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ON INTEGER SEQUENCES GENERATED BY LINEAR MAPS

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Abstract. Let $x_0 < x_1 < x_2 < \cdots$ be an increasing sequence of positive integers given by the formula $x_n = \lfloor \beta x_{n-1} + \gamma \rfloor$ for $n = 1, 2, 3, \dots$, where $\beta > 1$ and γ are real numbers and x_0 is a positive integer. We describe the conditions on integers $b_d, \dots, b_0$, not all zero, and on a real number $\beta > 1$ under which the sequence of integers $w_n = b_d x_{n+d} + \cdots + b_0 x_n$, $n = 0, 1, 2, \dots$, is bounded by a constant independent of n. The conditions under which this sequence can be ultimately periodic are also described. Finally, we prove a lower bound on the complexity function of the sequence $qx_{n+1} - px_n \in \{0, 1, \dots, q-1\}$, $n = 0, 1, 2, \dots$, where x_0 is a positive integer, $p > q > 1$ are coprime integers and $x_n = \lceil px_{n-1}/q \rceil$ for $n = 1, 2, 3, \dots$. A similar speculative result concerning the complexity of the sequence of alternatives ($F : x \mapsto x/2$ or $S : x \mapsto (3x + 1)/2$) in the $3x + 1$ problem is also given.

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1. Introduction. For a given real number y, let $\{y\}$, $\lfloor y \rfloor$ and $\lceil y \rceil$ be the fractional part, the integral part and the ceiling function of y, respectively. For any real numbers y and $\beta > 1$, one can study the sequence of so-called β-transformations, given by $y_0 = y$ and $y_n = \{\beta y_{n-1}\}$ for $n = 1, 2, 3, \dots$. This sequence was first investigated by Rényi [**18**] and Parry [**17**]. In particular, the sequence $y_0 = 1$, $y_n = \{\beta y_{n-1}\}$ for $n = 1, 2, 3, \dots$ is called the *Rényi development of unity*.

In fact, $y \in [0, 1)$ can be expressed as

$$y = \sum_{k=1}^{\infty} \varepsilon_k(y) \beta^{-k},$$

where $\varepsilon_k(y) = \lfloor \beta y_{k-1} \rfloor \in \{0, 1, \dots, \lfloor \beta \rfloor\}$. This expression is called the β-expansion of y. In general, if $y = \sum_{k=1}^{\infty} \varepsilon_k \beta^{-k}$ with some $\varepsilon_1, \varepsilon_2, \varepsilon_3, \dots \in \{0, 1, \dots, \lfloor \beta \rfloor\}$ then this is not necessarily the β-expansion of y (see [**12**, **17**]). Clearly, the β-expansion of y is ultimately periodic if and only if the sequence y_n, $n = 0, 1, 2, \dots$, is ultimately periodic. Schmidt [**20**] showed that if the β-expansion of every number $y \in \mathbb{Q} \cap [0, 1)$ is ultimately periodic then $\beta > 1$ must be either a Pisot number or a Salem number. Recall that $\beta > 1$ is a *Pisot number* (resp. *Salem number*) if it is an algebraic integer whose conjugates over $\mathbb{Q}$ (if any) all lie in the open unit disc $|z| < 1$ (resp. closed unit disc $|z| \leqslant 1$ with at least one conjugate lying on the circle $|z| = 1$). Finite β-expansions have been studied in [**9**] and [**11**]; those results are also related to Pisot numbers.

In this paper, in contrast to the fractional β-transformations, we shall study a kind of integral β-transformations. Let x_0 be a positive integer, and let $\beta > 1$ and γ be two real numbers such that $(\beta - 1)x_0 \geqslant 1 - \gamma$. Consider an increasing sequence of positive integers $x_0 < x_1 < x_2 < \cdots$ generated by the map

$$T_{\beta,\gamma} : x \mapsto \lfloor \beta x + \gamma \rfloor,$$

namely,

$$x_n = \lfloor \beta x_{n-1} + \gamma \rfloor = T^n_{\beta,\gamma}(x_0)$$

for each $n \geqslant 1$. Indeed, $x_n = \lfloor \beta x_{n-1} + \gamma \rfloor > \beta x_{n-1} + \gamma - 1 \geqslant x_{n-1}$ for $n \geqslant 1$, because $(\beta - 1)x_{n-1} \geqslant (\beta - 1)x_0 \geqslant 1 - \gamma$. For this sequence x_n, $n = 0, 1, 2, \ldots,$ we prove the following:

THEOREM 1. *Let $b_d, \ldots, b_1, b_0$ be some integers, not all zero. Then the sequence*

$$w_n = b_d x_{n+d} + \cdots + b_1 x_{n+1} + b_0 x_n, \quad n = 0, 1, 2, \ldots,$$

is bounded by an absolute constant $B = B(b_0, \ldots, b_d, \beta, \gamma, x_0)$ independent of n if and only if $\beta > 1$ is an algebraic number and the polynomial $b_d X^d + \cdots + b_1 X + b_0$ is divisible by the minimal polynomial of β in $\mathbb{Z}[X]$.

For the sequence $x_0 < x_1 < x_2 < \cdots$, where $x_n = \lfloor \beta x_{n-1} + \gamma \rfloor$ with an algebraic number $\beta > 1$, we prove the following:

THEOREM 2. *Let $\beta > 1$ be an algebraic number with minimal polynomial $b_d X^d + \cdots + b_1 X + b_0 \in \mathbb{Z}[X]$. If the sequence*

$$w_n = b_d x_{n+d} + \cdots + b_1 x_{n+1} + b_0 x_n, \quad n = 0, 1, 2, \ldots,$$

is ultimately periodic, then β must be either a Pisot number or a Salem number.

Note that the conclusion of Theorem 2 is the same as that of Schmidt [**20**] and as that of the author [**5**], where x_n was defined as $x_n = \lfloor \xi \beta^n \rfloor$ with $\xi \neq 0$. We remark that the same statements as those of Theorems 1 and 2 hold if we replace the map $T_{\beta,\gamma} : x \mapsto \lfloor \beta x + \gamma \rfloor$ by the map

$$U_{\beta,\gamma} : x \mapsto \lceil \beta x + \gamma \rceil,$$

where $(\beta - 1)x_0 + \gamma > 0$. (See the proofs of these two theorems in Sections 2 and 3.) The condition $(\beta - 1)x_0 + \gamma > 0$ implies that the sequence $x_n = \lceil \beta x_{n-1} + \gamma \rceil = U^n_{\beta,\gamma}(x_0)$ is strictly increasing, i.e. $x_0 < x_1 < x_2 < \cdots$. This sequence with $\gamma = 0$ was considered by Odłyzko and Wilf [**16**]. They proved that if $\beta \geqslant 2$ or $\beta = 2 - 1/q$ with some integer $q \geqslant 2$ then $x_n = \lfloor c(\beta)\beta^n \rfloor$ for each $n \geqslant 0$ and some constant $c(\beta)$.

Clearly, if $\beta > 1$ is a rational integer then $w_n = x_{n+1} - \beta x_n = 0$, so the sequence considered in Theorem 2 is purely periodic. In Section 6 we shall consider the sequence $x_0 \in \mathbb{N}$, $x_n = \lceil \beta x_{n-1} \rceil$, $n = 1, 2, 3, \ldots,$ with a quadratic Pisot number β. We will show that in this case the sequence w_n, $n = 0, 1, 2, \ldots,$ considered in Theorem 2 is also purely periodic.

Finally, let β be a rational number which is not an integer, i.e. $\beta = p/q$, where $p > q > 1$ are some coprime integers. Consider the map

$$U_{p/q} : x \mapsto \lceil px/q \rceil.$$

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The sequence of iterations

$$x_n = \lceil px_{n-1}/q \rceil = U_{p/q}^n(x_0), \quad n = 0, 1, 2, \ldots,$$

where x_0 is a positive integer, is strictly increasing $x_0 < x_1 < x_2 < \cdots$. We have

$$w_n = qx_{n+1} - px_n = q\lceil px_n/q \rceil - px_n \in \{0, 1, \ldots, q-1\},$$

so the sequence w_n, $n = 0, 1, 2, \ldots,$ is bounded. Since $\beta = p/q$ is neither a Pisot number nor a Salem number, by Theorem 2 (see also the remark above concerning its application to $U_{\beta,\gamma}$), this sequence is not ultimately periodic. In case $p < q^2$, we can prove much more than that.

THEOREM 3. *Let $w = w_0, w_1, w_2, \ldots$ be a sequence given by $w_n = qx_{n+1} - px_n$, $n = 0, 1, 2, \ldots,$ where $p > q > 1$ are two coprime integers, x_0 is a positive integer and $x_n = \lceil px_{n-1}/q \rceil$ for each $n \geqslant 1$. Then $\liminf_{n\to\infty} P(w, n)/n \geqslant \log q/\log(p/q)$.*

Here, $P(w, n)$ is the *complexity function* (or *block-complexity function*) of the sequence $w = w_0, w_1, w_2, \ldots,$ which, for every positive integer n, is defined as the number of distinct vectors $(w_j, w_{j+1}, \ldots, w_{j+n-1})$ of length n, where j runs through all non-negative integers $0, 1, 2, \ldots$. Clearly, the function $P(w, n)$ is non-decreasing in n. It is bounded from above by an absolute constant independent of n if and only if the sequence w is ultimately periodic; otherwise, $P(w, n) \geqslant n + 1$ for each positive integer n (see [**14**] or [**15**]). The sequences w for which equality $P(w, n) = n + 1$ holds for each positive integer n are called *Sturmian sequences* (see [**3**, **4**, **14**]). They have the lowest possible complexity among all sequences which are not ultimately periodic.

Note that in case $p < q^2$ the constant $\log q/\log(p/q)$ is greater than 1. So, by Theorem 3, $\liminf_{n\to\infty} P(w, n)/n > 1$. In particular, this implies that the sequence w considered in Theorem 3 is not Sturmian. If $p < q^{3/2}$ then $\log q/\log(p/q) > 2$, so the sequence w cannot belong to the class of *Arnoux–Rauzy sequences* which have complexity $2n + 1$. Since $w_n \pmod q = -px_n \pmod q$ and $\gcd(p, q) = 1$, the complexity $P(w, n)$ of w is equal to the complexity $P(\mathcal{X}, n)$ of the sequence $\mathcal{X} = x_n \pmod q$, $n = 0, 1, 2, \ldots$.

For $p/q = 3/2$, the map $U_{3/2}$ is given by

$$U_{3/2}(x) = \begin{cases} 3x/2, & \text{if } x \text{ is even,} \\ (3x+1)/2, & \text{if } x \text{ is odd.} \end{cases}$$

This map was studied in [**8**, p. 127]. It is related to the distribution of the fractional parts $\{\xi(3/2)^n\}$, $n = 0, 1, 2, \ldots$. The sequence given by $x_0 = 1$ and $x_n = \lceil 3x_{n-1}/2 \rceil = U_{3/2}^n(x_0)$ for $n \geqslant 1$ is exactly the sequence A061419 of [**21**]. See also [**2**], where similar sequences are used for expansions of integers in rational non-integer base. A corresponding $w_n = 2x_{n+1} - 3x_n = 2U_{3/2}(x_n) - 3x_n$ is equal to 0 if x_n is even, and to 1 if x_n is odd. So $w_n = x_n \pmod 2$. Theorem 3 implies the following:

COROLLARY 4. *Let $0 < x_0 < x_1 < x_2 < \cdots$ be a sequence of integers given by $x_n = \lceil 3x_{n-1}/2 \rceil$, $n = 1, 2, 3, \ldots$. Set $X_n = x_n \pmod 2 \in \{0, 1\}$ for $n \geqslant 0$, and let $\mathcal{X} = X_0, X_1, X_2, \ldots$. Then $P(\mathcal{X}, n) > 1.70951129n$ for each sufficiently large n.*

This corollary is the first result which claims something more than just non-periodicity of the sequence of iterations given by the map $U_{3/2}$. The famous unsolved

$3x+1$ problem asserts that the sequence of iterations given by a very similar map

$$U(x) = \begin{cases} x/2, & \text{if } x \text{ is even}, \\ (3x+1)/2, & \text{if } x \text{ is odd}, \end{cases}$$

which starts at a positive integer must end up with the cycle $2 \mapsto 1 \mapsto 2 \mapsto 1 \mapsto \cdots$. Let us write the letter F for the first alternative $x \mapsto x/2$ and the letter S for the second alternative $x \mapsto (3x+1)/2$. Starting from 15, we have

$$15 \mapsto 23 \mapsto 35 \mapsto 53 \mapsto 80 \mapsto 40 \mapsto 20 \mapsto 10 \mapsto 5 \mapsto 8 \mapsto 4 \mapsto 2 \mapsto 1 \mapsto 2 \mapsto 1 \cdots.$$

A corresponding sequence of letters is $SSSSFFFFSFFFSFS\ldots = S^4F^4SF^2(FS)^\infty$. Of course, the sequence of F, S is the following sequence of 0, 1

$$x_0, x_1, x_2, x_3, \ldots \pmod 2,$$

where F corresponds to 0 and S corresponds to 1. Assume that the $3x+1$-conjecture is false. Then there is either a non-trivial cycle or the sequence x_n, $n = 1, 2, 3, \ldots$, is unbounded. In the latter case (sometimes this is called the case of divergent trajectories), we shall prove the following speculative result:

THEOREM 5. *Let $x_0, x_1, x_2, \ldots$ be a sequence of positive integers given by $x_n = U(x_{n-1})$, $n = 1, 2, 3, \ldots$. Assume that $x_n \to \infty$ as $n \to \infty$. Set $X_n = x_n \pmod 2 \in \{0, 1\}$ for $n \geqslant 0$, and let $\mathcal{X} = X_0, X_1, X_2, \ldots$. Then $P(\mathcal{X}, n) > 1.70951129n$ for each sufficiently large n.*

For $\mathcal{X}$ given in Corollary 4, we conjecture that $P(\mathcal{X}, n) = 2^n$ for every positive integer n. More generally, we conjecture that $P(w, n) = q^n$ for every sequence w considered in Theorem 3. (See also [**16**], where an even stronger statement is conjectured in case $q = p - 1$.) It seems very likely that this conjecture is as difficult as a corresponding conjecture claiming that the complexity function $P(\alpha, n)$ of the expansion of an algebraic irrational number α in base $q \geqslant 2$, i.e.

$$\alpha = \lfloor \alpha \rfloor + \sum_{k=1}^{\infty} g_k(\alpha) q^{-k},$$

$g_k(\alpha) \in \{0, 1, \ldots, q-1\}$, defined as the complexity of the sequence $g_k(\alpha)$, $k = 1, 2, 3, \ldots$, is equal to q^n. So far the equality $P(\alpha, n) = q^n$ is out of reach. By a result of Adamczewski and Bugeaud [**1**], we know that $P(\alpha, n)/n \to \infty$ as $n \to \infty$ for each algebraic irrational number α. One among earlier results [**7**] implies that $P(\alpha, n) - n \to \infty$ as $n \to \infty$. Analogously, in our problem, Theorem 3 implies that $P(w, n) - n \to \infty$ as $n \to \infty$ in case $p < q^2$.

The sequence considered in Corollary 4 is related to the so-called Josephus problem (see, e.g., [**13**, **16**, **19**]). There are N places arranged around a circle and numbered clockwise $1, 2, \ldots, N$. Each of N people takes one of the places. Then the pth is executed. If some place is just vacated, then the pth one of the remaining survivors clockwise will be executed next and so on, until just one remains. Which is the initial place $J_p(N)$ of the last survivor? The answer is given in terms of one of the above sequences. Given integer $p \geqslant 2$, consider the sequence $x_0, x_1, x_2, \ldots$ defined by $x_0 = 1$ and $x_n = \lceil p x_{n-1}/(p-1) \rceil$ for $n \geqslant 1$. Then

$$J_p(N) = pN + 1 - x_k,$$

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where k is the least integer such that $x_k > (p-1)N$ (see, e.g., Section 3.3 in **[10]** or **[16]**).

Note that $w_n = (p-1)x_{n+1} - px_n$ modulo $p-1$ is equal to $x_n \pmod{(p-1)}$. Put $X_n = x_n \pmod{(p-1)} \in \{0, 1, \dots, p-2\}$. The constant

$$K(p) = 1 + \frac{1}{p}\sum_{k=0}^{\infty} X_k \left(\frac{p-1}{p}\right)^k$$

appears in **[16]**, where the exact formula for $J_3(n)$ was obtained. In particular, $K(2) = 1$, $K(3) = 1.6222705028\dots$. Theorem 3 implies that, for every integer $p \geqslant 3$ and every $\varepsilon > 0$, the complexity function $P(\mathcal{X}, n)$ of the sequence $\mathcal{X} = X_0, X_1, X_2, \dots$ is at least $(1/(\log p/\log(p-1) - 1) - \varepsilon)n$ for each sufficiently large n.

2. Proof of Theorem 1. Write $x_{n+m} = \lfloor \beta x_{n+m-1} + \gamma \rfloor = \beta x_{n+m-1} + \tau_{n+m-1}$ for each $n \geqslant 0$ and each $m \geqslant 1$, where $\tau_0, \tau_1, \tau_2, \dots \in (\gamma - 1, \gamma]$. Then

$$x_{n+m} = \beta^m x_n + \beta^{m-1}\tau_n + \beta^{m-2}\tau_{n+1} + \dots + \tau_{n+m-1}.$$

Applying this formula to $m = 1, 2, \dots, d$ and putting corresponding values into

$$w_n = b_d x_{n+d} + \dots + b_1 x_{n+1} + b_0 x_n,$$

we find that

$$w_n = (b_d\beta^d + \dots + b_1\beta + b_0)x_n + \sum_{j=0}^{d-1} \tau_{n+j} \sum_{i=0}^{d-j-1} b_{i+j+1}\beta^i.$$

Since $|\sum_{i=0}^{d-j-1} b_{i+j+1}\beta^i| \leqslant \beta^{d-1}(|b_d| + \dots + |b_1|)$ and $|\tau_{n+j}| < |\gamma| + 1$, the modulus of the double sum is bounded from above by $B_0 = d\beta^{d-1}(|b_d| + \dots + |b_1|)(|\gamma| + 1)$. Hence the sequence w_n, $n = 0, 1, 2, \dots$, is bounded by a constant B independent of n if and only if the term $(b_d\beta^d + \dots + b_1\beta + b_0)x_n$, $n = 0, 1, 2, \dots$, is bounded. However, $x_n \to \infty$ as $n \to \infty$, because $x_0 < x_1 < x_2 < \cdots$ is strictly increasing. Evidently, if there is a constant B_1 independent of n such that $|(b_d\beta^d + \dots + b_1\beta + b_0)x_n| \leqslant B_1$ for each $n \geqslant 0$ then $b_d\beta^d + \dots + b_1\beta + b_0 = 0$. Hence $b_dX^d + \dots + b_1X + b_0$ is divisible by the minimal polynomial of β in $\mathbb{Z}[X]$.

On the other hand, if $b_dX^d + \dots + b_1X + b_0$ is divisible by the minimal polynomial of an algebraic number β then $b_d\beta^d + \dots + b_1\beta + b_0 = 0$. Thus $|w_n| = |\sum_{j=0}^{d-1} \tau_{n+j} \sum_{i=0}^{d-j-1} b_{i+j+1}\beta^i|$ is bounded from above by the constant $B_0 = d\beta^{d-1}(|b_d| + \dots + |b_1|)(|\gamma| + 1)$.

3. Proof of Theorem 2. Below, we shall use the following lemma (which is a special case of Lemma 1 in **[6]**):

LEMMA 6. *Let $a_dX^d + \dots + a_1X + a_0 = a_d(X - \alpha_1)\cdots(X - \alpha_d) \in \mathbb{Z}[X]$ be an irreducible polynomial, $a_d > 0$, and let z_n, $n = 0, 1, 2, \dots$ be a sequence of integers satisfying $a_dz_{n+d} + \dots + a_1z_{n+1} + a_0z_n = 0$ for each $n \geqslant n_0$. Then $\alpha = \alpha_1$ is an algebraic integer, namely $a_d = 1$, and there is a polynomial $Q(X) \in \mathbb{Q}[X]$ such that*

$$z_n = Q(\alpha_1)\alpha_1^n + \dots + Q(\alpha_d)\alpha_d^n$$

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for each $n \geqslant n_0$.

Suppose that there is a positive integer t such that $w_n = w_{n+t}$ for each $n \geqslant n_0$. Then

$$b_d(x_{n+d+t} - x_{n+d}) + \cdots + b_1(x_{n+1+t} - x_{n+1}) + b_0(x_{n+t} - x_n) = 0$$

for each $n \geqslant n_0$. Here, each difference $x_{n+j+t} - x_{n+j}$, where $j = 0, 1, \ldots, d$, is a positive integer. Hence, by Lemma 6, there exists a polynomial $G(X)$ with rational coefficients such that

$$x_{n+t} - x_n = G(\beta_1)\beta_1^n + \cdots + G(\beta_d)\beta_d^n$$

for each $n \geqslant n_0$, where $\beta_1 = \beta, \beta_2, \ldots, \beta_d$ are the conjugates of $\beta > 1$ over $\mathbb{Q}$, and β must be an algebraic integer, i.e. $b_d = 1$. If $d = 1$, namely, $\beta > 1$ is a rational number, then β must be a positive integer greater than 1. So it is a Pisot number.

Suppose that $d \geqslant 2$. Using the inequality

$$|x_{n+1} - \beta x_n| = |\lfloor \beta x_n + \gamma \rfloor - \beta x_n| = |\gamma - \{\beta x_n + \gamma\}| < |\gamma| + 1,$$

which holds for each $n \geqslant 0$, we deduce that

$$\left| \sum_{j=1}^{d} G(\beta_j)\beta_j^{n+1} - \beta_1 \sum_{j=1}^{d} G(\beta_j)\beta_j^n \right| = |x_{n+t+1} - x_{n+1} - \beta(x_{n+t} - x_n)| < 2(|\gamma| + 1).$$

So the modulus of

$$\delta_n = \sum_{j=1}^{d} G(\beta_j)\beta_j^{n+1} - \beta_1 \sum_{j=1}^{d} G(\beta_j)\beta_j^n = \sum_{j=2}^{d} (\beta_j - \beta_1) G(\beta_j)\beta_j^n$$

is smaller than $2(|\gamma| + 1)$ for every $n \geqslant n_0$.

Taking $d - 1$ consecutive equations for $\delta_n, \ldots, \delta_{n+d-2}$, where $n \geqslant n_0$ and

$$\delta_{n+i} = \sum_{j=2}^{d} (\beta_j - \beta_1)\beta_j^i G(\beta_j)\beta_j^n, \quad i = 0, 1, \ldots, d-2,$$

we see that the vector $(\beta_2^n, \ldots, \beta_d^n)$ is a solution of a non-homogeneous linear system. By Cramer's rule, this linear system has a unique solution, because the corresponding matrix $A = ((\beta_j - \beta_1)G(\beta_j)\beta_j^i)_{0 \leqslant i \leqslant d-2,\ 2 \leqslant j \leqslant d}$ is non-singular. Indeed, its determinant is equal to the Vandermonde determinant $\prod_{2 \leqslant k < j \leqslant d}(\beta_j - \beta_k)$ multiplied by the factor $\prod_{j=2}^{d}(\beta_j - \beta_1)G(\beta_j)$. Here $(\beta_j - \beta_1)G(\beta_j) \neq 0$ for $j = 2, \ldots, d$, because $G(\beta_j) \neq 0$ for each j. Hence the matrix A is non-singular.

Now, using the fact that $|\delta_n|, \ldots, |\delta_{n+d-2}| < 2(|\gamma| + 1)$, by Cramer's rule, we deduce that each $|\beta_j^n|$, where $j = 2, \ldots, d$ and $n \geqslant n_0$, is bounded from above by a constant C independent of n. The inequality $|\beta_j^n| \leqslant C$, where $n = n_0, n_0 + 1, n_0 + 2, \ldots$, shows that $|\beta_j| \leqslant 1$. Thus $|\beta_j| \leqslant 1$ for every $j = 2, \ldots, d$. Since $\beta = \beta_1 > 1$ is an algebraic integer, we conclude that β must be either a Pisot number or a Salem number.

4. Proof of Theorem 3. Since $w_n = qx_{n+1} - px_n \in \{0, 1, \dots, q-1\}$ for each $n \geqslant 0$, expressing x_{n+m} as $px_{n+m-1}/q + w_{n+m-1}/q$ and so on, we obtain

$$x_{n+m} = (p/q)^m x_n + q^{-1}((p/q)^{m-1} w_n + (p/q)^{m-2} w_{n+1} + \cdots + w_{n+m-1}).$$

Suppose that the limit $\liminf_{n\to\infty} P(w, n)/n$ is strictly smaller than $\log q/\log(p/q)$. Then there is an infinite sequence of positive integers $m_1 < m_2 < m_3 < \cdots$ such that $P(w, m_k) \leqslant m_k(\log q/\log(p/q) - \varepsilon)$ for some $\varepsilon > 0$ and each $k \geqslant 1$.

Set $m = m_k$ for some fixed $k \geqslant 1$ which is so large that

$$\varepsilon m_k \log(p/q) > \log(x_0 + q - 1).$$

Consider the vectors $(w_n, w_{n+1}, \dots, w_{n+m-1})$ for $n = 0, 1, \dots, \lfloor m(\log q/\log(p/q) - \varepsilon)\rfloor$. There are more than $m(\log q/\log(p/q) - \varepsilon) \geqslant P(w, m)$ of such vectors, so at least two of them must be equal, say $(w_s, \dots, w_{s+m-1}) = (w_n, \dots, w_{n+m-1})$, where $0 \leqslant s < n \leqslant \lfloor m(\log q/\log(p/q) - \varepsilon)\rfloor$. Subtracting

$$x_{s+m} = (p/q)^m x_s + q^{-1}((p/q)^{m-1} w_s + (p/q)^{m-2} w_{s+1} + \cdots + w_{s+m-1})$$

from x_{n+m}, we deduce that

$$x_{n+m} - x_{s+m} = (p/q)^m (x_n - x_s).$$

Hence q^m divides $x_n - x_s$. Since $x_n > x_s > 0$, this implies that q^m must be smaller than x_n. But

$$x_n = (p/q)^n x_0 + q^{-1}((p/q)^{n-1} w_0 + (p/q)^{n-2} w_1 + \cdots + w_{n-1}),$$

so, using $w_j \leqslant q - 1$, we deduce that

$$q^m < x_n \leqslant (p/q)^n x_0 + ((p/q)^n - 1)(q-1)/(p-q) < (p/q)^n (x_0 + q - 1).$$

By taking the logarithms of both sides and using

$$n \leqslant \lfloor m(\log q/\log(p/q) - \varepsilon)\rfloor \leqslant m(\log q/\log(p/q) - \varepsilon),$$

we obtain

$$m \log q < \log x_n < n \log(p/q) + \log(x_0 + q - 1) \leqslant m \log q - \varepsilon m \log(p/q) + \log(x_0 + q - 1).$$

It follows that $\varepsilon m_k \log(p/q) = \varepsilon m \log(p/q) < \log(x_0 + q - 1)$, contrary to our assumption on m_k.

5. Proof of Theorem 5. Note that $x_{n+1} = (u_n x_n + v_n)/2$, where $(u_n, v_n) = (1, 0)$ if $X_n = x_n \pmod 2 = 0$ and $(u_n, v_n) = (3, 1)$ if $X_n = x_n \pmod 2 = 1$. Let $n \geqslant 0$ and $m \geqslant 1$ be two integers. Expressing x_{n+m} by x_{n+m-1} and so on up to x_n, we obtain

$$x_{n+m} = \frac{u_{n+m-1} \cdots u_n x_n}{2^m} + \frac{u_{n+m-1} \cdots u_{n+1} v_n}{2^m} + \frac{u_{n+m-1} \cdots u_{n+2} v_{n+1}}{2^{m-1}} + \cdots + \frac{v_{n+m-1}}{2}.$$

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Suppose that the limit $\liminf_{n\to\infty} P(\mathcal{X}, n)/n$ is strictly smaller than $\log 2/\log(3/2)$. Then there is an infinite sequence of positive integers $m_1 < m_2 < m_3 < \cdots$ such that $P(\mathcal{X}, m_k) \leqslant m_k(\log 2/\log(3/2) - \varepsilon)$ for some $\varepsilon > 0$ and each $k \geqslant 1$.

Fix any $m \in \{m_1, m_2, m_3, \cdots\}$ satisfying

$$m \log(3/2) > \varepsilon^{-1} \log(x_0 + 1).$$

Consider the vectors $(X_n, X_{n+1}, \ldots, X_{n+m-1})$ for $n = 0, 1, \ldots, \lfloor m(\log 2/\log(3/2) - \varepsilon)\rfloor$. There are more than $m(\log 2/\log(3/2) - \varepsilon) \geqslant P(\mathcal{X}, m)$ of such vectors. Hence, at least two of them must be equal, for instance $(X_s, \ldots, X_{s+m-1}) = (X_n, \ldots, X_{n+m-1})$, where $0 \leqslant s < n \leqslant \lfloor m(\log 2/\log(3/2) - \varepsilon)\rfloor$. Subtracting

$$x_{s+m} = \frac{u_{s+m-1}\cdots u_s x_s}{2^m} + \frac{u_{s+m-1}\cdots u_{s+1} v_s}{2^m} + \frac{u_{s+m-1}\cdots u_{s+2} v_{s+1}}{2^{m-1}} + \cdots + \frac{v_{s+m-1}}{2}$$

from a corresponding expression for x_{n+m} and using $u_{n+j} = u_{s+j}$, $v_{n+j} = v_{s+j}$ for $j = 0, 1, \ldots, m-1$, we derive that

$$x_{n+m} - x_{s+m} = \frac{u_{n+m-1}\cdots u_n}{2^m}(x_n - x_s).$$

Recall that $u_k \in \{1, 3\}$, so $\gcd(u_{n+m-1}\cdots u_n, 2^m) = 1$. Hence 2^m divides $|x_n - x_s|$. We claim that $x_n \neq x_s$. Indeed, if $x_n = x_s$ then the sequence $x_s, x_{s+1}, x_{s+2}, \ldots$ is an infinite repetition of the string $x_s, \ldots, x_{n-1}$. So the sequence $x_0, x_1, x_2, \ldots$ is bounded, contrary to the condition of the theorem. From

$$x_n = \frac{u_{n-1}\cdots u_0 x_0}{2^n} + \frac{u_{n-1}\cdots u_1 v_0}{2^n} + \frac{u_{n-1}\cdots u_2 v_1}{2^{n-1}} + \cdots + \frac{v_{n-1}}{2},$$

using $u_k \in \{1, 3\}$, $v_k \in \{0, 1\}$, we derive that $x_n < (3/2)^n(x_0 + 1)$. Similarly, $x_s < (3/2)^s(x_0 + 1)$. Hence,

$$2^m \leqslant |x_n - x_s| < (3/2)^n(x_0 + 1),$$

because $n > s$.

By taking the logarithms and using

$$n \leqslant \lfloor m(\log 2/\log(3/2) - \varepsilon)\rfloor \leqslant m(\log 2/\log(3/2) - \varepsilon),$$

we obtain

$$m \log 2 < n \log(3/2) + \log(x_0 + 1) \leqslant m \log 2 - \varepsilon m \log(3/2) + \log(x_0 + 1).$$

Consequently, $m \log(3/2) < \varepsilon^{-1} \log(x_0 + 1)$, contrary to our assumption on m.

Therefore, $\liminf_{n\to\infty} P(\mathcal{X}, n)/n \geqslant \log 2/\log(3/2)$, giving $P(\mathcal{X}, n) > 1.70951129n$ for each sufficiently large n.

6. Examples. Let us take $\beta = (1 + \sqrt{5})/2$. Consider the map $x \mapsto \lceil \beta x\rceil$ and a sequence of iterations $x_0 = 1$, $x_n = \lceil \beta x_{n-1}\rceil$ associated to it. Clearly, the golden mean $(1 + \sqrt{5})/2$ is a Pisot number, because its conjugate $\theta = (1 - \sqrt{5})/2$ lies in $(-1, 0)$. We claim that

$$x_n = F_{n+2} - 1 \quad \text{for each} \quad n \geqslant 0.$$

Here F_n is the nth Fibonacci number, given by $F_0 = F_1 = 1$, $F_{n+2} = F_{n+1} + F_n$.

We will show first that

$$x_{n+2} = x_{n+1} + x_n + 1$$

for each $n \geqslant 0$. Indeed, writing $x_{n+1} = \beta x_n + \tau_n$ and $x_{n+2} = \beta x_{n+1} + \tau_{n+1}$, where $\tau_n, \tau_{n+1} \in (0, 1)$, we obtain

$$\begin{aligned} x_{n+2} - x_{n+1} - x_n &= \beta x_{n+1} + \tau_{n+1} - x_{n+1} - x_n \\ &= (\beta^2 - \beta - 1)x_n + (\beta - 1)\tau_n + \tau_{n+1} = (\beta - 1)\tau_n + \tau_{n+1}. \end{aligned}$$

Since $x_{n+2} - x_{n+1} - x_n \in (0, \beta)$ is an integer, it is equal to 1. Hence $x_{n+2} = x_{n+1} + x_n + 1$, as claimed. In particular, we see that, for the sequence $x_0, x_1, x_2, \ldots$, a corresponding sequence $w_n = x_{n+2} - x_{n+1} - x_n$, $n = 0, 1, 2, \ldots$, considered in Theorem 2 is purely periodic.

Next, using $x_0 = 1 = F_2 - 1$ and $x_1 = \lceil \beta \rceil = 2 = F_3 - 1$, by induction on n, we find that

$$x_{n+2} = x_{n+1} + x_n + 1 = F_{n+3} - 1 + F_{n+2} - 1 + 1 = F_{n+3} + F_{n+2} - 1 = F_{n+4} - 1,$$

so the formula $x_n = F_{n+2} - 1$ holds for each $n \geqslant 0$.

More generally, let β be a quadratic Pisot number with minimal polynomial $X^2 - aX + b$. Consider the sequence which starts with an arbitrary positive integer x_0 and is given by the formula $x_n = \lceil \beta x_{n-1} \rceil$ for $n \geqslant 1$. Let β' be the conjugate of β, i.e. $X^2 - aX + b = (X - \beta)(X - \beta')$. Writing $x_{n+1} = \beta x_n + \tau_n$, we find that

$$\begin{aligned} w_n = x_{n+2} - a x_{n+1} + b x_n &= (\beta - a)x_{n+1} + \tau_{n+1} + b x_n \\ &= ((\beta - a)\beta + b)x_n + \tau_{n+1} + (\beta - a)\tau_n = \tau_{n+1} + (\beta - a)\tau_n. \end{aligned}$$

Since $0 < \tau_n, \tau_{n+1} < 1$ and $\beta - a = -b/\beta = -\beta'$, where $-1 < \beta' < 1$, we see that $w_n \in (0, 1 - \beta')$ if β' is negative and $w_n \in (-\beta', 1)$ if β' is positive. It follows that, for each $n \geqslant 0$, we have $w_n = 1$ if $\beta' < 0$ and $w_n = 0$ if $\beta' > 0$. In both cases, the sequence w_n, $n = 0, 1, 2, \ldots$, is purely periodic.

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