ON REDUCIBILITY OF TRINOMIALS

by ANDREW BREMNER

(Received 12 November, 1979)

In Schinzel [1] the following interesting question is asked: does there exist an absolute constant K such that every trinomial in $\mathbb{Q}[x]$ has a factor irreducible in $\mathbb{Q}[x]$ which has at most K terms? The only known result appears to be that of Mrs. H. Smyczek, given in the above paper, that if K exists, then $K \geqslant 6$. We here extend this bound to $K \geqslant 8$ by exhibiting a trinomial in $\mathbb{Z}[x]$ which splits into the product of two irreducible factors, each having 8 terms.

We consider for suitable integers a, b, c, d, e, f, g and prime p, the product

$$(x^7+2pax^6+2pbx^5+2pcx^4+2pdx^3+2pex^2+2pfx+pg)\\ \times(x^7-2pax^6+2pbx^5-2pcx^4+2pdx^3-2pex^2+2pfx-pg) \quad (1)$$

where the two factors are $\mathbb{Q}$-irreducible by Eisenstein's criterion if $p \nmid g$. We automatically have that the coefficients of odd powers of x in the product are zero; and we impose restrictions on $a, b, \ldots, g$ by demanding that the coefficients of $x^{12}, x^{10}, \ldots, x^4$ be also zero, so that the product (1) is indeed a trinomial, of the form $x^{14}+Ax^2+B$.
This requires

$$b = pa^2,$$
$$d = pa(2c-p^2a^3),$$
$$f = pc^2+2pae-2p^3a^3(2c-p^2a^3),$$
$$g = 6p^2ac^2-12p^4a^4c+5p^6a^7+4p^2a^2e-\frac{2ce}{a},$$

and

$$e^2-e\left(4p^2a^2c-4p^4a^5+\frac{2c^2}{a}\right)+(2p^2ac^3+6p^4a^4c^2-11p^6a^7c+4p^8a^{10})=0.$$

The latter equation is just

$$e = 2p^2a^2c-2p^4a^5+\frac{c^2}{a}\pm D, \quad (2)$$

where

$$D^2 = \frac{1}{a^2}(c-p^2a^3)c(c^2+3p^2a^3c-3p^4a^6);$$

equivalently,

$$Y^2 = X(X-1)(X^2+3X-3) \quad (3)$$

Glasgow Math. J. **22** (1981) 155–156.

https://doi.org/10.1017/S0017089500004602 Published online by Cambridge University Press

with

$$Y = D/p^4a^5, \qquad X = c/p^2a^3. \tag{4}$$

Now (3) is just the equation of an elliptic curve; if we transform by $s = Y/X^2$, and $t = 1-(1/X)$, we obtain the curve

$$s^2 = t(1+3t-3t^2), \tag{5}$$

and it is straightforward to show by standard methods that the group of rational points on (5) is generated by $(0, 0)$ of order 2 and $(1, 1)$ of infinite order.

Calculation gives

$$7(1, 1) = \left(\frac{16351^2}{28489^2}, \ldots\right)$$

with corresponding value of X equal to

$$\frac{31^2 . 919^2}{2^4 . 3 . 5 . 7 . 17^2 . 19 . 59};$$

and we can thus see from (4) how to choose a, c, p—indeed we can take $p = 17$, $a = 2^2 . 3 . 5 . 7 . 19 . 59$, $c = 2^2 . 3^2 . 5^2 . 7^2 . 19^2 . 59^2 . 31^2 . 919^2$. Since $a \mid c$ all the coefficients are integers; and since $17 \nmid c^2/a$ we can ensure, by appropriate choice of sign for D in (2), that $17 \nmid e$, whence $17 \nmid g$ as required for applying the Eisenstein criterion. The remaining coefficients can now be calculated explicitly, but are rather large.

REFERENCE

1. A. Schinzel, Some unsolved problems on polynomials, *Mat. Biblioteka* **25** (1963), 63–70.

Emmanuel College
Cambridge CB2 3AP