G-normal imitation* or a *G-almost identical imitation* of a good G-manifold pair (M, L) with *G-imitation map* $q: (M_*, L_*) \to (M, L)$ if q is a G-map and the G-orbit map $(M_*, \bar{F}(G, M_*) \cup L_*)/G \to (M, \bar{F}(G, M) \cup L)/G$ defined by q is a spherical completion of a normal imitation map or an almost identical imitation map to a good manifold pair, respectively. A G-normal imitation or a G-almost identical imitation is a normal imitation. First, we shall show the following:

LEMMA 3.1. *For any good G-manifold pair (M, L) with $L - \bar{F}(G, M) \neq \emptyset$ and any positive integer N and any positive number C, there are C-hyperbolic* (3,1)*-families $\mathfrak{F}^{(n)}$, $n = 1, 2, \ldots, 2^N$, with the same index set $I_{\mathbf{Z}}$ whose member $(M, L_m^{(n)}) \in \mathfrak{F}^{(n)}$ has a G-almost identical imitation map $q_m^{(n)}: (M, L_m^{(n)}) \to (M, L)$ with* Isom $E(L_m^{(n)}, M) \cong G$ *such that $(M, L_m^{(n)})$ and $(M, L_m^{(n')})$ are mutative and have non-diffeomorphic exteriors for all n, n' with $n \neq n'$ and all $m \in I_{\mathbf{Z}}$. Further, we have the following mutative reduction property:*

For any proper subgroup H of G, there are C-hyperbolic (3,1)-families $\mathfrak{I}^{(n)}_{(H)}$, $n = 1, 2, \ldots, 2^N$, with the index set $I_{\mathbf{Z}}$ such that $(M, L^{(n)}_{m(H)}) \in \mathfrak{I}^{(n)}_{(H)}$ has an H-normal imitation map $q^{(n)}_{m(H)} : (M, L^{(n)}_{m(H)}) \to (M, L)$ with $\operatorname{Isom} E(L^{(n)}_{m(H)}, M) \cong H$ and any two of $(M, L^{(n)}_{m(H)})$, $(M, L^{(n')}_{m(H)})$ and $(M, L^{(n)}_{m})$ are mutative and have non-diffeomorphic exteriors for all n, n' with $n \neq n'$ and all $m \in I_{\mathbf{Z}}$.

PROOF. We use the notations in the proof of Theorem 2.4 by considering the present good G-manifold pair (M, L) as a branch-missing good manifold pair $(\breve{M}, \breve{L})$ of a normal covering $(\tilde{M}, \tilde{L}) \to (M, L)$ of a good manifold pair (M, L) with $G(\tilde{M} \to M) \cong G$ and taking $J = |G|$. By Theorem 2.4, we have C-hyperbolic (3,1)-families $\breve{\mathfrak{I}}^{(n)}$, $n = 1, 2, \ldots, 2^N$, with the same index set $I_{\mathbf{Z}}$ whose member $(\breve{M}, \breve{L}^{(n)}_m) \in \breve{\mathfrak{I}}^{(n)}$ has a G-almost identical imitation map $\breve{q}^{(n)}_m : (\breve{M}, \breve{L}^{(n)}_m) \to (\breve{M}, \breve{L})$ with $\operatorname{Isom} E(\breve{L}^{(n)}_m, \breve{M}) \cong G$ such that $(M, \breve{L}^{(n)}_m)$ and $(M, \breve{L}^{(n')}_m)$ are mutative and have non-diffeomorphic exteriors for all n, n' with $n \neq n'$ and all $m \in I_{\mathbf{Z}}$, where we note that $\breve{M}^{(n)}_m = \breve{M}_\# = \breve{M}$ by the assumption that $L - \bar{F}(G, M) \neq \emptyset$. We show that these families have the mutative reduction property. For this purpose, we consider the trivial lift $(\breve{B}^{(1)}, \breve{t}^{(1)}_*)$ of $(B^{(1)}, t^{(1)}_*)$ to $(\breve{M}, \breve{L}_\# \cup \breve{O}^{(n)})$. For a component $(B^{(1)}, t^{(1)}_*)_0$ of $(\breve{B}^{(1)}, \breve{t}^{(1)}_*)$ and the left cosets $Hh_0, Hh_1, \ldots, Hh_s$ $(h_0 = 1)$ of H in G, the sets of tangles $(\breve{B}^{(1)}, \breve{t}^{(1)}_*)_i = Hh_i(B^{(1)}, t^{(1)}_*)_0 = \{hh_i((B^{(1)}, t^{(1)}_*)_0 | h \in H\}$, $i = 0, 1, \ldots, s$, form a mutually disjoint H-invariant division of $(\breve{B}^{(1)}, \breve{t}^{(1)}_*)$. Let $(\breve{M}, \breve{L}_\# \cup \breve{O}^{(n)}_{(H)})$ be the (3,1)-manifold pair obtained from $(\breve{M}, \breve{L}_\# \cup \breve{O}^{(n)})$ by the Conway mutation along each of the Conway spheres in $(\breve{B}^{(1)}, \breve{t}^{(1)}_*)_0$ using the symmetry ρ_1. We have a C-hyperbolic (3,1)-family $\breve{\mathfrak{I}}^{(n)}_{(H)} = \{(\breve{M}, \breve{L}^{(n)}_{m(H)}) | m \in I_{\mathbf{Z}}\}$ by using $(\breve{M}, \breve{L}_\# \cup \breve{O}^{(n)}_{(H)})$ instead of $(\breve{M}, \breve{L}_\# \cup \breve{O}^{(n)})$, whose member has an H-normal imitation $\breve{q}^{(n)}_{m(H)} : (\breve{M}, \breve{L}^{(n)}_{m(H)}) \to (\breve{M}, \breve{L})$. Since m_0 is taken sufficiently large, we can assume that $\breve{E}^{(n)}_{m(H)} = E(\breve{L}^{(n)}_{m(H)}, \breve{M})$ and $\breve{E}_\#$ do not have the same volume if $\breve{E}_\#$ is hyperbolic and the core $\breve{O}^{(n)}_{m(H)} \subset \breve{E}^{(n)}_{m(H)}$ of the solid tori used for the $1/m$-Dehn surgery of $\breve{E}_\#$ along $\breve{O}^{(n)}_{(H)}$ is invariant under any isometry of $\breve{E}^{(n)}_{m(H)}$. Let $H' = \operatorname{Isom} \breve{E}^{(n)}_{m(H)}$. Note that there is an action of H on $E = E(\breve{O}^{(n)}_{m(H)}, \breve{E}^{(n)}_{m(H)})$ induced by the action of H on $(\breve{E}^{(n)}_{m(H)}, \breve{O}^{(n)}_{m(H)})$. By Mostow rigidity, H' is isotopic (in E) to the subgroup of $\operatorname{Isom} E$ consisting of elements preserving the meridian system of $\breve{O}^{(n)}_{m(H)}$ (up to orders and signs). By [3, Lemma 5.3, Proof of Main Theorem] and the proof of Theorem 2.4, H' acts on $(\breve{B}^{(1)}, \breve{t}^{(1)}_*)_0$. H translates the components of the sublink of $\breve{O}^{(n)}_{m(H)}$ meeting $(\breve{B}^{(1)}, \breve{t}^{(1)}_*)_0$ transitively and hence by Mostow rigidity applied to the action H

on E, there is a monomorphism $\phi : H \to H'$ so that $\phi(H)$ translates the tangles of $(\breve{B}^{(1)}, \breve{t}_*^{(1)})_0$ transitively. If $|H| < |H'|$, then $(B^{(1)}, t_*^{(1)})$ must have a non-trivial periodic map, a contradiction. Hence $H \cong H'$. Any two of $(\breve{M}, \breve{L}^{(n)}_{m(H)})$, $(\breve{M}, \breve{L}^{(n')}_{m(H)})$ and $(\breve{M}, \breve{L}^{(n)}_m)$ are mutative and can be seen from the proof of Theorem 2.4 to have non-diffeomorphic exteriors for all n, n' with $n \neq n'$ and all $m \in I_{\mathbf{Z}}$. This completes the proof.

The following theorem is a main result of this section.

THEOREM 3.2. *For any good G-manifold pair (M, L) and any positive integer N and any positive number C, there are C-hyperbolic good families $\mathfrak{I}^{(n)}$, $n = 1, 2, \dots, 2^N$, with the same index set $I_{\mathbf{Z}}$ whose member $(M_m^{(n)}, L_m^{(n)}) \in \mathfrak{I}^{(n)}$ has a G-normal (or almost G-identical if $L \cup \bar{F}(G, M) \neq \emptyset$) imitation map $q_m^{(n)} : (M_m^{(n)}, L_m^{(n)}) \to (M, L)$ with* $\operatorname{Isom} E(L_m^{(n)}, M_m^{(n)}) \cong G$ *such that $(M_m^{(n)}, L_m^{(n)})$ and $(M_m^{(n')}, L_m^{(n')})$ are mutative and have non-diffeomorphic exteriors for all n, n' with $n \neq n'$ and all $m \in I_{\mathbf{Z}}$. Further, we have the following mutative reduction property:*

For any proper subgroup H of G, there are C-hyperbolic (3,1)*-families $\mathfrak{I}^{(n)}_{(H)}$, $n = 1, 2, \dots, 2^N$, with the index set $I_{\mathbf{Z}}$ such that $(M^{(n)}_{m(H)}, L^{(n)}_{m(H)}) \in \mathfrak{I}^{(n)}_{(H)}$ has an H-normal imitation map $q^{(n)}_{m(H)} : (M^{(n)}_{m(H)}, L^{(n)}_{m(H)}) \to (M, L)$ with* $\operatorname{Isom} E(L^{(n)}_{m(H)}, M^{(n)}_{m(H)}) \cong H$ *and any two of $(M^{(n)}_{m(H)}, L^{(n)}_{m(H)})$, $(M^{(n')}_{m(H)}, L^{(n')}_{m(H)})$ and $(M_m^{(n)}, L_m^{(n)})$ are mutative and have non-diffeomorphic exteriors for all n, n' with $n \neq n'$ and all $m \in I_{\mathbf{Z}}$.*

PROOF. When $L - \bar{F}(G, M) \neq \emptyset$, the result is direct from Lemma 3.1. We assume that $L \subset \bar{F}(G, M)$. Let O be a G-invariant trivial link in $\operatorname{int} M$ such that the G-orbit O/G is a trivial knot in the G-orbit space of $M - \bar{F}(G, M)$. By Lemma 3.1, we have mutually mutative good G-manifold pairs $(M, L \cup O^{(n)})$, $n = 1, 2, \dots, 2^N$, with G-almost identical imitation maps $q^{(n)} : (M, L \cup O^{(n)}) \to (M, L \cup O)$ such that the exteriors $E^{(n)} = E(L \cup O^{(n)}, M)$ are mutually non-diffeomorphic hyperbolic 3-manifolds with $\operatorname{Vol} E^{(n)} > C$ and $\operatorname{Isom} E^{(n)} \cong G$ and we have the following mutative reduction property: For any proper subgroup H of G, there are mutative good H-manifold pairs $(M, L \cup O^{(n)}_{(H)})$ with H-normal imitation maps $q^{(n)}_{(H)} : (M, L \cup O^{(n)}_{(H)}) \to (M, L \cup O)$, $n = 1, 2, \dots, 2^N$, such that $(M, L \cup O^{(n)}_{(H)})$ and $(M, L \cup O^{(n)}_{(H)})$ are mutative for any n and the exteriors $E^{(n)}_{(H)} = E(L \cup O^{(n)}_{(H)}, M)$ are mutually non-diffeomorphic hyperbolic 3-manifolds with $\operatorname{Isom} E^{(n)}_{(H)} \cong H$ for all n. Let $q_m^{(n)} : (M_m^{(n)}, L_m^{(n)}) \to (M, L)$ be a G-normal (or G-almost identical if $\bar{F}(G, M) \neq \emptyset$) imitation map, obtained

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from $q^{(n)}$ by the $1/m$-Dehn surgery along each component of $O^{(n)}$ and O. Let $O_m^{(n)} \subset M_m^{(n)}$ be the core of the solid tori used for the $1/m$-Dehn surgery along $O^{(n)}$. Then the families $\mathfrak{I}^{(n)} = \{(M_m^{(n)}, L_m^{(n)}) | m \in I_{\mathbf{Z}}\}$, $n = 1, 2, \ldots, 2^N$, for an index set $I_{\mathbf{Z}}$ ($\subset \mathbf{Z}$), are C-hyperbolic good families whose member's exteriors $E_m^{(n)} = (M_m^{(n)}, L_m^{(n)})$ for all n and each $m \in I_{\mathbf{Z}}$ are mutually non-diffeomorphic hyperbolic 3-manifolds with Isom $E_m^{(n)} \cong G$, (cf. proof of Theorem 2.5). Let $q_{m(H)}^{(n)} : (M_{m(H)}^{(n)}, L_{m(H)}^{(n)}) \to (M, L)$ be an H-normal imitation map obtained from $q_{(H)}^{(n)}$ by the $1/m$-Dehn surgery along $O_{(H)}^{(n)}$, O. By replacing $I_{\mathbf{Z}}$ with an infinite subset, also denoted by $I_{\mathbf{Z}}$, we see that $\mathfrak{I}_{(H)}^{(n)} = \{(M_{m(H)}^{(n)}, L_{m(H)}^{(n)}) | m \in I_{\mathbf{Z}}\}$, $n = 1, 2, \ldots, 2^N$, are C-hyperbolic good families whose member's exteriors $E_{m(H)}^{(n)} = E(M_{m(H)}^{(n)}, L_{m(H)}^{(n)})$ for all n and each $m \in I_{\mathbf{Z}}$ are mutually non-diffeomorphic hyperbolic 3-manifolds with Isom$E_{m(H)}^{(n)} \cong H$, (cf. proof of Theorem 2.5). Since any two of $(M_{m(H)}^{(n)}, L_{m(H)}^{(n)})$, $(M_{m(H)}^{(n')}, L_{m(H)}^{(n')})$ and $(M_m^{(n)}, L_m^{(n)})$ are mutative, the proof is completed.

The following is direct from Theorem 3.2 and Corollary 2.3:

COROLLARY 3.3. *For any good G-manifold pair (M, L) and any (possibly identical) subgroups $G^{(n)}$, $n = 1, 2, \ldots, N$, of G and any positive number C, we have $G^{(n)}$-normal imitations $(M^{(n)}, L^{(n)}) \to (M, L)$, (with $M^{(n)} = M$ if $L - \bar{F}(G, M) \neq \emptyset$), $n = 1, 2, \ldots, N$, whose exteriors $E(L^{(n)}, M^{(n)})$ for all n are mutually non-diffeomorphic hyperbolic 3-manifolds with the same volume $> C$ and with* Isom $E(L^{(n)}, M^{(n)}) \cong G^{(n)}$.

References

[1] J. S. Birman, *Braids, links and mapping class groups*, Annals of Math. Studies **82** (Princeton Univ. Press, Princeton, N. J., 1974).

[2] F. Bonahon and L. C. Siebenmann, 'The characteristic toric splitting of irreducible compact 3-orbifolds', *Math. Ann.* **278** (1987), 441–497.

[3] A. Kawauchi, 'Almost identical imitations of (3,1)-dimensional manifold pairs and the branched coverings', *Osaka J. Math.* **29** (1992), 299–327.

[4] D. Ruberman, 'Mutation and volumes of knots in S^3', *Invent. Math.* **90** (1987), 189–215.

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