A PROPERTY OF A TRIANGLE INSCRIBED IN A CONVEX CURVE

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The purpose of this paper is to prove the following theorem:

THEOREM. *Given a convex curve C, of perimeter length l and three points M, N, and P which divide the perimeter of C into three parts of equal length, the perimeter length of the triangle MNP is never less than* $\frac{1}{2}l$. *Equality holds only in the case where C is an equilateral triangle and M, N, and P are the mid-points of the three sides.*

To prove this theorem we observe first that by the Blaschke selection theorem, there is a set C that is either a segment or a convex curve of perimeter length l, for which the perimeter length or the corresponding (possibly degenerate) triangle MNP is the least possible. We shall show that C must be a triangle.

In what follows, C denotes the extremal figure and M, N, and P are points on C, dividing the perimeter into arcs of equal length and such that the perimeter of MNP is the least possible.

Now C cannot be a line segment, because in that case $MN + NP + PM = \frac{2}{3}l$, and we already know another case where $MN + NP + MP = \frac{1}{2}l$. That is the case when C is an equilateral triangle and M, N, and P are the mid-points of the three sides.

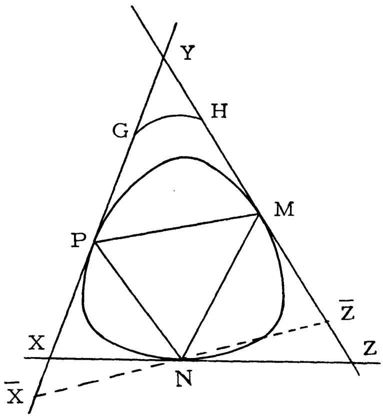

FIGURE 1

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Therefore C is a convex curve. M, N, and P divide the perimeter of C into three equal parts. (See Figure 1.) Draw three support lines, one each through M, N, and P. First we consider the case where these three lines form the triangle XYZ which contains C. If $PY + YM = MZ + ZN = NX + XP$, then XYZ is a triangle whose perimeter is larger than that of C (unless C is this triangle itself) and which is such that the three points M, N, and P divide its perimeter into three equal parts. Now if we draw a figure similar to this figure and such that the perimeter of the triangle corresponding to XYZ is equal to the perimeter of C, then the perimeter of a triangle inscribed in this triangle and dividing the perimeter of the triangle into three equal parts is less than, or equal to, the perimeter of MNP, and is actually less unless C is XYZ. Thus, in this case C is extremal only if it is a triangle.

If $PY + YM$, $MZ + ZN$, and $NX + XP$ are not equal, then let us assume that $PY + YM$ is the largest and $MZ + ZN$ the smallest of the three. First consider the case when $PX + XN > MZ + ZN$. Rotate the line XZ about N such that MZ increases. This means that the corresponding value for $MZ + ZN$ increases while the corresponding value for $PX + XN$ decreases. For a certain position of Z, say $\bar{Z}$, we shall have

$$M\bar{Z} + \bar{Z}N = N\bar{X} + \bar{X}P < PY + YM,$$

and all three of them are greater than the lengths of the three equal arcs of C, MN, NP, and PM. Therefore, we can construct a convex arc $PGHM$, having a length equal to $M\bar{Z} + \bar{Z}N$ and lying inside triangle PYM, and composed of the following parts: PG is part of the segment PY, MH is part of the segment MY, and GH is a convex arc (concave towards N). This arc, together with $M\bar{Z} + \bar{Z}N$ and $N\bar{X} + \bar{X}P$, forms a convex curve whose perimeter is greater than that of C, and which is divided by P, M, and N into three equal parts. As before, this leads to a contradiction with the extremal property of C. If $MZ + ZN = NX + XP$ we proceed exactly as above except that we need no longer rotate XZ about N in order to produce a curve which contradicts the extremal property of C. Therefore, if C is the extremal curve, it must be of the shape shown in Figure 2, where the length of the arc PHM is equal to $PX + XN = NZ + ZM$. The length $PY + YM$ is, of course, greater than that of arc PHM. Let their difference be equal to ϵ; $\epsilon > 0$.

Let us consider two points E and F, on XZ, such that

$$PE + EN = MF + FN = PHM + \rho.$$

It is clear that for any given $\rho > 0$ we can find the corresponding points E and F, and that ρ could be chosen as small as we wish, provided that it is positive.

If ρ is sufficiently small ($\rho < \epsilon$), EP and FM intersect at G and $PG + GM < PY + YM$. Now if ρ increases, E and F go farther away, that is $MF + FN = PE + EN$ increase while $PG + GM$ decreases, and vice versa. But for sufficiently small values of ρ the corresponding $MF + NF$ will be smaller

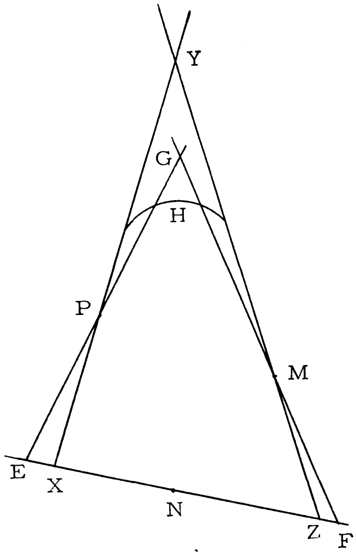

FIGURE 2

than the corresponding $PG + GM$, while for sufficiently large values of ρ the corresponding $MF + FN$ will be larger than the corresponding $PG + GM$. Since all these distances change continuously, it follows that for some particular value of ρ the two will be equal. Thus we shall have a triangle whose perimeter is larger than that of $PHMZX$ and is divided into three equal parts by M, N, and P. Therefore, if C is extremal, it must be a triangle.

So far we have assumed that the three support lines (see Figure 1) intersect and form a triangle XYZ which includes C. If two of these support lines are parallel, still all the previous arguments hold.

It is possible that the three support lines form a triangle, but this triangle does not contain C (Fig. 3). Using the notation of Figure 2, $NX + XP$ and $NZ + ZM$ are both greater than the arcs MN and NP. Assume that $PX + XN > NZ + ZM$. Revolve XZ about N. X and Z will move on PY and MY. For a certain position of the line XZ, say $\bar{X}\bar{Z}$, $P\bar{X} + \bar{X}N = N\bar{Z} + \bar{Z}M$ will be greater than the length of the arc NM. It is possible to move $\bar{X}$ and $\bar{Z}$, on the line $\bar{X}\bar{Z}$ and away from N in such a way that for the new positions of $\bar{X}$ and $\bar{Z}$, say $\tilde{X}$ and $\tilde{Z}$, $P\tilde{X} + \tilde{X}N = P\bar{X} + \bar{X}N + \sigma$. For any given $\sigma > 0$ there exists a unique pair of points $\tilde{X}$ and $\tilde{Z}$, and it is possible to choose σ such that the corresponding $P\tilde{X}$ and $M\tilde{Z}$ become parallel. Then the problem reduces to the above case and all the previous arguments hold.

Thus, in all cases, the extremal convex curve C has to be a triangle. We shall prove that this triangle must be equilateral.

Before this, however, we must prove certain lemmas. The following terminology will simplify the statements and the proofs of these lemmas.

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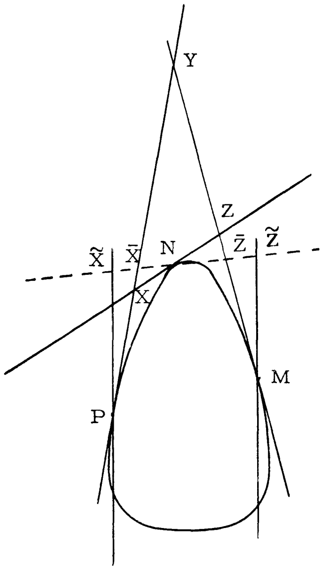

FIGURE 3

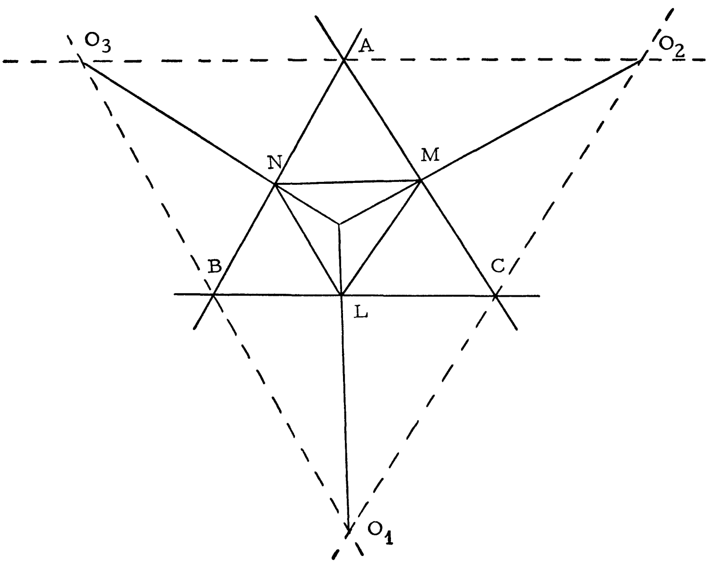

FIGURE 4

Definition. A triangle ABC and three points O_1, O_2, and O_3, centres of the three escribed circles of this triangle, are given (Fig. 4). O_1, O_2, and O_3 are opposite to A, B, and C, respectively. Consider three points L, M, and N lying on BC, AC, and AB, respectively, and such that $O_1 L$, $O_2 M$, and $O_3 N$

are bisectors of the three angles $\angle NLM$, $\angle NML$, and $\angle LNM$. By definition, the points L, M, and N have the "tricentre property" with respect to the triangle ABC.

For any given triangle there exists a unique set of points L, M, and N, such that L, M, and N have the tricentre property with respect to the triangle. But, since this fact will not be used here, the proof is omitted.

The next three lemmas are needed for the following developments:

LEMMA 1. *If ABC is an isosceles triangle and BC is its base, there exists a set L, M, and N which has the tricentre property with respect to ABC, and is such that L is the mid-point of BC and MN is parallel to BC. The set L, M, N is unique. Furthermore, if $\angle ABC < \frac{1}{3}\pi$, then $NA < NB$; if $\angle ABC > \frac{1}{3}\pi$, then $NA > NB$. If $\angle ABC = \frac{1}{3}\pi$, then $NA = NB$.*

The analytical proof of this lemma is very simple and will be omitted here.

LEMMA 2. *If the points L, M, and N have the tricentre property with respect to the triangle ABC, and L is the mid-point of BC, then $AB = AC$.*

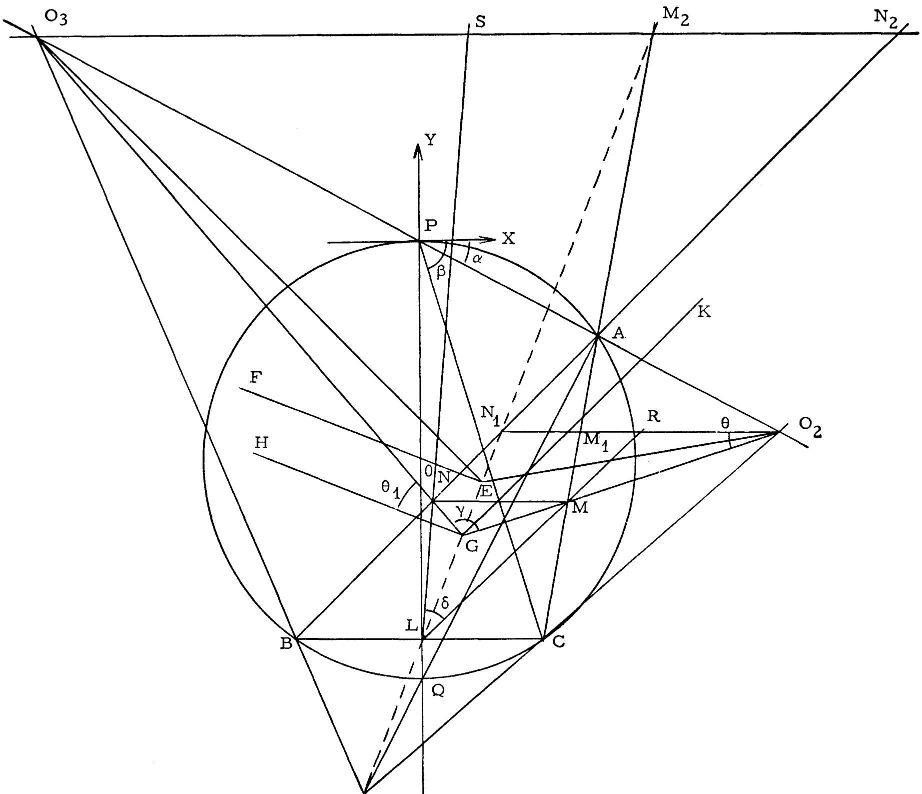

FIGURE 5

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Proof. Let ABC be the given triangle (Fig. 5) and O be the circumcentre. Draw OL perpendicular to BC and find P and Q, the points of intersection of OL with the circumcircle. Since $\angle ABC = \angle O_2PC$, $\angle O_2BC = \frac{1}{2} O_2PC$, and since P lies on the perpendicular bisector of BC and is not collinear with BO_2, it must be the centre of the circle passing through B, C, and O_2. A similar argument shows that O_3 also belongs to this circle.

Take P to be the origin and assign a cartesian co-ordinate system to the figure, assuming that PC is equal to one unit. If PO_2 and PC make angles α and β, respectively, with the x-axis, then we can evaluate the co-ordinates of the following points:

$$O_2(\cos\alpha, -\sin\alpha), \qquad O_3(-\cos\alpha, \sin\alpha),$$

$$B(-\cos\beta, -\sin\beta), \qquad C(\cos\beta, -\sin\beta),$$

$$L(0, -\sin\beta), \qquad O_1\left(-\sin\alpha\cot\beta, -\frac{1+\cos\alpha\cos\beta}{\sin\beta}\right),$$

$$A\left(\frac{\sin\alpha\cos\alpha}{\sin\beta}, -\frac{\sin^2\alpha}{\sin\beta}\right).$$

The equations of the lines OL, AC, and AB will be

$$(1) \qquad \frac{y+\sin\beta}{x} = \frac{\cos\alpha+\cos\beta}{\sin\alpha},$$

$$(2) \qquad y(\sin\alpha\cos\alpha - \sin\beta\cos\beta) = x(\sin^2\beta - \sin^2\alpha) + \sin\alpha\sin(\alpha-\beta),$$

$$(3) \qquad y(\sin\alpha\cos\alpha + \sin\beta\cos\beta) = x(\sin^2\beta - \sin^2\alpha) - \sin\alpha\sin(\alpha+\beta).$$

Next consider a point G on O_1L and draw the lines GO_2 and GO_3 to intersect AC and AB in M and N respectively. We shall prove that MN is always parallel to BC.

It follows from the construction of NM that M and N are related to each other by a projectivity; hence, either all the lines MN are concurrent, or they are tangent to a conic. But BC, N_1M_1, and N_2M_2 (see Fig. 5) are members of this family of lines and they are parallel to each other (since the ordinates of N_1 and M_2 are $-\sin\alpha$ and $+\sin\alpha$ respectively). Therefore, the family of the lines MN are all parallel to each other.

Now let us assume that L, M, and N have the tricentre property with respect to the triangle ABC. Since GM bisects $\angle NML$ and since NM is parallel to O_2N_1, $\angle N_1RM = 2\angle N_1O_2M$. Similarly $\angle M_2SN = 2\angle M_2O_3N$. Therefore,

$$\pi - \delta = 2(\pi-\gamma) \quad \text{or} \quad \gamma - \tfrac{1}{2}\delta = \tfrac{1}{2}\pi,$$

where $\gamma = \angle NGM$ and $\delta = \angle NLM$. Draw GK parallel to LM. Let

$$\angle KGM = \theta \quad \text{and} \quad \angle N_1GK = \angle N_1LM = \tfrac{1}{2}\delta.$$

(LMN has the tricentre property with respect to ABC.) Hence

$$\gamma - \tfrac{1}{2}\delta = \angle\, NGN_1 + \theta = \tfrac{1}{2}\pi.$$

Draw HG perpendicular to $O_1 L$. Then $\angle\, NGH + \angle\, NGN_1 = \frac{1}{2}\pi$ and hence $\angle\, NGH = \theta$. We shall prove that this is impossible unless $AB = AC$.

Consider a point E on $O_1 L$. Let

$$\left(h, \frac{h(\cos\alpha + \cos\beta)}{\sin\alpha} - \sin\beta\right)$$

be the co-ordinates of this point. We must find the particular location of E for which $\angle\, N_1 O_2 E = \angle\, FEO_3$, where EF is perpendicular to $O_1 L$. We have

$$\text{(4)} \qquad \text{slope of } O_2 E = \frac{h(\cos\alpha + \cos\beta) + \sin\alpha(\sin\alpha - \sin\beta)}{\sin\alpha(h - \cos\alpha)},$$

$$\text{(5)} \qquad \text{slope of } O_3 E = \frac{h(\cos\alpha + \cos\beta) - \sin\alpha(\sin\alpha + \sin\beta)}{\sin\alpha(h + \cos\alpha)},$$

and

$$\text{(6)} \qquad \text{slope of } O_1 L = \frac{\cos\alpha + \cos\beta}{\sin\alpha}.$$

(7)

$$\tan \angle\, O_3 EN_1 = \frac{\dfrac{h(\cos\alpha + \cos\beta) - \sin\alpha(\sin\alpha + \sin\beta)}{\sin\alpha(h + \cos\alpha)} - \dfrac{\cos\alpha + \cos\beta}{\sin\alpha}}{1 + \dfrac{h(\cos\alpha + \cos\beta) - \sin\alpha(\sin\alpha + \sin\beta)}{\sin\alpha(h + \cos\alpha)} \cdot \dfrac{\cos\alpha + \cos\beta}{\sin\alpha}}.$$

If $N_1 O_2 E = FEO_3$, then (7) must be equal to the inverse of (4). When this new equation is simplified, it reduces to the following quadratic equation in h:

$$\text{(8)} \qquad h^2(1 + \cos^2\beta + 2\cos\alpha\cos\beta) + \sin^2\alpha\cos\beta(\cos\beta + 2\cos\alpha) = 0.$$

Since α and β are both less than $\frac{1}{2}\pi$, equation (8) does not have a real root unless $\alpha = 0$, which implies $h = 0$. Therefore $\angle\, N_1 O_2 E$ cannot be equal to $\angle\, FEO_3$ unless $AB = AC$. This completes the proof of the lemma.

LEMMA 3. *If L, M, N have the tricentre property with respect to the triangle ABC, and if L, M, and N divide the perimeter of the triangle into three equal parts, then ABC is equilateral and L, M, and N are mid-points of BC, AC, and AB respectively.*

Proof. Assume that ABC is not equilateral and let $BAC < \frac{1}{3}\pi$ be the smallest angle of the triangle, and let $AB > AC$ (Fig. 6). Circumscribe a circle about ABC and let O be the centre of this circle. Draw OH perpendicular to BC and let it intersect the circle at A', on the opposite side of H from O. Also consider a point A'' on the circle, between A and C, such that $A''C = BC$.

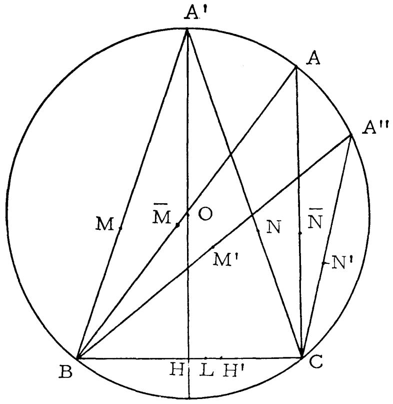

FIGURE 6

Let M, H, N and M', H', N' be the two sets having the tricentre property with respect to $A'BC$ and $A''BC$, respectively. Because of Lemma 1, these sets are unique, M' is the mid-point of BA'', $A'M = A'N > MB = NC$ and $A''N' > N'C$.

Since $BAC < \frac{1}{3}\pi$, $AB + AC > \frac{2}{3}p$, where p is the perimeter of triangle ABC.

Let L, $\bar{M}$, $\bar{N}$ be a set having the tricentre property with respect to ABC, and assume that L, $\bar{M}$, and $\bar{N}$ divide the perimeter of the triangle into three equal parts, that is $\bar{M}A + A\bar{N} = \frac{1}{3}p$. We shall prove that this is impossible unless ABC is equilateral.

As AB moves from position $A'B$ to position $A''B$, $\bar{M}$ moves on AB and the ratio $A\bar{M}/\bar{M}B = r$ changes. For position $A'B$, $r > 1$, and for position $A''B$, $r = 1$. Between these two positions, r cannot be equal to one. If $r = 1$ for some position of AB other than $A''B$, it follows from Lemma 2 that the corresponding triangle has to be isosceles and that is impossible.

Because of the construction of $\bar{M}$, $\bar{M}$ moves continuously on AB and r is continuous. This means that between positions $A'B$ and $A''B$, $r > 1$. If $r < 1$ for some position of AB, then for some position between $A'B$ and AB, r had to be equal to 1 and this was impossible. Thus, we have proved that $\bar{M}A > \bar{M}B$. A similar argument shows that $A\bar{N} > \bar{N}C$. Thus,

$$A\bar{M} + A\bar{N} > \tfrac{1}{2}(AB + AC) > \tfrac{1}{2}(\tfrac{2}{3}p) = \tfrac{1}{3}p.$$

This contradicts the assumption that $\bar{M}A + A\bar{N} = \frac{1}{3}p$. Therefore, BAC, the smallest angle of the triangle, cannot be less than $\frac{1}{3}\pi$ and ABC must be equilateral. The rest of the proof follows from Lemma 1.

Now we are in a position to complete the proof of our main theorem. We have already shown that the extremal curve C must be a triangle. Let triangle ABC represent this extremal curve C, and M, N, and P divide its perimeter into three equal parts. Draw the escribed circle opposite vertex A and let O

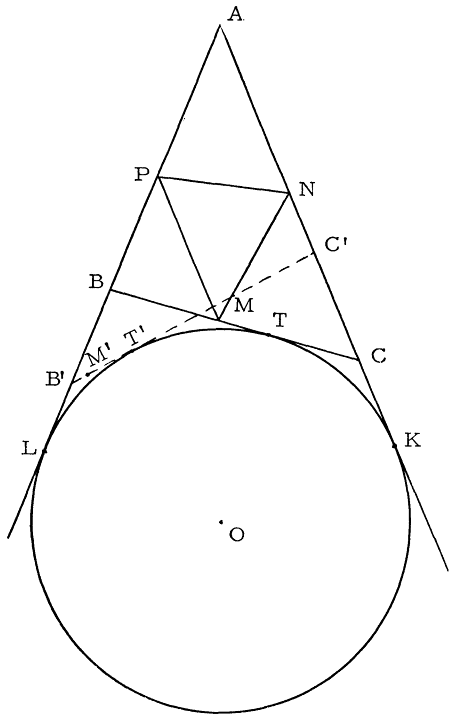

FIGURE 7

denote the centre of this circle (Fig. 7). Let AB, BC, and CA be tangent to this circle at the points L, T, and K, respectively.

Draw a line tangent to the escribed circle O at T', T' lying on the same arc LK of the circle as T, and let this line intersect AB and AC at B' and C', respectively. The perimeter of triangle $AB'C'$ is equal to that of triangle ABC.

Let M' be a point on $B'C'$ such that $T'M' = TM$ and M and M' are on the same side of T and T', respectively. The points M', P, and N will divide the perimeter of $AB'C'$ into three equal parts, and since PMN has the least possible perimeter,

$$PM' + M'N + NP > PM + MN + NP.$$

This is true, of course, for any position of the line $B'C'$. It is also clear that the locus of M' is a circle whose radius is OM and its centre is O. Let us call this circle Θ.

Now consider an ellipse whose focal points are P and N, and which passes through M. This ellipse must be tangent to Θ at M; otherwise there will exist a point $\bar{M}$ on Θ such that

$$P\bar{M} + \bar{M}N + NP < PM + MN + NP,$$

and this is impossible. Therefore, OM is normal to the ellipse, at M, and

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bisects the angle PMN. Similarly, the bisectors of the angles NPM and PNM must pass through the centres of the other two escribed circles, respectively. This means that M, P, N must have the tricentre property with respect to ABC. But since M, P, and N divide the perimeter of ABC into three equal parts, it follows from Lemma 3 that ABC must be equilateral and M, N, P be the mid-points of the three sides.

This completes the proof of the theorem.

Remark. It would be interesting to know what are the analogues of the main theorem of this paper, if the convex curve is divided into n arcs of equal length rather than three. The following conjecture is an answer to this question:

Given a convex curve C, of perimeter length l, and n (where $n > 1$) points $A_1, A_2, \ldots, A_n$ which divide the perimeter of C into n parts of equal length, the perimeter length of the polygon $A_1A_2 \ldots A_n$ is never less than $l(n-2)/n$ if n is even and

$$l\left(\frac{(n-2)^2 + \sqrt{2(n-1)(n-2)}}{n(n-1)}\right)$$

if n is odd. Furthermore, equality holds only in the case where C is a line segment (if n is even) or an isosceles triangle, the base being of length l/n and the sides being of length $l(n-1)/2n$ (if n is odd).

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