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ON FACTORS OF A GRAPH

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Let G be a graph with multiple edges. Let f be a function from the vertex set $V(G)$ of G to the non-negative integers. An *f-factor* of G is a spanning subgraph F of G such that the degree (valence) of each vertex x in F is $f(x)$. A theorem of Fulkerson, Hoffman and McAndrew [**1**] gives necessary and sufficient conditions to have an f-factor for a graph G with the *odd-cycle property*; i.e., if G has the property that either any two of its odd (simple) cycles have a common vertex, or there exists a pair of vertices, one from each cycle, which is joined by an edge. They proved this theorem using integer programming techniques, with a rather long proof. We show that this is a corollary of Tutte's f-factor theorem.

The f-factor theorem of Tutte with a slight modification in notations and formulation is as follows.

THEOREM [**2**]. *Let G be a graph with multiple edges, and let f be a non-negative function defined on $V(G)$. G contains an f-factor if and only if for every partition (S, T, U) of vertices of G, we have*

$$(1)\quad \sum_{a\in T} f(a) \leqq \sum_{a\in S} f(a) + \sum_{\substack{a\in T\\ b\in T\cup U}} c_{ab} - q(S, T)$$

where c_{ab} is the number of edges joining a to b, and $q(S, T)$ is the number of components C of $\langle U\rangle$ (the induced subgraph of G on the vertices U) such that

$$(2)\quad B(C, T) = \sum_{a\in C} f(a) - \sum_{\substack{a\in C\\ b\in T}} c_{ab}$$

is odd. (For simplicity we write $a \in C$ instead of $a \in V(C)$.)

COROLLARY [**1**]. *Assume that G has the odd-cycle property. Then G has an f-factor if and only if*

i) $\sum_{a\in V(G)} f(a)$ *is even, and*

ii) *for every partition (S, T, U) of $V(G)$*

$$(3)\quad \sum_{a\in T} f(a) \leqq \sum_{a\in S} f(a) + \sum_{\substack{a\in T\\ b\in T\cup U}} c_{ab}.$$

Proof. The necessity of the conditions is trivial.

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Define $\delta(S, T)$ to be the difference of both sides in (1), i.e.

$$\delta(S, T) = \sum_{a \in S} f(a) - \sum_{a \in T} f(a) + \sum_{\substack{a \in T \\ b \in T \cup U}} c_{ab} - q(S, T).$$

Substituting from (2)

$$\delta(S, T) = \sum_{a \in S} f(a) - \sum_{a \in T} f(a) + \sum_{\substack{a \in T \\ b \in T}} c_{ab}$$

$$+ \sum_{C \subset U} \left(-B(C, T) + \sum_{a \in C} f(a) \right) - q(S, T)$$

$$= \sum_{a \in S \cup U} f(a) - \sum_{a \in T} f(a) + \sum_{\substack{a \in T \\ b \in T}} c_{ab} - \sum_{C \subset U} B(C, T) - q(S, T)$$

or

$$(4) \quad \delta(S, T) = \sum_{a \in V(G)} f(a) - 2 \sum_{a \in T} f(a) + \sum_{\substack{a \in T \\ b \in T}} c_{ab} - \sum_{C \subset U} 2 \left\lceil \frac{B(C, T)}{2} \right\rceil$$

where $\lceil X \rceil$ = minimal integer $\geqq x$.

To prove sufficiency, we show that if G satisfies the hypothesis, then there exists a partition (S, T, U) for which $\delta(S, T)$ is minimal and $q(S, T) \leqq 1$. If $\sum_{a \in V(G)} f(a)$ is even, then (4) implies that $\delta(S, T)$ is even; hence (1) is satisfied.

Let (S, T, U) be any partition of $V(G)$ for which $\delta(S, T)$ is minimal. Then at most one of the components of $\langle U \rangle$ can have any odd cycles; all the other components are bipartite graphs. Let C be one such component; $V(C) = C_1 \cup C_2$, where $\langle C_1 \rangle$ and $\langle C_2 \rangle$ are totally disconnected subgraphs.

Let C' be any component of $\langle U \rangle$, $C' \neq C$; then

$$B(C', T \cup C_1) - B(C', T) = - \sum_{\substack{a \in C' \\ b \in C_1}} c_{ab} = 0.$$

Hence,

$$\delta(S \cup C_2, T \cup C_1) - \delta(S, T) = -2 \sum_{a \in C_1} f(a) + \sum_{\substack{a \in C_1 \\ b \in T}} c_{ab} + \sum_{\substack{a \in T \\ b \in C_1}} c_{ab}$$

$$+ \sum_{a, b \in C_1} c_{ab} + 2\lceil \tfrac{1}{2} B(C, T) \rceil.$$

Since C_1 is totally disconnected, $\sum_{a,b \in C_1} c_{ab} = 0$. A similar relation holds with C_1 replaced by C_2. Adding those two we find

$$[\delta(S \cup C_2, T \cup C_1) - \delta(S, T)] + [\delta(S \cup C_1, T \cup C_2) - \delta(S, T)]$$

$$= -2B(C, T) + 4\lceil \tfrac{1}{2} B(C, T) \rceil.$$

The right side is 0 if $B(C, T)$ is even, and 2 if $B(C, T)$ is odd. As all δ's are even, either $\delta(S \cup C_1, T \cup C_2)$ or $\delta(S \cup C_2, T \cup C_1)$ equals $\delta(S, T)$, i.e., is also minimal.

In this manner all bipartite components of $\langle U \rangle$ can be removed, leaving a partition (S^*, T^*, U^*) in which U^* has at most one component. Hence $q(S^*, T^*) \leqq 1$, while $\delta(S^*, T^*)$ is minimal.

There are further applications of Tutte's f-factor theorem in [**3**].

References

1. D. R. Fulkerson, A. J. Hoffman and M. H. McAndrew, *Some properties of graphs with multiple edges*, Can. J. Math. *17* (1965), 166–177.

2. W. T. Tutte, *A short proof of the factor theorem for finite graphs*, Can. J. Math. *6* (1954), 347–352.

3. ——— *Spanning subgraphs with specified valencies*, Discrete Math. *9* (1974), 97–108.

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