BOUNDED MEAN CURVATURE ISOMETRIC IMMERSIONS OF A COMPACT RIEMANNIAN MANIFOLD WITH IMAGES CONTAINED IN A TUBE

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(Received May 31, 1996)

Abstract. We characterize some isometric immersions of a compact Riemannian manifold into a tube of $S^n(\lambda)$ or $\mathbb{C}P^n(\lambda)$ (in fact, in some more general spaces in the real case) around a totally geodesic $S^q(\lambda)$ or $\mathbb{C}P^q(\lambda)$ respectively, with the norm of the mean curvature of the immersion bounded from above. This bound depends on the radius of the tube, and is related with the mean curvature of its boundary.

1. Introduction. There are some known theorems stating that a compact submanifold in a space form with large mean curvature cannot be included in a small geodesic ball. Among these, there are the results of Jorge and Xavier [**7**], getting estimates for the mean curvature of complete submanifolds included in a geodesic ball, Markvorsen [**11**], giving a rigidity theorem for compact hypersurfaces with bounded mean curvature and contained in a geodesic ball, and the authors [**2**], giving results analogous to those of Markvorsen, as well as some new results for Riemannian submanifolds of a complex space form which are included in a geodesic ball. The problem of the immersibility of a complete Riemannian manifold into a tube of the euclidean space was considered by Hasanis and Koutroufiotis [**6**] and generalized by Kitagawa [8] to immersions into tubes of Riemannian manifolds with sectional curvature bounded from above. They got lower bounds for the supremum of the length of the mean curvature. In the first part of this paper (Theorems 1.1 and 1.2) we consider this problem for immersions of compact manifolds. This stronger condition gives also stronger results: we have that if the supremum of the length of the mean curvature attains its lower bound, then the immersion is contained in the boundary of the tube; moreover, if the codimension is one, then the immersed manifold is a Riemannian covering of the boundary of the tube, and the tube must be an Eschenburg's tube (see its definition in section 2, and [**3**]). For bigger codimension, and for immersions in the sphere, we also get an splitting theorem (Corollary 1.2.1) which reduces the problem of immersions into tubes to immersions into balls. In the second part (Theorem 1.3) we consider the problem of the immersibility of a compact Riemannian manifold of dimension m into a tube of $\mathbb{C}P^n(\lambda)$ around a totally geodesic $\mathbb{C}P^q(\lambda)$, and show that the supremum of the length of the mean curvature of the immersion is bounded from below by a number which depends on the radius of the tube, and is related with the mean curvature of the tubular hypersurface. We show that the problem of getting immersions where the bound is attained is equivalent to the problem of getting minimal immersions of compact manifolds of dimension $m-2q-1$ into a $\mathbb{C}P^{n-q-1}$. In particular, geodesics in $\mathbb{C}P^{n-q-1}$ correspond to immersions of compact manifolds M of dimension $2q+2$ into a tube in $\mathbb{C}P^n(\lambda)$ around $\mathbb{C}P^q(\lambda)$ where

* Work partially supported by a DGICYT Grant No. PB94-0972 and by the E. C. Contract CHRX-CT94-0661 "G.A.D.G.E.T. III".

the supremum of the length of its mean curvature attains its bound, and the universal covering of these manifolds M is a product of a geodesic sphere in a $\mathbb{C}P^{q+1}$ and $\mathbb{R}$.

All the manifolds considered in this paper will be assumed to be connected. In order to state with more precision our results we give some notation.

Given a real number λ, let s_λ be the real function which is the solution of the problem

$$s''+\lambda s=0, \qquad s(0)=0, \qquad s'(0)=1,$$

and let c_λ, ta_λ and co_λ be the real functions

$$c_\lambda=s'_\lambda, \qquad \mathrm{ta}_\lambda=\lambda\frac{s_\lambda}{c_\lambda}, \qquad \mathrm{co}_\lambda=\frac{c_\lambda}{s_\lambda}.$$

They satisfy

$$c'_\lambda=-\lambda s_\lambda, \qquad c_\lambda^2+\lambda s_\lambda^2=1, \qquad s_{4\lambda}=s_\lambda c_\lambda, \qquad c_{4\lambda}=c_\lambda^2-\lambda s_\lambda^2.$$

Let $(\bar{M},\langle,\rangle)$ be an n-dimensional Riemannian manifold, and let P be a connected compact totally geodesic Riemannian submanifold of $\bar{M}$ of dimension $q\geq 1$. Let $r:\bar{M}\to\mathbb{R}$ be the distance to the submanifold P in $\bar{M}$. Let us denote by ∂_r the gradient of r in $\bar{M}$. If $\mathrm{cut}(P)$ is the cut locus of P in $\bar{M}$, for every point $x\in\bar{M}-(\mathrm{cut}(P)\cup P)$, the *$P$-radial sectional curvature at x* is the sectional curvature of any plane of $T_x\bar{M}$ containing $\partial_r(x)$ (see [5]).

Given any positive real number ρ, the *tube P_ρ of centre P and radius ρ* in $\bar{M}$ is the set of those points $x\in\bar{M}$ such that $\mathrm{distance}(P,x)\leq\rho$. The *tubular hypersurface ∂P_ρ of centre P and radius ρ* in $\bar{M}$ is the set of the points $x\in\bar{M}$ such that $\mathrm{distance}(P,x)=\rho$.

If $\rho<\mathrm{distance}(P,\mathrm{cut}(P))$, then P_ρ is diffeomorphic, by the exponential map, to the set of the vectors in the normal bundle $\mathcal{N}P$ to P (in $\bar{M}$) of length $\leq\rho$, and ∂P_ρ is actually a smooth hypersurface of $\bar{M}$ and is also the boundary of P_ρ. This will be the situation in this paper.

Note that if $\bar{M}$ is a manifold with boundary $\partial\bar{M}$ and $P\subset\mathrm{Int}(\bar{M})$, then the above definitions are still valid. In particular, when $\bar{M}$ is the Eschenburg manifold $\mathscr{E}_\rho^\lambda$ associated to a vector bundle $\pi:\mathscr{E}\to P$ (see Section 2), then $\mathscr{E}_\rho^\lambda=P_\rho$ and $\partial\mathscr{E}_\rho^\lambda=\partial P_\rho$, where P is identified with the 0-section of $\pi:\mathscr{E}\to P$. So, we call $\mathscr{E}_\rho^\lambda$ an Eschenburg tube. The λ in $\mathscr{E}_\rho^\lambda$ reflects the fact that $\mathscr{E}_\rho^\lambda$ has constant P-radial sectional curvature λ.

Our main results for the Riemannian case are the two following theorems and its corollary.

THEOREM 1.1. *Let P be a compact q-dimensional submanifold of a Riemannian n-dimensional manifold $\bar{M}$ with P-radial sectional curvature bounded from above by λ. Let $\psi:M\to\bar{M}$ be an isometric immersion of a compact m-dimensional manifold, $1\leq q<m<n$ ($m\geq 2q$ if $\lambda\leq 0$), with mean curvature H satisfying $m\,|H|\leq|(q-m)\mathrm{co}_\lambda(\rho)+q\,\mathrm{ta}_\lambda(\rho)|$, where ρ is a real number with $0<\rho<\mathrm{distance}(P,\mathrm{cut}(P))$ and, if $\lambda>0$, $0<\rho\leq\frac{1}{\sqrt{\lambda}}\arccos\sqrt{\frac{q}{m}}<\frac{\pi}{2\sqrt{\lambda}}$. If $\psi(M)\subset P_\rho$, then*

$$\psi(M)\subset\partial P_\rho, \qquad mH=((q-m)\mathrm{co}_\lambda(\rho)+q\,\mathrm{ta}_\lambda(\rho))\partial_r \quad \textit{and} \quad \tau_tT_pP\subset\psi_{*x}T_xM$$

for every $x\in M$, where p is the starting point of the geodesic γ realizing the distance from P to $\psi(x)$, t is the distance from P to $\psi(x)$, and τ_t is the parallel transport along γ from p to $\gamma(t)=\psi(x)$.

Moreover, if $m = n-1$, then P_ρ is isometric to an Eschenburg tube $\mathscr{E}^\lambda_\rho$ (see Section 2 for its definition), ψ is an embedding up to a covering map, and M is isometric (up to a Riemannian covering) to $\partial\mathscr{E}^\lambda_\rho$.

Let us remark that the hypothesis $\rho \le \rho_0 := \frac{1}{\sqrt{\lambda}} \arccos \sqrt{\frac{q}{m}}$, for $\lambda > 0$, is necessary, because for $\rho > \rho_0$ we have $\partial\mathscr{E}^\lambda_{\rho_0} \subset \mathscr{E}^\lambda_\rho$, and the mean curvature of $\partial\mathscr{E}^\lambda_{\rho_0}$ is 0.

Moreover, if $m > \max\{q, n-q-1\}$, by applying Theorem 1.1 to the tube $S^{n-q-1}(\lambda)_{(\pi/2\sqrt{\lambda})-\rho} = S^n(\lambda) - \mathrm{Int}(S^q(\lambda)_\rho)$ for $\rho > \rho_0$, we get that if $\psi: M \to S^n(\lambda)$ is an immersion such that $\psi(M) \subset S^n(\lambda) - \mathrm{Int}(S^q(\lambda)_\rho)$ and $m\,|H| \le |(q-m)\mathrm{co}_\lambda(\rho) + q\,\mathrm{ta}_\lambda(\rho)|$, then $\psi(M) \subset \partial S^q(\lambda)_\rho$, that is, for $\rho \ge \rho_0$ the obstruction is to be contained in $S^n(\lambda) - \mathrm{Int}(S^q(\lambda)_\rho)$, not in $S^q(\lambda)_\rho$.

Any of these results, applied to the case $\lambda > 0$ and $\rho = \rho_0$, proves that, for $m = n-1$, the only minimal immersion in $S^q(\lambda)_{\rho_0}$ is the boundary of this tube, or a Riemannian covering of it, that is, a Riemannian covering of the generalized Clifford torus. There are many examples of minimally immersed tori in S^3 (cf. [**12**]). The above argument shows that the only one contained in one of the "halves" of S^3 determined by the Clifford torus is the Clifford torus itself.

Given any Riemannian manifold $\mathcal{M}$, we shall denote by $\pi_{\mathcal{M}}$: $\hat{\mathcal{M}} \to \mathcal{M}$ its universal covering, with the induced Riemannian metric on $\hat{\mathcal{M}}$. In the next Theorem, p_1 and p_2 will denote the projections of $(P, c_\lambda(t)^2 g_P) \times S^{n-q-1}(1/s_\lambda(t)^2)$ onto the first and the second factor, respectively.

THEOREM 1.2. *Let P be a compact q-dimensional Riemannian manifold. Let $\mathscr{E}^\lambda_\rho$ be an Eschenburg tube of radius $\rho > 0$ (with $\rho \le \frac{1}{\sqrt{\lambda}} \arccos \sqrt{\frac{q}{m}} < \frac{\pi}{2\sqrt{\lambda}}$ if $\lambda > 0$) associated to a trivial vector bundle $\mathscr{E} = P \times \mathbb{R}^{n-q} \to P$ with a trivial connection D. Let M be an m-dimensional compact Riemannian manifold, $1 \le q < m < n$ ($m \ge 2q$ if $\lambda \le 0$), and let $\psi: M \to \mathscr{E}^\lambda_\rho$ be an isometric immersion with mean curvature H satisfying $m\,|H| \le |(q-m)\mathrm{co}_\lambda(\rho) + q\,\mathrm{ta}_\lambda(\rho)|$. Then there exist a compact Riemannian manifold G of dimension $m-q$, a Riemannian submersion $\pi: M \to G$ and a minimal isometric immersion $\phi: G \to S^{n-q-1}(1/s^2_\lambda(\rho))$ such that $p_2 \circ \psi = \phi \circ \pi$. Moreover $\hat{M}$ is isometric to $(\hat{P}, c_\lambda(\rho)^2 g_P) \times \hat{G}$.*

We remark that when $\lambda > 0$ and $P = S^q(\lambda)$, the tube $\mathscr{E}^\lambda_\rho$ of Theorem 1.2 is isometric to the tube $S^q(\lambda)_\rho$ of radius ρ around $S^q(\lambda)$ in $S^n(\lambda)$. In this case we have the following stronger result, a splitting theorem which, for the sphere, reduces immersions into a tube to immersions into a ball (actually, into a sphere).

COROLLARY 1.2.1. *Let M be an m-dimensional compact Riemannian manifold, and let $\psi: M \to S^n(\lambda)$ be an isometric immersion with mean curvature H satisfying $m\,|H| \le \sqrt{\lambda}\,|(q-m)\cot(\sqrt{\lambda}\rho) + q\tan(\sqrt{\lambda}\rho)|$, where $1 < q < m < n$, ρ being a real number with $0 < \rho \le \frac{1}{\sqrt{\lambda}} \arccos \sqrt{\frac{q}{m}}$. If $\psi(M) \subset S^q(\lambda)_\rho$, then*

$$\psi(M) \subset \partial S^q(\lambda)_\rho = S^q\left(\frac{\lambda}{\cos^2(\sqrt{\lambda}\,\rho)}\right) \times S^{n-q-1}\left(\frac{\lambda}{\sin^2(\sqrt{\lambda}\,\rho)}\right)$$

and there exists a compact Riemannian manifold G of dimension $m-q$ such that $M = S^q(\lambda/\cos^2(\sqrt{\lambda}\,\rho)) \times G$, and there is a minimal isometric immersion $\phi: G \to S^{n-q-1}(\lambda/\sin^2(\sqrt{\lambda}\,\rho))$ such that $\psi = \mathrm{id} \times \phi$.

Let us now consider the complex projective space $\mathbb{C}P^n(\lambda)$ of real dimension $2n$ and constant holomorphic sectional curvature 4λ, and $\mathbb{C}P^q(\lambda) \subset \mathbb{C}P^n(\lambda)$, and let $\mathcal{J}$, $\mathcal{H}$ and $\mathcal{V}$ be the distributions defined by $\mathcal{J} = \langle J\partial_r \rangle$, $\mathcal{H}_x = \tau_t T\mathbb{C}P^q(\lambda)$ (where $t = \text{distance}\,(x, \mathbb{C}P^q(\lambda))$), and $T_x\mathbb{C}P^n(\lambda) = \mathcal{H}_x \oplus \mathcal{V}_x \oplus \mathcal{J}_x \oplus \langle \partial_r \rangle$.

In Section 5 we shall see that the distribution $\mathcal{H} \oplus \mathcal{J}$ is integrable with leaves isometric to $\partial\mathcal{B}^{q+1}_{c,\lambda,\rho'}$ (the geodesic sphere of radius ρ' in $\mathbb{C}P^{q+1}(\lambda)$), with $\rho' = \dfrac{\pi}{2\sqrt{\lambda}} - \rho$ and the quotient $\partial\mathbb{C}P^q(\lambda)/\mathcal{H}\oplus\mathcal{J}$ is isometric to $\mathbb{C}P^{n-q-1}(\lambda/\sin^2(\sqrt{\lambda}\,\rho))$. We shall denote by $\Pi: \partial\mathbb{C}P^q(\lambda)_\rho \to \mathbb{C}P^{n-q-1}(\lambda/\sin^2(\sqrt{\lambda}\,\rho))$ the quotient map, which is a Riemannian submersion.

Then we shall prove

THEOREM 1.3. *Let m, n and q be integer numbers satisfying $2n > m > 2q$. Let M be a compact Riemannian manifold of dimension m and let $\psi: M \to \mathbb{C}P^n(\lambda)$ be an isometric immersion with mean curvature H satisfying*

$$m\,|H| \le \sqrt{\lambda}\,|(2q+1)\tan(\sqrt{\lambda}\,\rho) + (2q-m)\cot(\sqrt{\lambda}\,\rho)|,$$

where $\rho \le \dfrac{1}{\sqrt{\lambda}} \arccos\sqrt{\dfrac{2q+1}{m+1}} < \dfrac{\pi}{2\sqrt{\lambda}}$. If $\psi(M) \subset \mathbb{C}P^q(\lambda)_\rho$, then

$$\psi(M) \subset \partial\mathbb{C}P^q(\lambda)_\rho, \qquad mH = \sqrt{\lambda}\,((2q+1)\tan(\sqrt{\lambda}\,\rho) + (2q-m)\cot(\sqrt{\lambda}\,\rho))\partial_r.$$

Moreover,

(a) *if $m = 2n-1$, M is isometric to $\mathbb{C}P^q(\lambda)_\rho$ and ψ is an embedding.*

(b) *if $m < 2n-1$, then $\psi: M \to \partial\mathbb{C}P^q(\lambda)_\rho$ is minimal and there exist a compact manifold G of dimension $m-2q-1$, a Riemannian submersion $\pi: M \to G$ and a minimal isometric immersion $\phi: G \to \mathbb{C}P^{n-q-1}(\lambda/\sin^2(\sqrt{\lambda}\,\rho))$ such that the diagram*

$$\begin{array}{ccc} M & \xrightarrow{\ \psi\ } & \partial\mathbb{C}P^q(\lambda)_\rho \\ {\scriptstyle\pi}\downarrow & & \downarrow{\scriptstyle\Pi} \\ G & \xrightarrow{\ \phi\ } & \mathbb{C}P^{n-q-1}(\lambda/\sin^2(\sqrt{\lambda}\,\rho)), \end{array}$$

is commutative.

(b1) *If $m = 2q+1$, then ψ is an embedding and M is isometric to $\partial B^{q+1}_{c,\lambda,\rho'}$.*

(b2) *If $2n = 2q+4$, and $m = 2q+2$, then $G = S^1$, ϕ is a geodesic of $S^2(4\lambda/\sin^2(\sqrt{\lambda}\,\rho))$, and the universal covering $\hat{M}$ of M is isometric to the product $\partial B^{q+1}_{c,\lambda,\rho'} \times \mathbb{R}$.*

We thank K. Grove for showing us the reference [**1**], which is a necessary tool for our proofs of Theorems 1.2 and part (b2) of 1.3.

2. Eschenburg's tubes. Eschenburg's tubes were defined in [**3**] as a generalization of tubes around $S^q(\lambda)$ in $S^n(\lambda)$ that were appropriate to get some more general versions of the classical Heintze-Karcher's comparison theorems for the quotient of the volume of a tube by the volume of its centre. Here we shall recall the definitions and the main properties that we shall need in our context.

Let $\pi:\mathscr{E}\to P$ be a vector bundle of rank $n-q$ over a compact Riemannian manifold P of dimension q, with a fibre metric and a metric connection D. Let $\mathscr{E}_\rho=\{v\in\mathscr{E}\mid |v|\leq\rho\}$ (with $\rho\leq\dfrac{\pi}{2\sqrt{\lambda}}$ if $\lambda>0$). A metric on $\mathscr{E}_\rho$ was defined in [3] as follows: on the tangent to the fibres $\mathscr{E}_{\rho,p}$ we take the metric $dt^2+s_\lambda(t)^2 g_{S_{n-q-1}}$, where $g_{S^{n-q-1}}$ is the standard metric of the sphere S^{n-q-1} in $\mathbb{R}^{n-q}$ of radius 1, and on the horizontal distribution defined by the connection D we take $|X_v|_\lambda=c_\lambda(|v|)\,|X'_{\pi v}|$, where $X'_{\pi v}=\pi_{*v}(X)$. Moreover we declare the horizontal and vertical distributions to be perpendicular.

The manifold $\mathscr{E}_\rho$ with this metric is the model space $\mathscr{E}^\lambda_\rho$. It is a Riemannian manifold with boundary $\partial\mathscr{E}^\lambda_\rho=\{v\in\mathscr{E}\mid |v|=\rho\}$, a tubular hypersurface of radius ρ around P, which is totally geodesic in $\mathscr{E}^\lambda_\rho$. Moreover, the map $\pi:\partial\mathscr{E}^\lambda_\rho\to(P,c_\lambda(\rho)g_P)$ is a Riemannian submersion. In fact, for any $t<\rho$, the tubular hypersurface ∂P_t of radius t around P in $\mathscr{E}^\lambda_\rho$ is precisely $\partial\mathscr{E}^\lambda_t$.

For every n-dimensional Riemannian manifold $\bar M$ and every q-dimensional compact totally geodesic submanifold P of $\bar M$, $S(r)$ will denote the $(1,1)$-tensor field defined on $\bar M-(P\cup\mathrm{cut}(P))$ by $S(r)=-\bar\nabla_A\partial_r$, where $\bar\nabla$ denotes the covariant derivative in $\bar M$. The restriction of $S(r)$ to ∂P_t is the Weingarten map of ∂P_t, and $S(r)\partial_r=0$.

Let us consider the distribution $\mathscr{H}$ on $\bar M$ defined by $\mathscr{H}_x=\tau_t T_p P$, where p is the starting point of the geodesic $\gamma(t)$ realizing the distance from P to x, and $t=\mathrm{distance}(P,x)$, and τ_t is the parallel transport along this geodesic. Let $\mathscr{V}$ be the distribution given at each point x by the subspace $\mathscr{V}_x$ of $T_x\bar M$ defined by the orthogonal decomposition $T_x\bar M=\mathscr{H}_x\oplus\mathscr{V}_x\oplus\langle\partial_r\rangle$.

Let $S_\lambda(r)$ be the tensor field on $\bar M-(P\cup\mathrm{cut}(P))$ defined by

$$(S_\lambda(r)A)_x=\begin{cases}\mathrm{ta}_\lambda(t)A & \text{if } A\in\mathscr{H}_x,\\ -\mathrm{co}_\lambda(t)A & \text{if } A\in\mathscr{V}_x \text{ where } t=\mathrm{distance}(P,x),\\ 0 & \text{if } A=\partial_r.\end{cases}\tag{2.1}$$

When $\bar M=\mathscr{E}^\lambda_\rho$, then $R(r)=R_\lambda(r)$, and $S(r)=S_\lambda(r)$ (see 6.3 in [3] and formula (2.5) in [10]).

The following lemma of J. H. Eschenburg will be a key step in the proof of Theorem 1.1.

LEMMA 2.1. *Let $\bar M$ be a Riemmanian n-manifold, and P a totally geodesic compact submanifold of dimension q. If the radial sectional curvature on P_t is at most λ for some $t\leq\mathrm{distance}(P,\mathrm{cut}(P))$, if $\lambda\leq 0$, and $t\leq\min\left\{\mathrm{distance}(P,\mathrm{cut}(P)),\dfrac{\pi}{2\sqrt{\lambda}}\right\}$, if $\lambda>0$, then*

$$S(r)\leq S_\lambda(r)\ \textit{for every}\ x\in P_t.$$

Moreover, if the equality holds on ∂P_ρ for some $\rho\leq t$, then there is a vector bundle $\mathscr{E}\to P$ of rank $n-q$ and an isomorphism of vector bundles $\psi:\mathscr{N}P\to\mathscr{E}$ such that the map $\phi:P_\rho\to\mathscr{E}^\lambda_\rho$ given by $\phi(x)=\exp\circ\psi\circ\exp_P^{-1}(x)$ is an isometry.

3. Proof of Theorem 1.1. Given the isometric immersion $\psi: M \to \bar{M}$, $r \circ \psi$ will also be denoted by r. By ∂_r^{T} we shall denote the vector field in M defined as the preimage by ψ_* of (the component of) ∂_r (tangent to $\psi(M)$). If $f: \mathbb{R} \to \mathbb{R}$ is any function, we shall denote by $f(r)$ the function $f \circ r: M \to \mathbb{R}$.

From now on, ∇, Δ and H will denote, respectively, the covariant derivative of M, the laplacian of M and the mean curvature vector of the immersion $\psi: M \to \bar{M}$. We have

$$\Delta f(r) = -f''(r)\,|\partial_r^{\mathrm{T}}|^2 + f'(r)\left\{\sum_{i=1}^{m} \langle S(r)e_i, e_i\rangle - m\langle H, \partial_r\rangle\right\}. \tag{3.1}$$

This formula can be obtained by the same computations that in the case of P being a point (see, for instance [2]). Now, from the hypothesis $R(t) \le R_\lambda(t)$ and Lemma 2.1, one gets

$$\Delta f(r) \le -f''(r)\,|\partial_r^{\mathrm{T}}|^2 + f'(r)\left\{\sum_{i=1}^{m} \langle S_\lambda(r)e_i, e_i\rangle - m\langle H, \partial_r\rangle\right\}. \tag{3.2}$$

We compute the value of $\langle S_\lambda(r)e_i, e_i\rangle$ for points in $\partial P_t \cap M$, using (2.1), and orthonormal bases $\{h_1, \dots, h_q\}$ of $\mathcal{H}$ and $\{v_{q+1}, \dots, v_{n-1}\}$ of $\mathcal{V}$:

$$\langle S_\lambda(r)e_i, e_i\rangle = \left\langle S_\lambda(r)\left(\sum_{j=1}^{q} \langle e_i, h_j\rangle h_j + \sum_{k=q+1}^{n-1} \langle e_i, v_k\rangle v_k + \langle e_i, \partial_r\rangle \partial_r\right),\right.$$

$$\left.\sum_{j=1}^{q} \langle e_i, h_j\rangle h_j + \sum_{k=q+1}^{n-1} \langle e_i, v_k\rangle v_k + \langle e_i, \partial_r\rangle\partial_r\right\rangle = \sum_{j=1}^{q} \langle e_i, h_j\rangle^2 \mathrm{ta}_\lambda(t) - \sum_{h=q+1}^{n-1} \langle e_i, v_k\rangle^2 \mathrm{co}_\lambda(t)$$

$$= \sum_{j=1}^{q} \langle e_i, h_j\rangle^2 (\mathrm{ta}_\lambda(t) + \mathrm{co}_\lambda(t)) - \left(\sum_{j=1}^{q} \langle e_i, h_j\rangle^2 + \sum_{k=q+1}^{n-1} \langle e_i, v_k\rangle^2\right)\mathrm{co}_\lambda(t).$$

Then, if $m > q$,

$$\sum_{i=1}^{m} \langle S_\lambda(r)e_i, e_i\rangle = \left(\sum_{i=1}^{m}\sum_{j=1}^{q} \langle e_i, h_j\rangle^2\right)(\mathrm{ta}_\lambda(t) + \mathrm{co}_\lambda(t))$$

$$- \sum_{i=1}^{m} (1 - \langle e_i, \partial_r\rangle^2)\mathrm{co}_\lambda(t) \le q(\mathrm{ta}_\lambda(t) + \mathrm{co}_\lambda(t)) - m\,\mathrm{co}_\lambda(t) + |\partial_r^{\mathrm{T}}|^2\,\mathrm{co}_\lambda(t)$$

$$= (q - m)\mathrm{co}_\lambda(t) + q\,\mathrm{ta}_\lambda(t) + |\partial_r^{\mathrm{T}}|^2\,\mathrm{co}_\lambda(t).$$

By substitution of this inequality in (3.2), we get

$$\Delta f(r) \le (-f''(r) + \mathrm{co}_\lambda f'(r))\,|\partial_r^{\mathrm{T}}|^2 + f'(r)(\phi - m\langle H, \partial_r\rangle),$$

where ϕ is the function defined by

$$\phi(t) = (q - m)\mathrm{co}_\lambda(t) + q\,\mathrm{ta}_\lambda(t).$$

It can be easily checked that

$$\lim_{t \to 0} \phi(t) = -\infty$$

and

$$\phi' = (q-m)\frac{-\lambda s_\lambda^2 - c_\lambda^2}{s_\lambda^2} + q\frac{\lambda}{c_\lambda^2} = -(q-m)\frac{1}{s_\lambda^2} + q\frac{\lambda}{c_\lambda^2}.$$

Since $m > q$, we have that $\lambda \geq 0$ implies $\phi' > 0$. Further, it is easy to see that

$$\phi' = \frac{(m-2q)c_\lambda^2 + q}{s_\lambda^2 c_\lambda^2}$$

which is positive if $\lambda < 0$ and $m \geq 2q$.

If $\lambda \geq 0$, then ϕ is negative and increasing in the interval $]0, z^+(\phi)[$, where $z^+(\phi) = \inf\{t > 0 \mid \phi(t) = 0\}$, $(z^+(\phi) = \infty$ if $\lambda = 0$ and $z^+(\phi) = \frac{1}{\sqrt{\lambda}} \arccos \sqrt{\frac{q}{m}}$ if $\lambda > 0)$.

Finally, if $\lambda < 0$, then ϕ is negative and increasing if $m \geq 2q$.

Our hypotheses imply that ϕ is negative and increasing, and that $m\,|H| \leq -\phi$, so that, taking a function f such that $f' = s_\lambda$, we have

$$\begin{aligned} \Delta f(r) &\leq s_\lambda(\phi - m\langle H, \partial_r\rangle) \leq s_\lambda(\phi(\rho) - m\langle H, \partial_r\rangle) \\ &\leq s_\lambda(\phi(\rho) + m\,|H|) \leq 0. \end{aligned} \tag{3.3}$$

Then, by the Hopf principle, we have that $\Delta f(r) = 0$, and all the inequalities we have used to get (3.3) must be inequalities, which implies $r = \text{constant} = \rho$, $\sum_{i=1}^{m} \sum_{j=1}^{q} \langle e_i, h_j\rangle = q$, i.e. $h_j \in \langle\{e_i, \ldots, e_m\}\rangle$, which means $\mathcal{H}_\rho \subset TM$, and $mH = \phi(\rho)\partial_r$. Moreover, if $m = n-1$, then $R(r) = R_\lambda(r)$ and $S(r) = S_\lambda(r)$ at every point of P_ρ, which, by Lemma 2.1, gives the last statement of the theorem.

4. Proof of Theorem 1.2. First, let us observe that the Eschenburg tube $\mathcal{E}_\rho^\lambda$ associated to a trivial vector bundle $\mathcal{E} = P \times \mathbb{R}^{n-q}$ with a trivial connection D, is the warped product $B_\rho^\lambda \times_{c_\lambda(t)} P$, where t is the distance to the centre of B_ρ^λ, the geodesic ball of radius ρ in the space form $\mathbb{K}^{n-q}(\lambda)$ of constant sectional curvature λ. Then, for each $t \in]0, \rho]$, the tubular hypersurface ∂P_t in this space is isometric to $(P, c_\lambda(t)^2 g_P) \times S^{n-q-1}(1/s_\lambda(t)^2)$, and the distribution $\mathcal{H}$ on ∂P_t is integrable and its leaves are isometric to $(P, c_\lambda(t)^2 g_P)$. The condition $\mathcal{H}_{\psi(x)} \subset \psi_{*x} T_x M$ proved in Theorem 1.1 implies that every such leaf containing a point of $\psi(M)$ is contained in $\psi(M)$. ψ induces a distribution $\mathcal{F}$ on M which will also be integrable with compact leaves. These leaves are covering spaces of $(P, c_\lambda(\rho)^2 g_P)$. Since $\mathcal{F}$ is a regular foliation, there exists a structure of differentiable manifold on the space of its leaves $G = M/\mathcal{F}$, and there is a commutative diagram

$$\begin{array}{ccc} M & \xrightarrow{\ \psi\ } & \partial P_\rho = \partial \mathcal{E}_\rho^\lambda \\ \downarrow{\scriptstyle \pi} & & \downarrow{\scriptstyle p_2} \\ G = M/\mathcal{F} & \xrightarrow{\ \phi\ } & S^{n-q-1}\left(\dfrac{1}{s_\lambda(\rho)^2}\right), \end{array}$$

where π is the natural projection, we have identified ∂P_t with $P \times S^{n-q-1}$, via the above isometry, and ϕ is the induced map between the quotients. Since ψ is an immersion, so also is ϕ, and we can consider on G the metric induced from that of $S^{n-q-1}(1/s_\lambda(\rho)^2)$ through ϕ. Further, it is easy to see that the map $M \to (P, c_\lambda(\rho)^2 g_P) \times G$, given by

$x \mapsto (p_1 \circ \psi(x), \pi(x))$ is a local isometry. In particular, π is a Riemannian submersion. A straightforward computation shows that the mean curvature of $\psi(M)$ in ∂P_ρ vanishes. Then, ϕ is also minimal (see Lemma 2 in [9]). Moreover, it follows from Theorem A of [1] that $\hat{M}$ is isometric to $(\hat{P}, c_\lambda(\rho)^2 g_P) \times \hat{G}$.

Corollary 1.2.1 follows from the above proof. First, note that the leaves have the form $\pi^{-1}(z)$ for $z \in G$. Now we observe that, since in this case $P = S^q(\lambda)$ is simply connected, the covering maps $(p_1 \circ \psi)|_{\pi^{-1}(z)}: \pi^{-1}(z) \to (P, c_\lambda(\rho)^2 g_p) = S^q(\lambda/c_\lambda(\rho)^2)$ are isometries, and hence the map $M \to S^q(\lambda/c_\lambda(\rho)^2) \times G$ defined above is bijective. In fact, it has an inverse defined by $(y, z) \mapsto (p_1 \circ \psi|_{\pi^{-1}(z)})^{-1}(y)$. From the definition of the maps, we have that, modulo this isometry, $\psi = id \times \phi$.

5. Proof of Theorem 1.3. In $\mathbb{C}P^n(\lambda)$, with $\mathbb{C}P^q(\lambda) \subset \mathbb{C}P^n(\lambda)$, the distributions $\mathcal{J}$, $\mathcal{H}$, and $\mathcal{V}$ are defined by:

$$\mathcal{J} = \langle J\partial_r \rangle, \qquad \mathcal{H} = \tau_t T\mathbb{C}P^q(\lambda), \qquad (t = d(x, \mathbb{C}P^q(\lambda))),$$
$$T_x\mathbb{C}P^n(\lambda) = \mathcal{H}_x \oplus \mathcal{V}_x \oplus \mathcal{J}_x \oplus \langle \partial_r \rangle. \tag{5.1}$$

If $\{h_1, \ldots, h_{2q}\}$, $\{v_{2q+1}, \ldots, v_{2n-2}\}$ are local orthonormal frames of $\mathcal{H}$ and $\mathcal{V}$, the Weingarten map $S_\lambda^c(t)$ of $\partial\mathbb{C}P^q(\lambda)_t$ satisfies

$$S_\lambda^c(t)|_{\mathcal{H}} = \mathrm{ta}_\lambda(t)\,\mathrm{id}, \qquad S_\lambda^c(t)|_{\mathcal{V}} = -\mathrm{co}_\lambda(t)\,\mathrm{id}, \qquad S_\lambda^c(t)J\partial_r = -\mathrm{co}_{4\lambda}(t)J\partial_r.$$

Now, if $\{e_1, \ldots, e_m\}$ is an orthonormal basis of M, we have

$$\langle S_\lambda^c(t)e_i, e_i \rangle = \sum_{j=1}^{2q} \langle e_i, h_j \rangle^2 \mathrm{ta}_\lambda(t) - \sum_{k=2q+1}^{2n-2} \langle e_i, v_k \rangle^2 \mathrm{co}_\lambda(t) - \langle e_i, J\partial_r \rangle^2 co_{4\lambda}(t).$$

Then,

$$\begin{aligned}\sum_{i=1}^{m} \langle S_\lambda^c(t)e_i, e_i \rangle &= 2q(\mathrm{ta}_\lambda(t) + \mathrm{co}_\lambda(t)) - \sum_{i=1}^{m} \left(\sum_{j=1}^{2q} \langle e_i, h_j \rangle^2 + \sum_{k=2q+1}^{2n-2} \langle e_i, v_k \rangle^2 \right) \mathrm{co}_\lambda(t) \\ &\quad - \sum_{i=1}^{m} \langle e_i, J\partial_r \rangle^2 \mathrm{co}_{4\lambda}(t) = 2q(\mathrm{ta}_\lambda(t) + \mathrm{co}_\lambda(t)) \\ &\quad - \sum_{i=1}^{m} (1 - \langle e_i, J\partial_r \rangle^2 - \langle e_i, \partial_r \rangle^2)\mathrm{co}_\lambda(t) - \sum_{i=1}^{m} \langle e_i, J\partial_r \rangle^2 \mathrm{co}_{4\lambda}(t) \\ &= 2q(\mathrm{ta}_\lambda(t) + \mathrm{co}_\lambda(t)) - m\,\mathrm{co}_\lambda(t) + (|(J\partial_r)^{\mathrm{T}}|^2 + |\partial_r^{\mathrm{T}}|^2)\mathrm{co}_\lambda(t) \\ &\quad - |(J\partial_r)^{\mathrm{T}}|^2\,\mathrm{co}_{4\lambda}(t).\end{aligned}$$

Then, from (3.1), if $f'(r) \geq 0$, we get

$$\begin{aligned}\Delta f(r) &\leq -f''(r)\,|\partial_r^{\mathrm{T}}|^2 + f'(r)\{2q(\mathrm{ta}_\lambda(r) + \mathrm{co}_\lambda(r)) - m\,\mathrm{co}_\lambda(r) + |\partial_r^{\mathrm{T}}|^2\,\mathrm{co}_\lambda(r) \\ &\quad + |(J\partial_r)^{\mathrm{T}}|^2\,(\mathrm{co}_\lambda(r) - \mathrm{co}_{4\lambda}(r)) - m\langle H, \partial_r \rangle\} \\ &= |\partial_r^{\mathrm{T}}|^2\,(-f''(r) + f'(r)\mathrm{co}_\lambda(r)) + f'(r)(\mathrm{co}_\lambda(r) - \mathrm{co}_{4\lambda}(r))\,|(J\partial_r)^{\mathrm{T}}|^2 \\ &\quad + f'(r)\{2q(\mathrm{ta}_\lambda(r) + \mathrm{co}_\lambda(r)) - m\,\mathrm{co}_\lambda(r) - m\langle H, \partial_r \rangle\}.\end{aligned}$$

If we take the function f so that $f'(r)=s_\lambda(r)$, then

$$\begin{aligned}\Delta f(r) &\le \lambda\frac{s_\lambda(r)^2}{c_\lambda(r)}|(J\partial_r)^{\mathrm{T}}|^2+s_\lambda(r)\{(2q-m)\mathrm{co}_\lambda(r)+2q\ \mathrm{ta}_\lambda(r)-m\langle H,\partial_r\rangle\}\\ &\le \lambda\frac{s_\lambda(r)^2}{c_\lambda(r)}+s_\lambda(r)\{(2q-m)\mathrm{co}_\lambda(r)+2q\ \mathrm{ta}_\lambda(r)-m\langle H,\partial_r\rangle\}\\ &= s_\lambda(r)\{(2q-m)\mathrm{co}_\lambda(r)+(2q+1)\mathrm{ta}_\lambda(r)-m\langle H,\partial_r\rangle\}.\end{aligned}$$

Since the function α defined by $\alpha(t)=(2q-m)\mathrm{co}_\lambda(t)+(2q+1)\mathrm{ta}_\lambda(t)$ behaves like the function ϕ in Section 3, we have

$$\Delta f(r)\le s_\lambda(r)\{(2q-m)\mathrm{co}_\lambda(\rho)+(2q+1)\mathrm{ta}_\lambda(\rho)+m\,|H|\}\le 0.$$

Then, $\Delta f(r)=0$, which implies $r=\rho$, $H=\alpha(\rho)\partial_r$, $\mathcal{H}_x\subset\psi_{*x}T_xM$, $\mathcal{J}_x\subset\psi_{*x}T_xM$, and, if $m=2n-1$, then $\psi(M)=\partial\mathbb{C}P^q(\lambda)_\rho$, which finishes the proof of part (a) of Theorem 1.3.

In order to prove part (b), let us recall that in $\mathbb{C}P^n(\lambda)$,

$$\partial\mathbb{C}P^q(\lambda)_\rho=\partial\mathbb{C}P^{n-q-1}(\lambda)_{\rho'},\quad\text{with}\quad \rho'=\frac{\pi}{2\sqrt{\lambda}}-\rho$$

and that if $\mathcal{H}'$, $\mathcal{V}'$, $\mathcal{J}'$ are the distributions on $\mathbb{C}P^n(\lambda)$ defined as in (5.1) but starting from $\mathbb{C}P^{n-q-1}(\lambda)$ instead of $\mathbb{C}P^q(\lambda)$, then

$$\mathcal{H}'=\mathcal{V},\qquad \mathcal{V}'=\mathcal{H},\qquad \mathcal{J}'=\mathcal{J}. \tag{5.2}$$

As we have proved above, the existence of the isometric immersion implies that $\mathcal{H}\oplus\mathcal{J}\subset\psi_*(TM)$, so that $\mathcal{V}'\oplus\mathcal{J}'\subset\psi_*(TM)$. In particular, if we consider the value of the distribution $\mathcal{V}'\oplus\mathcal{J}'$ at a point $\gamma_N(\rho)$ of the tubular hypersurface of radius ρ about P, $(\mathcal{V}'\oplus\mathcal{J}')_{\gamma_N(\rho)}$ is the tangent space to $\exp_p\{v\in\mathcal{N}_p\mathbb{C}P^{n-q-1}(\lambda)\mid |v|=\rho'\}$, where $p=\gamma_N\left(\frac{\pi}{2\sqrt{\lambda}}\right)$, which is a geodesic $(2q+1)$-sphere $\partial B^{q+1}_{\rho'}$ and a totally geodesic submanifold of $\partial\mathbb{C}P^q(\lambda)_\rho$. Thus, the distribution $\mathcal{H}\oplus\mathcal{J}$ defines a regular foliation $\mathcal{F}$ on M with totally geodesic leaves isometric to $\partial B^{q+1}_{c,\lambda,\rho'}$, and $G=M/\mathcal{F}$ is a differentiable manifold. The above argument shows that each leave of $\mathcal{V}'\oplus\mathcal{J}'$ contained in $\partial\mathbb{C}P^q(\lambda)_\rho$ determines a point of $\mathbb{C}P^{n-q-1}(\lambda)$. In fact, we can consider the map $\tilde{\Pi}:\partial\mathbb{C}P^q(\lambda)_\rho=\partial\mathbb{C}P^{n-q-1}(\lambda)_{\rho'}\to\mathbb{C}P^{n-q-1}(\lambda)$, given by $\gamma_N(\rho)\mapsto\gamma_N\left(\frac{\pi}{2\sqrt{\lambda}}\right)$. Now, since the derivative of the exponential map can be expressed in terms of the Jacobi fields, it can be easily seen, from the expressions of the Jacobi fields of $\mathbb{C}P^n(\lambda)$ (see [**4**]) that $\tilde{\Pi}$ is a Riemannian submersion up to a constant factor $s_\lambda(\rho)^2$ in the metric of $\partial\mathbb{C}P^q(\lambda)_\rho$. In this way we get a Riemannian submersion $\Pi:\partial\mathbb{C}P^q(\lambda)_\rho\to\mathbb{C}P^{n-q-1}\left(\frac{1}{s_\lambda(\rho)^2}\right)$ whose fibres are the leaves of $\mathcal{V}'\oplus\mathcal{J}'$ and are totally geodesic submanifolds of $\partial\mathbb{C}P^q(\lambda)_\rho$. Let ϕ be the induced map making the diagram

$$\begin{array}{ccc} M & \xrightarrow{\psi} & \partial\mathbb{C}P^q(\lambda)_\rho \\ \pi\downarrow & & \downarrow\Pi \\ G = M/\mathscr{F} & \xrightarrow{\phi} & \mathbb{C}P^{n-q-1}\left(\dfrac{1}{s_\lambda(\rho)^2}\right), \end{array}$$

commutative, and let us consider on G the metric induced through ϕ by that of $\mathbb{C}P^{n-q-1}\left(\dfrac{1}{s_\lambda(\rho)^2}\right)$. (From the definition it follows that ϕ_* is injective.) Then π is a Riemannian submersion. In fact, since Π is a Riemannian submersion, one has, for every $X, Y \in (\mathscr{H}\oplus\mathscr{J})^\perp$ (the horizontal distribution of the submersion π):

$$\langle \phi_*\pi_*X, \phi_*\pi_*Y\rangle = \langle \Pi_*\psi_*X, \Pi_*\psi_*Y\rangle = \langle X, Y\rangle.$$

A straightforward computation shows that the mean curvature of $\psi: M \to \partial\mathbb{C}P^q(\lambda)$ vanishes, and then, it follows from Lemma 2 in [**9**] that ϕ is also minimal.

When $m = 2q+1$, at each point $\mathscr{H}\oplus\mathscr{J}$ must be equal to the tangent space to $\psi(M)$, so that we must have $\psi(M) = \partial B^{q+1}_{c,\lambda,\rho'}$, M must be a geodesic sphere in $\mathbb{C}P^{q+1}(\lambda)$, and ψ is an embedding.

If $2n = 2q+4$ and $m = 2q+2$, it follows from above that $\psi(M^{2q+2}) = \Pi^{-1}\left(\text{a geodesic of } S^2\left(\dfrac{4}{s_\lambda(\rho)^2}\right)\right)$, which implies, Π being a Riemannian submersion, that M is a compact manifold admitting two orthogonal totally geodesic foliations, one of dimension 1, the other with leaves isometric to $\partial B^{q+1}_{c,\lambda,\rho'}$ (the fibre of Π). It follows from Theorem A in [**1**] that the universal cover $\hat{M}$ of M is the Riemannian product of $\partial B^{q+1}_{c,\lambda,\rho'}$ (which is simply connected) and $\mathbb{R}$, the universal cover of S^1.

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