ON THE CHANGES OF SIGN OF A CERTAIN ERROR FUNCTION

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1. Introduction. Though much effort has been expended in studying the mean values of arithmetic functions there is one case which has not yielded a great deal either to elementary or analytic methods. The case to which we refer is that of estimating

$$\Phi(x) = \sum_{n \leqslant x} \phi(n), \tag{1.1}$$

where $\phi(n)$ is the Euler function (i.e. $\phi(n)$ = the number of integers less than n which are relatively prime to n). If we define the error function $R(x)$ via

$$R(x) = \Phi(x) - \frac{3}{\pi^2} x^2, \tag{1.2}$$

the question reduces to studying the behaviour of $R(x)$. The first result is due to Dirichlet [1], who proved that

$$R(x) = O(x^{\delta}) \tag{1.3}$$

for some δ, $1 < \delta < 2$. This was improved by Mertens [2] to

$$R(x) = O(x \log x). \tag{1.4}$$

The proofs in both cases are very short and simple and may be found in various textbooks [1], [3]. It is therefore of particular interest that to date there has been no improvement in the estimate for $R(x)$ beyond (1.4).

In a different direction Pillai and Chowla [4] have proved that

$$R(x) \neq o(x \log \log \log x), \tag{1.5}$$

and

$$\sum_{n \leqslant x} R(n) \sim \frac{3}{2\pi^2} x^2. \tag{1.6}$$

Sylvester, [5], [6], conjectured among other things that for all integers $x > 0$, $R(x) > 0$. This was disproved by M. L. N. Sarma [7], by the simple expedient of showing that $R(820) < 0$.

In this paper we propose to prove that $R(x)$ changes sign for infinitely many integers x. More precisely, there exists a positive constant c and infinitely many integers x such that

$$R(x) > c\, x \log \log \log \log x, \tag{1.7}$$

and infinitely many integers x such that

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$$R(x) < -c\,x \log\log\log\log x. \tag{1.8}$$

2. The evaluation of certain sums. The proofs of the results mentioned in the introduction are obtained by first treating the error function

$$H(x) = \sum_{n \leqslant x} \frac{\phi(n)}{n} - \frac{6}{\pi^2}\,x.$$

The relationship between $H(x)$ and $R(x)$ is given by

LEMMA 2.1. *For integral* x,

$$\sum_{n \leqslant x} H(n) = \frac{3}{\pi^2}\,x + (x+1)\,H(x) - R(x). \tag{2.1}$$

Proof.

$$\begin{aligned}
\sum_{n \leqslant x} H(n) &= \sum_{n \leqslant x} \left\{ \sum_{n \leqslant x} \frac{\phi(n)}{n} - \frac{6}{\pi^2}\,n \right\} \\
&= \sum_{m \leqslant x} (x - n + 1)\,\frac{\phi(n)}{n} - \frac{3}{\pi^2}\,x(x+1) \\
&= (x+1)\left\{\frac{6}{\pi^2}x + H(x)\right\} - \sum_{n \leqslant x} \phi(n) - \frac{3}{\pi^2}\,x(x+1) \\
&= \frac{3}{\pi^2}\,x + (x+1)\,H(x) - R(x).
\end{aligned}$$

We will need estimates for certain sums which we now provide.

LEMMA 2.2.

$$\sum_{d \leqslant x} \frac{1}{d}\,H\left(\frac{x}{d}\right) = O(1), \tag{2.2}$$

$$\sum_{d \leqslant x} H\left(\frac{x}{d}\right) = O(x), \tag{2.3}$$

$$\sum_{d \leqslant x} R\left(\frac{x}{d}\right) = O(x). \tag{2.4}$$

Proof. (2.3) follows immediately from the fact that $H(x) = O(\log x)$. Next we consider (2.2):

$$\begin{aligned}
x + O(1) = \sum_{n \leqslant x} 1 &= \sum_{n \leqslant x} \frac{1}{n} \sum_{d \mid n} \phi(d) \\
&= \sum_{dd' \leqslant x} \frac{\phi(d)}{dd'} = \sum_{d \leqslant x} \frac{1}{d}\left\{\frac{6}{\pi^2}\frac{x}{d} + H\left(\frac{x}{d}\right)\right\} \\
&= x + O(1) + \sum_{d \leqslant x} \frac{1}{d}\,H\left(\frac{x}{d}\right),
\end{aligned}$$

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which yields (2.2). Similarly,

$$\frac{x^2}{2} + O(x) = \sum_{n \leqslant x} n = \sum_{n \leqslant x} \sum_{d|n} \phi(d)$$

$$= \sum_{d \leqslant x} \sum_{d' \leqslant x|d} \phi(d)' = \sum_{d \leqslant x} \left\{ \frac{3}{\pi^2} \frac{x^2}{d^2} + R\left(\frac{x}{d}\right) \right\};$$

whence (2.4) follows.

THEOREM 2.1.

$$\sum_{mn \leqslant x} H(n) = \frac{3}{\pi^2} x \log x + O(x). \tag{2.5}$$

Proof. From Lemma 2.1 we obtain for all $x > 0$ that

$$\sum_{n \leqslant x} H(n) = \frac{3}{\pi^2} x + x\, H(x) - R(x) + O(\log x). \tag{2.6}$$

Replacing x by x/m in (2.5) and summing over all integral $m \leqslant x$ we have

$$\sum_{m \leqslant x} \sum_{n \leqslant x|m} H(n) = \frac{3}{\pi^2} \sum_{m \leqslant x} \frac{x}{m} + x \sum_{m \leqslant x} \frac{1}{m} H\left(\frac{x}{m}\right) - \sum_{m \leqslant x} R\left(\frac{x}{m}\right) + O\,(\log x).$$

Then, taking into account the estimates of Lemma 2.2 we obtain (2.5).

Actually, Pillai and Chowla [4] have proved that

$$\sum_{n \leqslant x} H(n) \sim \frac{3}{\pi^2} x, \tag{2.7}$$

and we could use (2.7) instead of (2.5) in our development. However, the proof of (2.7) requires the prime number theorem, and we therefore introduce (2.5) for the sake of simplicity.

3. The average of $H(n)$ over arithmetic progressions. The main part of our proof consists of evaluating certain averages of $H(n)$ over arithmetic progressions. We begin with

LEMMA 3.1.

$$\sum_{\substack{m \leqslant z \\ m \equiv \beta(A)}} \frac{\phi(m)}{m} = \frac{C}{A} \sum_{d|(A,\beta)} \frac{\mu(d)}{d} z + O(\log z), \tag{3.1}$$

where

$$C = C(A) = \prod_{p \nmid A} \left(1 - \frac{1}{p^2}\right).$$

Proof.

$$\sum_{\substack{m \leqslant z \\ m \equiv \beta(A)}} \frac{\phi(m)}{m} = \sum_{\substack{dd' \equiv \beta(A) \\ dd' \leqslant z}} \frac{\mu(d)}{d}$$

$$= \sum_{\substack{d \leqslant z \\ (d,A)|\beta}} \frac{\mu(d)}{d} \left\{ \frac{(d,A)}{A} \frac{z}{d} + O(1) \right\}$$

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$$= \frac{z}{A} \sum_{\tau | (A,\beta)} \tau \sum_{(dA)=\tau} \frac{\mu(d)}{d^2} + O(\log z)$$

$$= \frac{z}{A} \sum_{\tau | (A,\beta)} \tau \frac{\mu(\tau)}{\tau} \sum_{(t,A)=1} \frac{\mu(t)}{t^2} + O(\log z)$$

$$= \frac{Cz}{A} \sum_{\tau | (A,\beta)} \frac{\mu(\tau)}{\tau} + O(\log z).$$

THEOREM 3.1. *For A, B any integers such that $A > B \geqslant 0$*

$$\sum_{n \leqslant x} H(An - B) = \frac{1}{A} \sum_{n \leqslant Ax} H(n) + \Lambda x + O(\log x) \tag{3.2}$$

where

$$\Lambda = \Lambda(A, B) = M(A, B) - 3/\pi^2,$$

$$M(A,B) = \begin{cases} \dfrac{6}{\pi^2} B - \dfrac{1}{2} \dfrac{\phi(A)C(A)}{A} - C(A) \displaystyle\sum_{c=1}^{B-1} \frac{\phi(A,c)}{(A,c)} & \text{for } B \neq 0; \\ \dfrac{1}{2} \dfrac{\phi(A)C(A)}{A} & \text{for } B = 0. \end{cases} \tag{3.3}$$

Proof. It clearly suffices to prove (3.2) for x integral, and so we assume x an integer. We have

$$\begin{aligned} \sum_{n \leqslant x} H(An - B) &= \sum_{n \leqslant x} \sum_{m \leqslant An - B} \frac{\phi(m)}{m} - \frac{6}{\pi^2} \sum_{n \leqslant x} (An - B) \\ &= \sum_{m \leqslant Ax - B} \frac{\phi(m)}{m} \sum_{\frac{m+B}{A} \leqslant n \leqslant x} 1 - \frac{3}{\pi^2} (Ax^2 + Ax - 2Bx) \\ &= \sum_{m \leqslant Ax - B} \frac{\phi(m)}{m} \left\{ x - \left[\frac{m+B}{A} \right] \right\} + \sum_{\substack{m \leqslant Ax - B \\ m \equiv -B(A)}} \frac{\phi(m)}{m} \\ &\quad - \frac{3}{\pi^2} (Ax^2 + Ax - 2Bx). \end{aligned} \tag{3.4}$$

Considering the first sum of (3.4) we have

$$\sum_{m \leqslant Ax - B} \frac{\phi(m)}{m} \left\{ x - \left[\frac{m+B}{A} \right] \right\} = x \sum_{m \leqslant Ax - B} \frac{\phi(m)}{m} - \sum_{\alpha=0}^{A-1} \sum_{\substack{m \leqslant Ax - B \\ m+B \equiv \alpha(A)}} \frac{\phi(m)}{m} \frac{m+B-\alpha}{A}$$

$$= \left\{ x \sum_{m \leqslant Ax - B} \frac{\phi(m)}{m} - \frac{1}{A} \sum_{m \leqslant Ax - B} \phi(m) \right\} - \sum_{\alpha=0}^{A-1} \frac{(B-\alpha)}{A} \sum_{\substack{m \leqslant Ax - B \\ m+B \equiv \alpha(A)}} \frac{\phi(m)}{m}$$

$$= \frac{1}{A} \left\{ (Ax - B + 1) \sum_{m \leqslant Ax - B} \frac{\phi(m)}{m} - \sum_{m \leqslant Ax - B} \phi(m) - \frac{3}{\pi^2} [(Ax - B)^2 + (Ax - B)] \right\}$$

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$$(3.5)\quad \begin{aligned} &+\frac{B-1}{A}\sum_{m\leqslant Ax-B}\frac{\phi(m)}{m}+\sum_{a=0}^{A-1}\frac{(a-B)}{A}\sum_{\substack{m\leqslant Ax-B\\ m+B\equiv a(A)}}\frac{\phi(m)}{m}\\ &+\frac{3}{\pi^2}\frac{[(Ax-B)^2+(Ax-B)]}{A}\end{aligned}$$

$$=\frac{1}{A}\sum_{n\leqslant Ax-B}H(n)+\frac{3}{\pi^2}Ax^2-\frac{3}{\pi^2}x+\sum_{a=0}^{A-1}\frac{(a-B)}{A}\sum_{\substack{m\leqslant Ax-B\\ m+B\equiv a(A)}}\frac{\phi(m)}{m}+O(\log x).$$

Next, using Lemma 3.1, we note that

$$(3.6)\quad \sum_{a=0}^{A-1}\frac{(a-B)}{A}\sum_{\substack{m\leqslant Ax-B\\ m+B\equiv a(A)}}\frac{\phi(m)}{m}=\frac{x}{A}\sum_{a=0}^{A-1}(a-B)\,C(A)\sum_{d|(A,\,a-B)}\frac{\mu(d)}{d}+O(\log x).$$

On the other hand,

$$(3.7)\quad \begin{aligned}\sum_{a=0}^{A-1}(a-B)\sum_{d|(A,\,a-B)}\frac{\mu(d)}{d}&=\sum_{c=-B}^{A-B-1}c\sum_{d|(A,\,c)}\frac{\mu(d)}{d}\\ &=\sum_{c=0}^{A-B-1}c\sum_{d|(A,\,c)}\frac{\mu(d)}{d}+\sum_{c=A-B}^{A-1}(c-A)\sum_{d|(A,\,c)}\frac{\mu(d)}{d}\\ &=\sum_{c=0}^{A-1}c\sum_{d|(A,\,c)}\frac{\mu(d)}{d}-A\sum_{c=A-B}^{A-1}\sum_{d|(A,\,c)}\frac{\mu(d)}{d}\\ &=\sum_{c=0}^{A-1}c\sum_{d|(A,\,c)}\frac{\mu(d)}{d}-A\sum_{c=1}^{B}\sum_{d|(A,\,c)}\frac{\mu(d)}{d}.\end{aligned}$$

For each term of (3.7) we have in turn

$$(3.8)\quad \begin{aligned}\sum_{c=0}^{A-1}c\sum_{d|(A,\,c)}\frac{\mu(d)}{d}&=\sum_{d|A}\frac{\mu(d)}{d}\sum_{\substack{1\leqslant c\leqslant A-1\\ c\equiv 0(d)}}c\\ &=\frac{1}{2}\sum_{d|A}\mu(d)\left\{\left(\frac{A}{d}\right)^2-\left(\frac{A}{d}\right)\right\}\\ &=\frac{1}{2}A^2\sum_{d|A}\frac{\mu(d)}{d^2}-\frac{1}{2}\phi(A);\end{aligned}$$

and

$$(3.9)\quad \sum_{c=1}^{B}\sum_{d|(A,\,c)}\frac{\mu(d)}{d}=\sum_{c=1}^{B}\frac{\phi(A,c)}{(A,c)},$$

where this last sum is 0 if $B=0$.

Combining (3.6), (3.7), (3.8), and (3.9) we get

$$(3.10)\quad \begin{aligned}\sum_{a=0}^{A-1}\left(\frac{a-B}{A}\right)\sum_{\substack{m\leqslant Ax-B\\ m+B\equiv a(A)}}\frac{\phi(m)}{m}=x\Bigg\{&\frac{C(A)A}{2}\sum_{d|A}\frac{\mu(d)}{d^2}-\frac{1}{2}\frac{\phi(A)C(A)}{\mathrm{A}}\\ &-C(A)\sum_{c=1}^{B}\frac{\phi(A,c)}{(A,c)}\Bigg\}+O(\log x).\end{aligned}$$

Finally, inserting this in (3.5), noting that $C(A) \sum_{d|A} \frac{\mu(d)}{d^2} = \frac{6}{\pi^2}$, and combining with (3.4) and Lemma 3.1 we obtain

$$
\begin{aligned}
\sum_{m \leqslant Ax} H(An-B) &= \frac{1}{A} \sum_{n \leqslant Ax} H(n) + \frac{3}{\pi^2} Ax^2 - \frac{3}{\pi^2} x \\
&\quad + \frac{3}{\pi^2} Ax - \frac{x}{2} \frac{\phi(A)C(A)}{A} - C(A)x \sum_{c=1}^{B} \frac{\phi(A,c)}{(A,c)} \\
&\quad + \frac{C(A)x\phi(A,B)}{(A,B)} - \frac{3}{\pi^2}[Ax^2+Ax-2Bx] + O(\log x) \\
&= \frac{1}{A} \sum_{n \leqslant Ax} H(n) + \Lambda x + O(\log x).
\end{aligned} \tag{3.11}
$$

THEOREM 3.2. *For A, B any integers, $A > B \geqslant 0$,*

$$\sum_{mn \leqslant x} H(An - B) = M(A,B)\, x \log x + O(x). \tag{3.12}$$

Proof. Replacing x by x/m in (3.2) and summing over all integers $m \leqslant x$, we have

$$\sum_{mn \leqslant x} H(An-B) = \frac{1}{A} \sum_{m \leqslant x} \sum_{n \leqslant Ax|m} H(n) + \Lambda x \log x + O(x).$$

Since

$$\sum_{x < m \leqslant Ax} \sum_{n \leqslant Ax|m} H(n) = O\Big(\sum_{m \leqslant Ax} 1 \Big) = O(x),$$

we get

$$\sum_{mn \leqslant x} H(An-B) = \frac{1}{A} \sum_{mn \leqslant Ax} H(n) + \Lambda\, x \log x + O(x), \tag{3.13}$$

so that via (2.5) this reduces to (3.12).

We note in passing that if we combine (3.2) with the deeper result (2.7) we have

THEOREM 3.3. *For A, B any integers, $A > B \geqslant 0$,*

$$\sum_{n \leqslant x} H(An - B) \backsim M(A, B)x. \tag{3.14}$$

4. On the changes of sign of $H(x)$. Merely to show that $H(x)$ changes sign infinitely often is easily deduced from (3.12). We note first that if $A = A_\kappa = \prod_{i=1}^{\kappa} p_i$, and κ is sufficiently large

$$\sum_{c=1}^{B-1} \frac{\phi(A,c)}{(A,c)} = \sum_{c=1}^{B-1} \frac{\phi(c)}{c} = \frac{6}{\pi^2}(B-1) + H(B-1).$$

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Thus we obtain easily for $B \neq 0$, and fixed, that

$$\lim_{\kappa\to\infty} \lim_{x\to\infty} \frac{1}{x \log x} \sum_{mn \leqslant x} H(A_\kappa n - B) = \frac{6}{\pi^2} - H(B-1).$$

Since

$$\frac{6}{\pi^2} - H(B-1) = \frac{\phi(B)}{B} - H(B),$$

this may be written as

$$\text{(4.1)} \qquad \lim_{\kappa\to\infty} \lim_{x\to\infty} \frac{1}{x \log x} \sum_{mn \leqslant x} H(A_\kappa n - B) = \frac{\phi(B)}{B} - H(B).$$

From (2.5) it follows that $H(n)$ is positive for infinitely many n, and we need only show that we cannot have $H(n) \geqslant 0$ for all sufficiently large n. For if this were so, for all sufficiently large B

$$\lim_{\kappa\to\infty} \lim_{\kappa\to\infty} \frac{1}{x \log x} \sum_{mn \leqslant x} H(A_\kappa n - B) \geqslant 0,$$

so that we would have

$$\frac{\phi(B)}{B} \geqslant H(B) \geqslant 0.$$

For $\epsilon > 0$, small, choosing a large odd number B such that $\dfrac{\phi(B)}{B} < \epsilon$, we see that

$$H(B+1) = H(B) - \frac{6}{\pi^2} + \frac{\phi(B+1)}{B+1} \leqslant \epsilon - \frac{6}{\pi^2} + \frac{1}{2} < 0,$$

which would provide a contradiction.

The above argument can be improved upon if we use the analogue of (1.5) for $H(x)$ in conjunction with (4.1). This analogue, also proved by Pillai and Chowla, asserts that

$$\text{(4.2)} \qquad H(x) \neq o(\log\log\log x).$$

Thus their exist infinitely many integral x such that

$$\text{(4.3)} \qquad |H(x)| > c \log\log\log x,$$

where k is some positive constant. From (4.3) we note that given any large number $N \geqslant 6$ we can find an integer B such that $|H(B)| > N$. We then examine two cases:

Case 1. $H(B) > N$.

In this case we obtain from (4.1) that

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$$\lim_{\kappa\to\infty}\ \lim_{x\to\infty}\ \frac{1}{x\log x}\sum_{mn\leqslant x} H(A_\kappa n - B) < -N+1;$$

and for all sufficiently large k, say $k \geqslant k_0$, we have

$$\lim_{x\to\infty}\ \frac{1}{x\log x}\sum_{mn\leqslant x} H(A_\kappa n - B) < -N+2.$$

Then for each such k there exists an $x_0 = x_0(k)$ such that, for all $x \geqslant x_0$,

$$\sum_{mn\leqslant x} H(A_\kappa n - B) < (-N+3)\,x\log x; \tag{4.4}$$

from (4.4) we see that for each $k \geqslant k_0$ we obtain an $n^* = n^*(k)$ such that

$$H(A_\kappa n^* - B) < -N + 3 \leqslant -\tfrac{1}{2}N.$$

Case 2. $H(B) < -N$.

In this case we proceed exactly as in Case (1), obtaining from (4.1) that

$$\lim_{\kappa\to\infty}\ \lim_{x\to\infty}\ \frac{1}{x\log x}\sum_{mn\leqslant x} H(A_\kappa n - B) > N.$$

This in turn yields a k_0 such that for each $k \geqslant k_0$ there is an $n^* = n^*(k)$ such that

$$H(A_\kappa n^* - B) \geqslant \tfrac{1}{2}N.$$

From the above we see that $H(x)$ assumes arbitrarily large positive and negative values. We may restate this and its implication for $R(x)$ as follows.

THEOREM 4.1. *For integral x, we have*

$$\overline{\lim}\, H(x) = \infty \quad \text{and} \quad \underline{\lim}\, H(x) = -\infty, \tag{4.5}$$

$$\overline{\lim}\, \frac{R(x)}{x} = \infty \quad \text{and} \quad \underline{\lim}\, \frac{R(x)}{x} = -\infty. \tag{4.6}$$

Proof. (4.5) is clear from the above remarks. From (2.1) and (2.7) (or the weaker estimate $\sum_{n\leqslant x} H(n) = O(x)$), we obtain

$$R(x) = x\,H(x) + O(x), \tag{4.7}$$

and (4.6) then follows from (4.5).

5. More precise results. By refining some of our estimates the arguments used above may be made to yield the still more precise result that for some $c > 0$, there exist infinitely many integers x such that

$$H(x) > c \log\log\log\log x, \tag{5.1}$$

and infinitely many such that

$$H(x) < -c \log\log\log\log x. \tag{5.2}$$

We shall now give a sketch of the proof of this.

We need to obtain the dependence of many of the estimates obtained above on the modulus A. To begin with, a glance at the proof of Lemma 3.1 yields

$$\sum_{\substack{m \leqslant z \\ m \equiv \beta(A)}} \frac{\phi(m)}{m} = \frac{C}{A} \sum_{d \mid (A, \beta)} \frac{\mu(d)}{d} z + O\left(\sum_{\tau \mid (A, \beta)} \mu^2(\tau) \log \frac{z}{\tau}\right). \tag{5.3}$$

Using (5.3) instead of (3.1) in the proof of Theorem 3.1 we obtain for integral x,

$$\sum_{n \leqslant x} H(An - B) = \frac{1}{A} \sum_{n \leqslant Ax} A(n) + \Lambda x + O(2^{\nu(A)} \log Ax), \tag{5.4}$$

where $\nu(A)$ = the number of distinct prime factors of A.

Combining (5.4) and (2.7) gives

$$\sum_{n \leqslant x} H(An - B) = M(A,B)x + O(2^{\nu(A)} \log Ax) + o(x), \tag{5.5}$$

where both the O and o are uniform in A. Then taking $x = A = \prod_{p \leqslant B} p$ and noting that then $1 - \frac{c_1}{B} < C(A) < 1 - \frac{c_2}{B}$ $(c_1 > 0,\ c_2 > 0)$, we obtain (for all sufficiently large B) that there is a constant l, independent of both A and B, such that

$$\left| \frac{1}{A} \sum_{n \leqslant A} H(An - B) + H(B) \right| \leqslant l. \tag{5.6}$$

The desired result now follows from (5.6). We know that for infinitely many B,

$$|H(B)| > c \log \log \log B.$$

There are then, two cases:

Case (a). $H(B) > k \log \log \log B$.

In this case (5.6) implies that there exists an $n^* \leqslant A$ such that

$$\begin{aligned} H(An^* - B) &\leqslant l - c \log \log \log B \\ &\leqslant -\tfrac{1}{2}c \log \log \log B \\ &\leqslant -c_1 \log \log \log \log (An^* - B), \end{aligned}$$

for large B, since for $A = \prod_{p \leqslant B} p$, $\log A \backsim B$.

Case (b). $H(B) < -c \log \log \log B$.

Then as in Case (a), (5.6) implies that there exists an $n^* \leqslant A$ such that

$$\begin{aligned} H(An^* - B) &\geqslant c \log \log \log B - l \\ &\geqslant \tfrac{1}{2}c \log \log \log B \\ &\geqslant c_1 \log \log \log \log (An^* - B). \end{aligned}$$

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Thus we see that there exist infinitely many integers x such that each of the inequalities (5.1), (5.2) hold. Combining this information with (4.7) we obtain the analogous result for the inequalities (1.7) and (1.8).

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