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THE MULTIPLIER ALGEBRA AND BSE PROPERTY OF THE DIRECT SUM OF BANACH ALGEBRAS

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Abstract

The notion of BSE algebras was introduced and first studied by Takahasi and Hatori and later studied by Kaniuth and Ülger. This notion depends strongly on the multiplier algebra $M(\mathcal{A})$ of a commutative Banach algebra $\mathcal{A}$. In this paper we first present a characterisation of the multiplier algebra of the direct sum of two commutative semisimple Banach algebras. Then as an application we show that $\mathcal{A} \oplus \mathcal{B}$ is a BSE algebra if and only if $\mathcal{A}$ and $\mathcal{B}$ are BSE. We also prove that if the algebra $\mathcal{A} \times_\theta \mathcal{B}$ with θ-Lau product is a BSE algebra and $\mathcal{B}$ is unital then $\mathcal{B}$ is a BSE algebra. We present some examples which show that the BSE property of $\mathcal{A} \times_\theta \mathcal{B}$ does not imply the BSE property of $\mathcal{A}$, even in the case where $\mathcal{B}$ is unital.

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1. Introduction

Let $\mathcal{A}$ be a commutative Banach algebra. Throughout this paper $\Delta(\mathcal{A})$ denotes the set of all nonzero multiplicative linear functionals on $\mathcal{A}$. Then $\Delta(\mathcal{A})$ is a topological space with the Gelfand topology, called the Gelfand spectrum of $\mathcal{A}$.

A bounded continuous function σ on $\Delta(\mathcal{A})$ is called a *BSE function* if there exists a constant $C > 0$ such that for every finite number of $\varphi_1, \ldots, \varphi_n$ in $\Delta(\mathcal{A})$ and the same number of complex numbers $c_1, \ldots, c_n$, the inequality

$$\left|\sum_{j=1}^{n} c_j \sigma(\varphi_j)\right| \le C \left\|\sum_{j=1}^{n} c_j \varphi_j\right\|_{\mathcal{A}^*}$$

holds. The BSE norm of σ, $\|\sigma\|_{BSE}$, is defined to be the infimum of all such C. The set of all BSE functions is denoted by $C_{BSE}(\Delta(\mathcal{A}))$. Takahasi and Hatori [14] showed that under the norm $\|\cdot\|_{BSE}$, $C_{BSE}(\Delta(\mathcal{A}))$ is a commutative semisimple Banach algebra.

A bounded linear operator on $\mathcal{A}$ is called a *multiplier* if it satisfies $xT(y) = T(xy)$ for all $x, y \in \mathcal{A}$. The set $M(\mathcal{A})$ of all multipliers of $\mathcal{A}$ is a closed unital commutative subalgebra of the operator algebra $B(\mathcal{A})$, called the *multiplier algebra* of $\mathcal{A}$.

For each $T \in M(\mathcal{A})$ there exists a unique continuous function $\widehat{T}$ on $\Delta(\mathcal{A})$ such that $\widehat{T(a)}(\varphi) = \widehat{T}(\varphi)\hat{a}(\varphi)$ for all $a \in \mathcal{A}$ and $\varphi \in \Delta(\mathcal{A})$. See [9] for a proof. Write

$$\widehat{M(\mathcal{A})} = \{\widehat{T} : T \in M(\mathcal{A})\}.$$

We say that a commutative Banach algebra $\mathcal{A}$ without order is a *BSE algebra* (or it is said to have the *BSE property*) if $\mathcal{A}$ satisfies the condition

$$C_{BSE}(\Delta(\mathcal{A})) = \widehat{M(\mathcal{A})}.$$

Remark 1.1. Let $\mathcal{A}$ be a semisimple Banach algebra and $\Phi : \Delta(\mathcal{A}) \to \mathbb{C}$ be a continuous function such that $\Phi \cdot \widehat{\mathcal{A}} \subseteq \widehat{\mathcal{A}}$. We call Φ a multiplier of $\mathcal{A}$. This is another definition of a multiplier of a Banach algebra. In the presence of semisimplicity this definition is equivalent to the above definition, by considering $\Phi = \widehat{T}$; see [9] for more details. Define

$$\mathcal{M}(\mathcal{A}) = \{\Phi : \Delta(\mathcal{A}) \to \mathbb{C} : \Phi \text{ is continuous and } \Phi\widehat{A} \subseteq \widehat{A}\}.$$

When $\mathcal{A}$ is a semisimple Banach algebra, $\widehat{M(A)} = \mathcal{M}(A)$.

The abbreviation BSE stands for Bochner–Schoenberg–Eberlein and refers to the famous theorem, proved by Bochner and Schoenberg [2, 13] for the additive group of real numbers and by Eberlein [3] for general locally compact abelian groups G, saying that, in the above terminology, the group algebra $L^1(G)$ is a BSE algebra. (See [11] for a proof.)

The notions of BSE algebra and BSE functions were introduced and studied by Takahasi and Hatori [14, 15] and later by Kaniuth and Ülger [8].

A bounded net $(e_\alpha)_\alpha$ in $\mathcal{A}$ is called a bounded approximate identity for $\mathcal{A}$ if it satisfies $\|e_\alpha a - a\| \to 0$ for all $a \in \mathcal{A}$. A bounded net $(e_\alpha)_\alpha$ in $\mathcal{A}$ is called a Δ-weak bounded approximate identity for $\mathcal{A}$ if it satisfies $\varphi(e_\alpha) \to 1$ (equivalently, $\varphi(e_\alpha a) \to \varphi(a)$ for every $a \in \mathcal{A}$) for all $\varphi \in \Delta(\mathcal{A})$. Such approximate identities were studied in [6], where the first example was given of a semisimple commutative Banach algebra which has a Δ-weak approximate identity but does not possess a bounded approximate identity. As is shown in [14, Corollary 5], $\mathcal{A}$ has a Δ-weak bounded approximate identity if and only if $\widehat{M(\mathcal{A})} \subseteq C_{BSE}(\Delta(\mathcal{A}))$.

In this paper we first present a characterisation of the multiplier algebra of the direct sum of two semisimple Banach algebras. Then as an application we show that for two semisimple Banach algebras $\mathcal{A}$ and $\mathcal{B}$, $\mathcal{A} \oplus \mathcal{B}$ is BSE if and only if $\mathcal{A}$ and $\mathcal{B}$ are.

We also prove that if $\mathcal{A}$ and $\mathcal{B}$ are Banach algebras such that $\mathcal{B}$ is unital and $\mathcal{A} \times_\theta \mathcal{B}$ is BSE, then $\mathcal{B}$ is a BSE algebra, and we present some examples showing that the BSE property of $\mathcal{A} \times_\theta \mathcal{B}$ does not imply that of $\mathcal{A}$.

2. Direct sum of Banach algebras

Let $\mathcal{A}$ and $\mathcal{B}$ be two commutative Banach algebras. The direct sum algebra $\mathcal{A} \oplus \mathcal{B}$ of $\mathcal{A}$ and $\mathcal{B}$ is defined as the Cartesian product $\mathcal{A} \times \mathcal{B}$ with the algebra multiplication

$$(a, a') \cdot (b, b') = (aa', bb') \quad (a, a' \in \mathcal{A}, b, b' \in \mathcal{B}),$$

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and with norm

$$\|(a, b)\| = \|a\| + \|b\| \quad (a \in \mathcal{A}, b \in \mathcal{B}).$$

In this section we will give a characterisation of the multiplier algebra of the direct sum algebra $\mathcal{A} \oplus \mathcal{B}$, $\mathcal{M}(\mathcal{A} \oplus \mathcal{B})$, and then prove that for two semisimple Banach algebras $\mathcal{A}$ and $\mathcal{B}$, $\mathcal{A} \oplus \mathcal{B}$ is a BSE algebra if and only if $\mathcal{A}$ and $\mathcal{B}$ are BSE algebras. First we need to prove the following lemma.

Lemma 2.1. *Let $\mathcal{A}$ and $\mathcal{B}$ be two Banach algebras and*

$$E = \{(\varphi, 0) : \varphi \in \Delta(\mathcal{A})\}, \qquad F = \{(0, \psi) : \psi \in \Delta(\mathcal{B})\}.$$

Then $\Delta(\mathcal{A} \oplus \mathcal{B}) = E \cup F$.

Proof. It is obvious that $E \cup F \subseteq \Delta(\mathcal{A} \oplus \mathcal{B})$. For the reverse inclusion suppose that $(\varphi, \psi) \in \Delta(\mathcal{A} \oplus \mathcal{B}) \subseteq \mathcal{A}^* \oplus \mathcal{B}^*$. Then, for every $a_1, a_2 \in \mathcal{A}$ and $b_1, b_2 \in \mathcal{B}$,

$$(\varphi, \psi)(a_1a_2, b_1b_2) = (\varphi, \psi)(a_1, b_1) \cdot (\varphi, \psi)(a_2, b_2).$$

This means that, for all $a_1, a_2 \in \mathcal{A}$ and $b_1, b_2 \in \mathcal{B}$,

$$\begin{aligned} \varphi(a_1a_2) + \psi(b_1b_2) &= (\varphi(a_1) + \psi(b_1)) \cdot (\varphi(a_2) + \psi(b_2)) \\ &= \varphi(a_1)\varphi(a_2) + \varphi(a_1)\psi(a_2) + \psi(b_1)\varphi(a_2) + \psi(b_1)\psi(b_2). \end{aligned} \tag{I}$$

If we take $b_1 = b_2 = 0$, it follows that $\varphi(a_1a_2) = \varphi(a_1)\varphi(a_2)$ for all $a_1, a_2 \in \mathcal{A}$. And similarly, if we take $a_1 = a_2 = 0$ it follows that $\psi(b_1b_2) = \psi(b_1)\psi(b_2)$. Then $\varphi \in \Delta(\mathcal{A}) \cup \{0\}$ and $\psi \in \Delta(\mathcal{B}) \cup \{0\}$. Now if $\varphi = 0$, then $(\varphi, \psi) = (0, \psi) \in F$. If $\varphi \neq 0$, then (I) implies that

$$\varphi(a_1)\psi(a_2) + \psi(b_1)\varphi(a_2) = 0,$$

for all $a_1, a_2 \in \mathcal{A}$ and $b_1, b_2 \in \mathcal{B}$. If we set $a_1 = 0$ and a_2 such that $\varphi(a_2) \neq 0$, it follows that $\psi(b_1) = 0$ for all $b_1 \in \mathcal{B}$ and then $\psi = 0$. This means that $(\varphi, \psi) = (\varphi, 0) \in E$. So $\Delta(\mathcal{A} \oplus \mathcal{B}) \subseteq E \cup F$. □

Remark 2.2. Since $E \cup F \subset (\mathcal{A} \oplus \mathcal{B})^* = \mathcal{A}^* \oplus \mathcal{B}^*$, its topology is the one induced from $\mathcal{A}^* \oplus \mathcal{B}^*$ and is precisely the Gelfand topology of $\Delta(\mathcal{A} \oplus \mathcal{B})$.

Note that Lemma 2.1 implies that $\mathcal{A} \oplus \mathcal{B}$ is semisimple if and only if both $\mathcal{A}$ and $\mathcal{B}$ are semisimple.

Theorem 2.3. *Let $\mathcal{A}$ and $\mathcal{B}$ be semisimple Banach algebras. Then*

$$\mathcal{M}(\mathcal{A} \oplus \mathcal{B}) = \{(\Phi, \Psi) : \Phi \in \mathcal{M}(\mathcal{A}), \Psi \in \mathcal{M}(\mathcal{B})\},$$

where $(\Phi, \Psi)(\varphi, 0) = \Phi(\varphi)$ and $(\Phi, \Psi)(0, \psi) = \Phi(\psi)$ for all $(\varphi, 0) \in E$ and $(0, \psi) \in F$.

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PROOF. Let $\Phi \in \mathcal{M}(\mathcal{A})$ and $\Psi \in \mathcal{M}(\mathcal{B})$. Since $\Phi\widehat{\mathcal{A}} \subseteq \widehat{\mathcal{A}}$ and $\Psi\widehat{\mathcal{B}} \subseteq \widehat{\mathcal{B}}$, then for all $(\varphi, 0) \in E$ and $(a, b) \in \mathcal{A} \oplus \mathcal{B}$, there are elements $a' \in \mathcal{A}$ and $b' \in \mathcal{B}$ such that

$$\begin{aligned}((\Phi, \Psi) \cdot \widehat{(a, b)})(\varphi, 0) &= (\Phi, \Psi)(\varphi, 0)\widehat{(a, b)}(\varphi, 0)\\ &= \Phi(\varphi)\widehat{a}(\varphi) = \widehat{a'}(\varphi),\end{aligned}$$

for all $\varphi \in \Delta(\mathcal{A})$, and

$$\begin{aligned}((\Phi, \Psi) \cdot \widehat{(a, b)})(0, \psi) &= (\Phi, \Psi)(0, \psi)\widehat{(a, b)}(0, \psi)\\ &= \Psi(\psi)\widehat{b}(\psi) = \widehat{b'}(\psi),\end{aligned}$$

for all $\psi \in \Delta(\mathcal{B})$. Then $((\Phi, \Psi) \cdot \widehat{(a, b)})(\varphi, 0) = \widehat{(a', b')}(\varphi, 0)$ and $((\Phi, \Psi) \cdot \widehat{(a, b)})(0, \psi) = \widehat{(a', b')}(0, \psi)$. This implies that

$$(\Phi, \Psi) \cdot \widehat{\mathcal{A} \oplus \mathcal{B}} \subseteq \widehat{\mathcal{A} \oplus \mathcal{B}}$$

and $(\Phi, \Psi) \in \mathcal{M}(\mathcal{A} \oplus \mathcal{B})$.

Now let $F \in \mathcal{M}(\mathcal{A} \oplus \mathcal{B})$. Define $\Phi(\varphi) = F(\varphi, 0)$ and $\Psi(\psi) = F(0, \psi)$, for all $\varphi \in \Delta(\mathcal{A})$ and $\psi \in \Delta(\mathcal{B})$. So $F = (\Phi, \Psi)$. It is enough to show that $\Phi \in \mathcal{M}(\mathcal{A})$ and $\Psi \in \mathcal{M}(\mathcal{B})$. For all $a \in \mathcal{A}$, there exists $(a', b') \in \mathcal{A} \oplus \mathcal{B}$ such that

$$\Phi(\varphi)\widehat{a}(\varphi) = (\Phi, \Psi)(\varphi, 0)\widehat{(a, 0)}(\varphi, 0) = \widehat{(a', b')}(\varphi, 0) = \widehat{a'}(\varphi).$$

Then $\Phi\widehat{\mathcal{A}} \subseteq \widehat{\mathcal{A}}$ and $\Phi \in \mathcal{M}(\mathcal{A})$. Similarly, $\Psi \in \mathcal{M}(\mathcal{A})$. □

THEOREM 2.4. *Let $\mathcal{A}$ and $\mathcal{B}$ be two semisimple Banach algebras. Then $\mathcal{A} \oplus \mathcal{B}$ is BSE if and only if $\mathcal{A}$ and $\mathcal{B}$ are BSE.*

PROOF. First suppose that $\mathcal{A}$ and $\mathcal{B}$ are BSE. Then by [14, Corollary 5] $\mathcal{A}$ and $\mathcal{B}$ have Δ-weak bounded approximate identities. Let $\{e_\alpha\}_\alpha$ and $\{f_\beta\}_\beta$ be Δ-weak bounded approximate identities of $\mathcal{A}$ and $\mathcal{B}$, respectively. Then $\{(e_\alpha, f_\beta)\}_{(\alpha,\beta)}$ is a Δ-weak bounded approximate identity for $\mathcal{A} \oplus \mathcal{B}$. Indeed, for all $\varphi \in \Delta(\mathcal{A})$ and $\psi \in \Delta(\mathcal{B})$,

$$\lim_{(\alpha,\beta)}(\varphi, 0)(e_\alpha, f_\beta) = \lim_\alpha \varphi(e_\alpha) = 1$$

and

$$\lim_{(\alpha,\beta)}(0, \psi)(e_\alpha, f_\beta) = \lim_\beta \psi(f_\beta) = 1,$$

so that, for all $\Phi \in E \cup F = \Delta(\mathcal{A} \oplus \mathcal{B})$,

$$\lim_{(\alpha,\beta)} F(e_\alpha, f_\beta) = 1,$$

and $\{(e_\alpha, f_\beta)\}_{(\alpha,\beta)}$ is a Δ-weak approximate identity for $\mathcal{A} \oplus \mathcal{B}$. Then by [14, Corollary 5]

$$\mathcal{M}(\mathcal{A} \oplus \mathcal{B}) \subseteq C_{BSE}(\Delta(\mathcal{A} \oplus \mathcal{B})).$$

For the reverse conclusion, let $\sigma \in C_{BSE}(\Delta(\mathcal{A} \oplus \mathcal{B}))$. Then, by [14, Theorem 4(ii)], $\sigma \in C_b(\Delta(\mathcal{A} \oplus \mathcal{B})) \cap (\mathcal{A} \oplus \mathcal{B})^{**}|_{\Delta(\mathcal{A}\oplus\mathcal{B})} = C_b(\Delta(\mathcal{A} \oplus \mathcal{B})) \cap (\mathcal{A}^{**} \oplus \mathcal{B}^{**})|_{\Delta(\mathcal{A}\oplus\mathcal{B})}$. Then

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there are $\sigma_1 \in \mathcal{A}^{**}$ and $\sigma_2 \in \mathcal{B}^{**}$ such that $\sigma_1|_{\Delta(\mathcal{A})} \in C_b(\Delta(\mathcal{A})) \cap \mathcal{A}^{**}|_{\Delta(\mathcal{A})}$, $\sigma_2|_{\Delta(\mathcal{B})} \in C_b(\Delta(\mathcal{B})) \cap \mathcal{B}^{**}|_{\Delta(\mathcal{B})}$ and $\sigma = (\sigma_1, \sigma_2)|_{\Delta(\mathcal{A}\oplus\mathcal{B})}$. On the other hand, since $\sigma \in C_{BSE}(\Delta(\mathcal{A}\oplus\mathcal{B}))$, there exists $\beta > 0$ such that, for every finite number of $c_1, \ldots, c_n \in \mathbb{C}$ and $(\varphi_1, \psi_1), \ldots, (\varphi_n, \psi_n) \in \Delta(\mathcal{A}\oplus\mathcal{B})$,

$$\bigg|\sum_{i=1}^n c_i\sigma(\varphi_i, \psi_i)\bigg| \le \beta\bigg\|\sum_{i=1}^n c_i(\varphi_i, \psi_i)\bigg\|_{\mathcal{A}^*\oplus\mathcal{B}^*}.$$

In particular, for every $(\varphi_1, 0), \ldots, (\varphi_n, 0) \in E$ and $c_1, \ldots, c_n \in \mathbb{C}$,

$$\begin{aligned}\bigg|\sum_{i=1}^n c_i\sigma(\varphi_i, 0)\bigg| &= \bigg|\sum_{i=1}^n c_i\sigma_1(\varphi_i)\bigg| \\ &\le \beta\bigg\|\sum_{i=1}^n c_i(\varphi_i, 0)\bigg\|_{\mathcal{A}^*\oplus\mathcal{B}^*} \\ &= \beta \sup\bigg\{\bigg|\sum_{i=1}^n c_i(\varphi_i, 0)(a, b)\bigg| : \|a\| + \|b\| \le 1\bigg\} \\ &\le \beta \sup\bigg\{\bigg|\sum_{i=1}^n c_i\varphi_i(a)\bigg| : \|a\| \le 1\bigg\} \\ &= \beta\bigg\|\sum_{i=1}^n c_i\varphi_i\bigg\|_{\mathcal{A}^*}.\end{aligned}$$

This means that $\sigma_1 \in C_{BSE}(\Delta(\mathcal{A}))$. Now since $\mathcal{A}$ is a semisimple BSE algebra, $\sigma_1 \in \mathcal{M}(\mathcal{A})$. In a similar way (by considering $(0, \psi_1), \ldots, (0, \psi_n)$) we conclude that $\sigma_2 \in \mathcal{M}(\mathcal{B})$. So $\sigma = (\sigma_1, \sigma_2) \in \mathcal{M}(\mathcal{A}\oplus\mathcal{B})$. Then $C_{BSE}(\Delta(A \oplus B)) \subset \mathcal{M}(\mathcal{A}\oplus\mathcal{B})$ and $\mathcal{A}\oplus\mathcal{B}$ is a BSE algebra.

Now suppose that $A \oplus B$ is BSE and let $\{(e_\alpha, f_\alpha)\}_\alpha$ be a Δ-weak bounded approximate identity for $\mathcal{A}\oplus\mathcal{B}$. Then, for all $\varphi \in \Delta(\mathcal{A})$,

$$\lim_\alpha \varphi(e_\alpha) = \lim_\alpha(\varphi, 0)(e_\alpha, f_\alpha) = 1.$$

So $\{e_\alpha\}_\alpha$ is a Δ-weak bounded approximate identity for $\mathcal{A}$, and similarly $\{f_\alpha\}_\alpha$ is a Δ-weak bounded approximate identity for $\mathcal{B}$, and [14, Corollary 5] implies that $\mathcal{M}(\mathcal{A}) \subseteq C_{BSE}(\Delta(\mathcal{A}))$ and $\mathcal{M}(\mathcal{B}) \subseteq C_{BSE}(\Delta(\mathcal{B}))$.

Now let $\sigma_1 \in C_{BSE}(\Delta(\mathcal{A}))$ and $\sigma_2 \in C_{BSE}(\Delta(\mathcal{B}))$. Then by [14, Theorem 4(i)] there are nets $\{x_\lambda\}_\lambda \subset \mathcal{A}$ and $\{y_\mu\}_\mu \in \mathcal{B}$ such that $\lim_\lambda \widehat{x_\lambda}(\varphi) = \sigma_1(\varphi)$ and $\lim_\mu \widehat{y_\mu}(\psi) = \sigma_2(\psi)$, for all $\varphi \in \Delta(\mathcal{A})$ and $\psi \in \Delta(\mathcal{B})$. If we consider the net $\{(x_\lambda, y_\mu)\}_{(\lambda,\mu)} \subset \mathcal{A}\oplus\mathcal{B}$, then

$$\begin{aligned}\lim_{(\lambda,\mu)} \widehat{(x_\lambda, y_\mu)}(\varphi, 0) &= \lim_{(\lambda,\mu)}(\varphi, 0)(x_\lambda, y_\mu) = \lim_{(\lambda,\mu)} \varphi(x_\lambda) + 0 \\ &= \lim_{(\lambda,\mu)} \widehat{x_\lambda}(\varphi) + 0 = \sigma_1(\varphi) + 0 \\ &= (\sigma_1, \sigma_2)(\varphi, 0),\end{aligned}$$

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for all $\varphi \in \Delta(\mathcal{A})$. And similarly, for all $\psi \in \Delta(\mathcal{B})$,

$$\lim_{(\lambda,\mu)} \widehat{(x_\lambda, y_\mu)}(0, \psi) = (\sigma_1, \sigma_2)(0, \psi).$$

This means that if we let $\sigma = (\sigma_1, \sigma_2)$ then

$$\lim_{(\lambda,\mu)} \widehat{(x_\lambda, y_\mu)}(\Phi) = \sigma(\Phi),$$

for all $\Phi \in E \cup F = \Delta(\mathcal{A} \oplus \mathcal{B})$. Then $\sigma = (\sigma_1, \sigma_2) \in C_{BSE}(\Delta(\mathcal{A} \oplus \mathcal{B}))$. Now since $\mathcal{A} \oplus \mathcal{B}$ is a BSE algebra, $\sigma = (\sigma_1, \sigma_2) \in \mathcal{M}(\mathcal{A} \oplus \mathcal{B})$ and, by Theorem 2.3, $\sigma_1 \in \mathcal{M}(\mathcal{A})$ and $\sigma_2 \in \mathcal{M}(\mathcal{B})$. So $\mathcal{A}$ and $\mathcal{B}$ are BSE algebras. □

Let G be a locally compact abelian group and $M(G)$ the Banach algebra of bounded regular measures on G. The set of continuous measures in $M(G)$ is denoted by $M_c(G)$. This is a closed ideal in $M(G)$, and

$$M(G) = M_d(G) \oplus M_c(G) = l^1(G) \oplus M_c(G).$$

When G is not discrete, $M_c(G) \neq \{0\}$. It is shown in [14] that $M(G)$ is a BSE algebra if and only if G is discrete. So we have the following result.

Corollary 2.5. *For a nondiscrete locally compact abelian group G, $M_c(G)$ is not a BSE algebra.*

Corollary 2.6. *Let G be a locally compact abelian group and $\mu \in M(G)$ be a measure which factors as a product of an invertible measure and an idempotent measure. Then $\mu * L^1(G)$ is a BSE algebra.*

Proof. Define $T_\mu(f) = \mu * f$. Then T_μ is a multiplier of the Banach algebra $L^1(G)$. It is shown in [16] that $T(L^1(G)) = \mu * L^1(G)$ is closed in $L^1(G)$. By [4, Theorem 3.4], since $L^1(G)$ is a commutative semisimple amenable Banach algebra, it factors as follows:

$$L^1(G) = T_\mu(L^1(G)) \oplus \operatorname{Ker}(T_\mu) = (\mu * L^1(G)) \oplus \operatorname{Ker}(T_\mu).$$

Since $L^1(G)$ is a BSE algebra, by Theorem 2.4, $\mu * L^1(G)$ is BSE as well. □

3. θ-Lau product of Banach algebras

The products $\mathcal{A} \times_\theta \mathcal{B}$ of Banach algebras $\mathcal{A}$ and $\mathcal{B}$ were first introduced and studied by Lau [10]. The Banach algebra $\mathcal{B}$ inherits some properties of $\mathcal{A} \times_\theta \mathcal{B}$. For instance, from [7, Proposition 2.1], for $n \in \mathbb{N}$, n-ideal amenability of $\mathcal{A} \times_\theta \mathcal{B}$ implies that of $\mathcal{B}$. In this section we prove that if the θ-Lau product of two Banach algebras $\mathcal{A}$ and $\mathcal{B}$, $\mathcal{A} \times_\theta \mathcal{B}$, with $\mathcal{B}$ unital, is BSE, then $\mathcal{B}$ is a BSE algebra. Before that we need to present some definitions and preliminaries. More results on this product can be found in [12].

Definition 3.1. Let $\mathcal{A}$ and $\mathcal{B}$ be two commutative Banach algebras for which $\Delta(\mathcal{B}) \neq \emptyset$. Let $\theta \in \Delta(\mathcal{B})$. The θ-Lau product $\mathcal{A} \times_\theta \mathcal{B}$ is defined as the Cartesian product $\mathcal{A} \times \mathcal{B}$ with the algebra multiplication

$$(a_1, b_1)(a_2, b_2) = (a_1a_2 + \theta(b_1)a_2 + \theta(b_2)a_1, b_1b_2)$$

and norm $\|(a, b)\| = \|a\| + \|b\|$.

Obviously if $\mathcal{B}$ is a unital Banach algebra, then $\mathcal{A} \times_\theta \mathcal{B}$ is a unital Banach algebra for any Banach algebra $\mathcal{A}$.

Remark 3.2. The space $\mathcal{A} \times_\theta \mathcal{B}$ is a Banach algebra. If we allow $\theta = 0$, we obtain the usual direct sum of Banach algebras. If $\mathcal{B} = \mathbb{C}$ and $\theta : \mathbb{C} \to \mathbb{C}$ is the identity map, then $\mathcal{A} \times_\theta \mathbb{C}$ coincides with the unitisation $\mathcal{A}_e$ of $\mathcal{A}$.

The dual of the space $\mathcal{A} \times_\theta \mathcal{B}$ can be identified with $\mathcal{A}^* \oplus \mathcal{B}^*$ in the natural way: $(\varphi, \psi)(a, b) = \varphi(a) + \psi(b)$. The dual norm on $\mathcal{A}^* \oplus \mathcal{B}^*$ is the maximum norm $\|(\varphi, \psi)\| = \max\{\|\varphi\|, \|\psi\|\}$. On $\mathcal{A}^* \oplus \mathcal{B}^*$, the weak* topology coincides with the product of the weak* topologies of $\mathcal{A}^*$ and $\mathcal{B}^*$. The following theorem, which is proved in [12], identifies the Gelfand spectrum $\Delta(\mathcal{A} \times_\theta \mathcal{B})$ of $\mathcal{A} \times_\theta \mathcal{B}$.

Theorem 3.3. *Let $\mathcal{A}$ and $\mathcal{B}$ be Banach algebras with $\Delta(\mathcal{B}) \neq \emptyset$. Let $\theta \in \Delta(\mathcal{B})$ and*

$$E = \{(\varphi, \theta) : \varphi \in \Delta(\mathcal{A})\}, \quad F = \{(0, \psi) : \psi \in \Delta(\mathcal{B})\}.$$

Set $E = \emptyset$ if $\Delta(A) = \emptyset$. Then $\Delta(A \times_\theta B) = E \cup F$.

Note that the topology on $E \cup F = \Delta(\mathcal{A} \times_\theta \mathcal{B})$ is the weak* topology induced from $(\mathcal{A} \times_\theta \mathcal{B})^* = \mathcal{A}^* \oplus \mathcal{B}^*$ and it is precisely the Gelfand topology on $\Delta(\mathcal{A} \times_\theta \mathcal{B})$.

Theorem 3.4. *Let $\mathcal{A}$ and $\mathcal{B}$ be two commutative Banach algebras. Suppose that $\mathcal{A} \times_\theta \mathcal{B}$ is a BSE algebra. Then $\mathcal{B}$ is a BSE algebra.*

Proof. Let $\sigma \in C_{BSE}(\Delta(\mathcal{B}))$. Then by [14, Theorem 4(i)], there exists a bounded net $\{y_\lambda\}_\lambda \subset \mathcal{B}$ such that $\lim_\lambda \widehat{y_\lambda}(\psi) = \sigma(\psi)$, for all $\psi \in \Delta(\mathcal{B})$. If we consider the bounded net $\{(0, y_\lambda)\}_\lambda \subset \mathcal{A} \times_\theta \mathcal{B}$,

$$\begin{aligned}\lim_\lambda \widehat{(0, y_\lambda)}(0, \psi) &= \lim_\lambda (0, \psi)(0, y_\lambda) = \lim_\lambda 0 + \widehat{y_\lambda}(\psi) \\ &= \sigma(\psi) = (0, \sigma)(0, \psi),\end{aligned}$$

for all $(0, \psi) \in E$. Also, for all $(\varphi, \theta) \in F$,

$$\begin{aligned}\lim_\lambda \widehat{(0, y_\lambda)}(\varphi, \theta) &= \lim_\lambda (\varphi, \theta)(0, y_\lambda) = \lim_\lambda \widehat{y_\lambda}(\theta) \\ &= \sigma(\theta) = (0, \sigma)(\varphi, \theta).\end{aligned}$$

Consequently, for all $\Phi \in E \cup F = \Delta(\mathcal{A} \times_\theta \mathcal{B})$,

$$\lim_\lambda \widehat{(0, y_\lambda)}(\Phi) = (0, \sigma)(\Phi).$$

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Then $(0,\sigma) \in C_{BSE}(\Delta(\mathcal{A} \times_\theta \mathcal{B}))$ and since $\mathcal{A} \times_\theta \mathcal{B}$ is BSE, $(0,\sigma) \in (\widehat{\mathcal{A} \times_\theta \mathcal{B}})$. So there exists $(a,b) \in \mathcal{A} \times_\theta \mathcal{B}$ such that $\widehat{(a,b)} = (0,\sigma)$. Then, for all $\psi \in \Delta(\mathcal{B})$,

$$\sigma(\psi) = (0,\sigma)(0,\psi) = \widehat{(a,b)}(0,\psi) = (0,\psi)(a,b) = \widehat{b}(\psi).$$

This means that $\sigma \in \widehat{\mathcal{B}}$ and since $\mathcal{B}$ is unital, $\mathcal{B}$ is a BSE algebra. □

The following examples show that if $\mathcal{A} \times_\theta \mathcal{B}$, for $\mathcal{B}$ unital, is BSE we cannot conclude that $\mathcal{A}$ is BSE in general. Before that we need to present a result proved by Kaniuth and Ülger, [8, Theorem 4.8].

Theorem 3.5. *Let $\mathcal{A}$ be a nonunital commutative Banach algebra. Then the unitisation $\mathcal{A}_e$ of $\mathcal{A}$ is a BSE algebra if and only if*

$$C_{BSE}(\Delta(\mathcal{A})) \cap C_0(\Delta(\mathcal{A})) = \widehat{\mathcal{A}}.$$

Example 3.6.

(1) Let G be a second countable noncompact locally compact group whose regular representation is not completely reducible and $A(G)$ be the Fourier algebra of G. Then $A(G) \neq B(G) \cap C_0(G)$ (see [1, 5, 8]). Thus if in addition G is amenable, then $A(G)$ is a BSE algebra [8], but $A(G)_e = A(G) \times_\theta \mathbb{C}$, such that $\theta : \mathbb{C} \to \mathbb{C}$ is the identity map, is not BSE, by Theorem 3.5

(2) Let $l^1(\mathbb{N})$ be the semigroup algebra of the additive semigroup of natural numbers. Then $l^1(\mathbb{N})$ is not a BSE algebra [14]. However, the semigroup algebra $l^1(\mathbb{N} \cup \{0\}) = l^1(\mathbb{N})_e = l^1(\mathbb{N}) \times_\theta \mathbb{C}$, such that $\theta : \mathbb{C} \to \mathbb{C}$ is the identity map, is a BSE algebra by [15, Theorem 6].

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