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Jonathan Brundan

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Abstract

We use the theory of finite W-algebras associated to nilpotent orbits in the Lie algebra $\mathfrak{g} = \mathfrak{gl}_N(\mathbb{C})$ to give another proof of Mœglin's theorem about completely prime primitive ideals in the enveloping algebra $U(\mathfrak{g})$. We also make some new observations about Joseph's Goldie rank polynomials in Cartan type A.

1. Introduction

The space $\operatorname{Prim} U(\mathfrak{g})$ of primitive ideals in the universal enveloping algebra of the Lie algebra $\mathfrak{g} := \mathfrak{gl}_N(\mathbb{C})$ has an unbelievably rich structure which has been studied intensively since the 1970s. In this article we revisit several of the foundational results about $\operatorname{Prim} U(\mathfrak{g})$ from the perspective of the theory of finite W-algebras that has been developed in the last few years by Premet [Pre02, Pre07a, Pre07b, Pre10, Pre11], Losev [Los10a, Los10b, Los11a, Los11b] and others [BG10, BG07, BGK08, BK06, BK08a, GG02, Gin09]. This article was inspired by the most recent breakthrough of Premet in [Pre11], so we start by discussing that in more detail.

Given a nilpotent element $e \in \mathfrak{g}$ there is associated a finite W-algebra $U(\mathfrak{g}, e)$, and Skryabin proved that the category of $U(\mathfrak{g}, e)$-modules is equivalent to a certain category of generalized Whittaker modules for $\mathfrak{g}$; see [Pre02, Skr02]. If L is any irreducible $U(\mathfrak{g}, e)$-module, we define $I(L) \in \operatorname{Prim} U(\mathfrak{g})$ by applying Skryabin's equivalence of categories to get an irreducible $\mathfrak{g}$-module and then taking the annihilator of that module. Premet's theorem [Pre11, Theorem B] can be stated for $\mathfrak{g} = \mathfrak{gl}_N(\mathbb{C})$ as follows.

Theorem 1.1 (Premet). *If L is a finite-dimensional irreducible $U(\mathfrak{g}, e)$-module and $I := I(L) \in \operatorname{Prim} U(\mathfrak{g})$, then the Goldie rank of $U(\mathfrak{g})/I$ is equal to the dimension of L.*

Premet actually worked with the finite W-algebra attached to a nilpotent element in an arbitrary reductive Lie algebra, showing in analogous notation in that general context that $\operatorname{rk} U(\mathfrak{g})/I$ always *divides* $\dim L$, with equality if the Goldie field of $U(\mathfrak{g})/I$ is isomorphic to the ring of fractions of a Weyl algebra. The fact that this condition for equality is satisfied for all $I \in \operatorname{Prim} U(\mathfrak{g})$ when $\mathfrak{g} = \mathfrak{gl}_N(\mathbb{C})$ follows from a result of Joseph from [Jos80b, § 10.3]. A key step in Joseph's proof involved showing in [Jos80b, § 9.1] that the ring of fractions of $U(\mathfrak{g})/\operatorname{Ann}_{U(\mathfrak{g})} M$ is isomorphic to the ring of fractions of $\mathscr{L}(M, M)$, the ad $\mathfrak{g}$-locally finite maps from M to itself, for all irreducible highest weight modules M. This is the weak form of Kostant's problem; see also [Jan83, 12.13].

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In the same article, Joseph proved an additivity principle for certain Goldie ranks, which, when combined with the solution of the weak form of Kostant's problem just mentioned, led Joseph to the discovery of a systematic method for computing the Goldie ranks of all primitive quotients of enveloping algebras in Cartan type A; see [Jos80b, § 8.1]. Soon afterwards, in [Jos80d, § 5.1], Joseph worked out a general approach to compute Goldie ranks of primitive quotients in arbitrary Cartan types via his remarkable theory of Goldie rank polynomials. These polynomials involve some mysterious constants which even today are only determined explicitly in Cartan type A; see the discussion in [Jos81b, § 1.5] and use [Jos87, Lemma 5.1] to treat Cartan type A. Much more recently, in [BK08a, § 8.5], we described a method for computing the dimensions of all finite-dimensional irreducible representations of finite W-algebras, again only in Cartan type A. As should come as no surprise given Theorem 1.1, these two methods, Joseph's method for computing Goldie ranks in Cartan type A and our method for computing dimensions, reduce after some book-keeping to performing exactly the same computation with Kazhdan–Lusztig polynomials. In the last section of the article, we will use this observation to give another proof of Theorem 1.1, quite different from Premet's argument in [Pre11] that involves reduction modulo p techniques.

Premet's theorem allows several other classical problems about $\mathrm{Prim}\, U(\mathfrak{g})$ to be attacked using finite W-algebra techniques. Perhaps our most striking accomplishment along these lines is a new proof of Mœglin's theorem from [Mœ87], asserting that all completely prime primitive ideals of $U(\mathfrak{g})$ are induced from one-dimensional representations of parabolic subalgebras. In the rest of the introduction, we will discuss this in more detail and formulate some other results about Goldie ranks of primitive quotients in Cartan type A obtained using the link to finite W-algebras. We will also make some other apparently new observations about Joseph's Goldie rank polynomials. Before we give any more details, we introduce some combinatorial language.

- A *tableau* A is a left-justified array of complex numbers with λ_1 entries in the bottom row, λ_2 entries in the next row up and so on, for some partition $\lambda = (\lambda_1 \geqslant \lambda_2 \geqslant \cdots)$ of N; we refer to λ as the *shape* of A.
- Two tableaux A and B are *row-equivalent*, denoted $A \sim B$, if one can be obtained from the other by permuting entries within rows.
- A tableau is *column-strict* if its entries are strictly increasing from bottom to top within each column with respect to the partial order $\geqslant$ on $\mathbb{C}$ defined by $a \geqslant b$ if $a - b \in \mathbb{Z}_{\geqslant 0}$.
- A tableau is *column-connected* if every entry in every row apart from the bottom row is one more than the entry immediately below it.
- A tableau is *column-separated* if it is column-strict and no two of its columns are linked, where we say that two columns are *linked* if the sets I and J of entries from the two columns satisfy the following:
  - if $|I| > |J|$, then $i > j > i'$ for some $i, i' \in I\backslash J$ and $j \in J\backslash I$;
  - if $|I| < |J|$, then $j' < i < j$ for some $i \in I\backslash J$ and $j, j' \in J\backslash I$;
  - if $|I| = |J|$, then either $i > j > i' > j'$ or $i' < j' < i < j$ for some $i, i' \in I\backslash J$ and $j, j' \in J\backslash I$.
- A tableau is *standard* if its entries are $1, \ldots, N$ and they increase from bottom to top in each column and from left to right in each row.

Now we go back to the Lie algebra $\mathfrak{g} = \mathfrak{gl}_N(\mathbb{C})$. Let $\mathfrak{t}$ and $\mathfrak{b}$ be the usual choices of Cartan and Borel subalgebras consisting of diagonal and upper triangular matrices in $\mathfrak{g}$, respectively. Let $W := S_N$ be the Weyl group of $\mathfrak{g}$ with respect to $\mathfrak{t}$, identified with the group of all permutation matrices in $G := GL_N(\mathbb{C})$. Let ℓ be the usual length function and $w_0 \in W$ be the longest element.

Let $\varepsilon_1, \dots, \varepsilon_N \in \mathfrak{t}^*$ be the dual basis to the basis $x_1, \dots, x_N \in \mathfrak{t}$ given by the diagonal matrix units. Given any tableau A, we attach a weight $\gamma(A) \in \mathfrak{t}^*$ by letting $a_1, \dots, a_N \in \mathbb{C}$ be the sequence obtained by reading the entries of A in order down columns starting with the leftmost column and then setting

$$\gamma(A) := \sum_{i=1}^{N} a_i \varepsilon_i. \tag{1.1}$$

Finally, let Φ^+ be the positive roots corresponding to $\mathfrak{b}$ and set

$$\rho := -\varepsilon_1 - 2\varepsilon_2 - \cdots - N\varepsilon_N, \tag{1.2}$$

which is the usual half-sum of positive roots up to a convenient normalization.

Given $\alpha \in \mathfrak{t}^*$, let $L(\alpha)$ denote the *irreducible* $\mathfrak{g}$-module generated by a $\mathfrak{b}$-highest weight vector of weight $\alpha - \rho$. By Duflo's theorem [Duf77], the map

$$I : \mathfrak{t}^* \to \operatorname{Prim} U(\mathfrak{g}), \quad \alpha \mapsto I(\alpha) := \operatorname{Ann}_{U(\mathfrak{g})} L(\alpha)$$

is surjective. In [Jos78a, Théorème 1] (see also [Jan83, 5.26(1)]), Joseph described the fibers of this map explicitly via the Robinson–Schensted algorithm, as follows. Take $\alpha \in \mathfrak{t}^*$ and set $a_i := x_i(\alpha)$. Construct a tableau $Q(\alpha)$ by starting from the empty tableau A_0 and then recursively inserting the numbers $a_1, \dots, a_N$ into the bottom row using the Schensted insertion algorithm. So, at the ith step, we are given a tableau A_{i-1} and need to insert a_i into the bottom row of A_{i-1}. If there is no entry $b > a_i$ on this row, then we simply add a_i to the end of the row; otherwise, we replace the leftmost $b > a_i$ on the row with a_i and then repeat the procedure to insert b into the next row up. It is clear from this construction that $Q(\alpha)$ is always row-equivalent to a column-strict tableau. Now Joseph's fundamental result is that

$$I(\alpha) = I(\beta) \Leftrightarrow Q(\alpha) \sim Q(\beta) \tag{1.3}$$

for any $\alpha, \beta \in \mathfrak{t}^*$.

Thus, we have a complete classification of the primitive ideals in $U(\mathfrak{g})$. Our first new result identifies the primitive ideals I in this classification that are completely prime, i.e. the ones for which the quotient $U(\mathfrak{g})/I$ is a domain.

Theorem 1.2. *For $\alpha \in \mathfrak{t}^*$, the primitive ideal $I(\alpha)$ is completely prime if and only if $Q(\alpha)$ is row-equivalent to a column-connected tableau.*

Of course, $I(\alpha)$ is completely prime if and only if $\operatorname{rk} U(\mathfrak{g})/I(\alpha) = 1$. So, in view of Theorem 1.1, the completely prime primitive ideals of $U(\mathfrak{g})$ are related to one-dimensional representations of the finite W-algebras $U(\mathfrak{g}, e)$. This is the basic idea for the proof of Theorem 1.2: we deduce it from a classification of one-dimensional representations of $U(\mathfrak{g}, e)$ obtained via another result of Premet [Pre10, Theorem 3.3] describing the maximal commutative quotient $U(\mathfrak{g}, e)^{\mathrm{ab}}$.

Our next theorem constructs a large family of primitive ideals which are *induced* in the spirit of [CD77, Théorème 8.6]; again our proof of this uses finite W-algebras in an essential way.

Theorem 1.3. *Suppose that we are given $\alpha \in \mathfrak{t}^*$ such that $Q(\alpha) \sim A$ for some column-separated tableau A. Let $\lambda' = (\lambda'_1 \geqslant \lambda'_2 \geqslant \cdots)$ be the transpose of the shape of A. Then we have that*

$$I(\alpha) = \operatorname{Ann}_{U(\mathfrak{g})}(U(\mathfrak{g}) \otimes_{U(\mathfrak{p})} F),$$
$$\operatorname{rk} U(\mathfrak{g})/I(\alpha) = \dim F,$$

where $\mathfrak{p}$ *is the standard parabolic subalgebra with diagonally embedded Levi factor* $\mathfrak{gl}_{\lambda'_1}(\mathbb{C}) \oplus \mathfrak{gl}_{\lambda'_2}(\mathbb{C}) \oplus \cdots$ *and* F *is the finite-dimensional irreducible* $\mathfrak{p}$*-module generated by a* $\mathfrak{b}$*-highest weight vector of weight* $\gamma(A) - \rho$*; cf. (1.1)–(1.2).*

Using these two results, we can already recover Mœglin's theorem.

COROLLARY (Mœglin). *Every completely prime primitive ideal* I *of* $U(\mathfrak{g})$ *is the annihilator of a module induced from a one-dimensional representation of a parabolic subalgebra of* $\mathfrak{g}$*.*

Proof. Take a completely prime $I \in \operatorname{Prim} U(\mathfrak{g})$ and represent it as $I(\alpha)$ for $\alpha \in \mathfrak{t}^*$. By Theorem 1.2, there exists a column-connected tableau $A \sim Q(\alpha)$. Since column-connected tableaux are obviously column-separated, we then apply Theorem 1.3 to deduce that $I = \operatorname{Ann}_{U(\mathfrak{g})}(U(\mathfrak{g}) \otimes_{U(\mathfrak{p})} F)$ for some parabolic $\mathfrak{p}$ and some $\mathfrak{p}$-module F. Finally, observe from its explicit description in Theorem 1.3 that F is actually one dimensional in the case that A is column-connected. □

We record another piece of folklore peculiar to Cartan type A; it justifies the decision to restrict attention for the remainder of the introduction just to weights from the lattice $P := \bigoplus_{i=1}^N \mathbb{Z}\varepsilon_i$ of *integral weights*. We will give a natural proof of this via finite W-algebras, though it also follows from more classical techniques.

THEOREM 1.4. *Suppose that we are given* $\alpha \in \mathfrak{t}^*$ *and set* $a_i := x_i(\alpha)$*. For fixed* $z \in \mathbb{C}$*, let* $\mathfrak{g}_z := \mathfrak{gl}_n(\mathbb{C})$*, where* $n := \#\{i = 1, \dots, N \mid a_i \in z + \mathbb{Z}\}$*; then set* $\alpha_z := \sum_{j=1}^n (a_{i_j} - z)\varepsilon_j$*, where* $i_1 < \cdots < i_n$ *are all the* $i \in \{1, \dots, N\}$ *such that* $a_i \in z + \mathbb{Z}$*. So,* α_z *is an integral weight for* $\mathfrak{g}_z$*. We have that*

$$\operatorname{rk} U(\mathfrak{g})/I(\alpha) = \prod_z \operatorname{rk} U(\mathfrak{g}_z)/I(\alpha_z),$$

where the product is over a set of representatives for the cosets of $\mathbb{C}$ *modulo* $\mathbb{Z}$*.*

In order to say more about Goldie ranks, we need some language related to the geometry of P. A weight $\alpha \in P$ is *anti-dominant* (respectively *regular anti-dominant*) if it satisfies $x_i(\alpha) \leqslant x_{i+1}(\alpha)$ (respectively $x_i(\alpha) < x_{i+1}(\alpha)$) for each $i = 1, \dots, N-1$. Given any $\alpha \in P$, we let δ be its *anti-dominant conjugate*, the unique anti-dominant weight in its W-orbit, and then define $d(\alpha) \in W$ to be the unique element of minimal length such that $\alpha = d(\alpha)\delta$. Note that the stabilizer W_δ of δ in W is a parabolic subgroup, and the element $d(\alpha)$ belongs to the set D_δ of minimal length W/W_δ-coset representatives. For $w \in W$, let

$$\widehat{C}_w := \{\alpha \in P \mid d(\alpha) = w\}, \tag{1.4}$$

which is the set of integral weights lying in the *upper closure* of the chamber containing $w(-\rho)$, i.e. we have $\alpha \in \widehat{C}_w$ if and only if the following hold for every $1 \leqslant i < j \leqslant N$:

$$\begin{aligned} w^{-1}(i) < w^{-1}(j) &\Rightarrow x_i(\alpha) \leqslant x_j(\alpha), \\ w^{-1}(i) > w^{-1}(j) &\Rightarrow x_i(\alpha) > x_j(\alpha). \end{aligned}$$

The upper closures $\widehat{C}_w$ for all $w \in W$ partition the set P into disjoint subsets.

Recall also the *left cells* of W, which in the case of the symmetric group can be defined in purely combinatorial terms as the equivalence classes of the relation $\sim_L$ on W defined by

$$x \sim_L y \Leftrightarrow Q(x) = Q(y).$$

The map Q here comes from the classical Robinson–Schensted bijection

$$w \mapsto (P(w), Q(w))$$

from W to the set of all pairs of standard tableaux of the same shape as in e.g. [Ful97, ch. 1]; so $P(w)$ is the *insertion tableau* and $Q(w)$ is the *recording tableau*. Comparing with our earlier notation, we have that

$$Q(w) = P(w^{-1}) = Q(w(-\rho)), \tag{1.5}$$

hence the connection between left cells in W and the Duflo–Joseph classification of primitive ideals from (1.3).

We say that $w \in W$ is *minimal in its left cell* if $P(w)$ has the entries $1, \ldots, N$ appearing in order up columns starting from the leftmost column. It is clear from the Robinson–Schensted correspondence that every left cell has a unique such minimal representative. Given any $\alpha \in \widehat{C}_w$, the Robinson–Schensted algorithm assembles the tableaux $Q(\alpha)$ and $Q(w(-\rho)) = P(w^{-1})$ in exactly the same order, i.e. they have the same recording tableau $Q(w^{-1}) = P(w)$. If w is minimal in its left cell, so that this recording tableau has entries $1, \ldots, N$ in order up columns, we therefore have that

$$\alpha = \gamma(Q(\alpha)) \tag{1.6}$$

for any $\alpha \in \widehat{C}_w$ and w that is minimal in its left cell. This is the reason that the minimal left cell representatives are particularly convenient to work with.

At last, we can resume the main discussion of Goldie ranks. In [Jos80c, § 5.12], Joseph made the striking discovery that for each $w \in W$ there is a unique polynomial $p_w \in \mathbb{C}[\mathfrak{t}^*]$ with the property that

$$\operatorname{rk} U(\mathfrak{g})/I(\alpha) = p_w(\delta) \tag{1.7}$$

for each $\alpha \in \widehat{C}_w$, where δ denotes the anti-dominant conjugate of α. The polynomials p_w are Joseph's *Goldie rank polynomials*, which have many remarkable properties. We recall in particular that p_w only depends on the left cell of w. To see this, take any regular anti-dominant $\delta \in P$. Assuming $w \sim_L w'$, we have that $Q(w\delta) = Q(w'\delta)$, so $I(w\delta) = I(w'\delta)$ by (1.3). Also, $w\delta$ and $w'\delta$ belong to (the interior of) $\widehat{C}_w$ and $\widehat{C}_{w'}$, respectively, by regularity. Hence, (1.7) gives that $p_w(\delta) = p_{w'}(\delta)$. Since the regular anti-dominant weights are Zariski dense, this implies $p_w = p_{w'}$ whenever $w \sim_L w'$.

The following theorem, which is ultimately deduced from Theorem 1.3, gives an explicit formula for Goldie rank polynomials in several important cases, e.g. it includes the extreme cases $w = 1$ (when $p_w = 1$) and $w = w_0$ (when it is essentially Weyl's dimension formula), as well as all situations when the tableau $Q(w)$ has just two rows.

Theorem 1.5. *Suppose that we are given $w \in W$ such that $Q(w) \sim A$ for some column-separated tableau A. Then we have that*

$$p_w = \prod_{(i,j)} \frac{x_i - x_j}{d(i,j)},$$

where the product is over all pairs (i, j) of entries from the tableau A such that i is strictly above and in the same column as j, and $d(i, j) > 0$ is the number of rows that i is above j.

For general w, the polynomials p_w are more complicated but can be written explicitly in terms of Kazhdan–Lusztig polynomials. To explain this, and for later use, we must make one more notational digression. Recall that the irreducible module $L(\alpha)$ is the unique irreducible quotient of the *Verma module* $M(\alpha) := U(\mathfrak{g}) \otimes_{U(\mathfrak{b})} \mathbb{C}_{\alpha-\rho}$, where $\mathbb{C}_{\alpha-\rho}$ is the one-dimensional $\mathfrak{b}$-module of weight $\alpha - \rho$. We have the usual *decomposition numbers* $[M(\alpha) : L(\beta)] \in \mathbb{Z}_{\geqslant 0}$ and the *inverse decomposition numbers* $(L(\alpha) : M(\beta)) \in \mathbb{Z}$ defined from

$$\mathrm{ch}\, L(\alpha) = \sum_{\beta} (L(\alpha) : M(\beta))\, \mathrm{ch}\, M(\beta). \tag{1.8}$$

For $w \in W$, we denote $L(w(-\rho))$ and $M(w(-\rho))$ simply by $L(w)$ and $M(w)$, respectively; in particular, $L(w_0)$ is the trivial module. By the translation principle (see [Jan83, 4.12]), we have that

$$[M(\alpha) : L(\beta)] = [M(x) : L(y)], \tag{1.9}$$

$$(L(\alpha) : M(\beta)) = \sum_{z \in W_\delta} (L(x) : M(yz)) \tag{1.10}$$

for any $\alpha, \beta \in P$ with the same anti-dominant conjugate δ, where $x := d(\alpha)$ and $y := d(\beta)$. Moreover, by the Kazhdan–Lusztig conjecture established in [BB81, BK81], it is known for $x, y \in W$ that

$$[M(x) : L(y)] = P_{xw_0, yw_0}(1), \tag{1.11}$$

$$(L(x) : M(y)) = (-1)^{\ell(x)+\ell(y)} P_{y,x}(1), \tag{1.12}$$

where $P_{x,y}(t)$ denotes the Kazhdan–Lusztig polynomial attached to $x, y \in W$ from [KL79].

The following theorem gives an explicit formula for the Goldie rank polynomials p_w. It is a straightforward consequence of Joseph's original approach for computing Goldie ranks in Cartan type A from [Jos80b], which we already mentioned in the discussion after Theorem 1.1. As was explained to me by Joseph, it can also be deduced from Joseph's general formula for Goldie rank polynomials (bearing in mind that all the scale factors are known in Cartan type A). We give yet another proof in the last section of the article via finite W-algebras, exploiting Theorem 1.1. Recall for the statement that p_w depends only on the left cell of w, so it is sufficient to compute p_w just for the minimal left cell representatives.

Theorem 1.6 (Joseph). *Suppose that $w \in W$ is minimal in its left cell. Let λ be the shape of the tableau $Q(w)$ with transpose $\lambda' = (\lambda'_1 \geqslant \lambda'_2 \geqslant \cdots)$. Let W^λ denote the parabolic subgroup $S_{\lambda'_1} \times S_{\lambda'_2} \times \cdots$ of $W = S_N$ and D^λ be the set of maximal length $W^\lambda \backslash W$-coset representatives. Then*

$$p_w = \sum_{z \in D^\lambda} (L(w) : M(z)) z^{-1}(h_\lambda), \tag{1.13}$$

where $h_\lambda := \prod_{(i\,j) \in W^\lambda} ((x_i - x_j)/(j - i))$ (product over all transpositions $(ij) \in W^\lambda$).

Joseph has directed a great deal of attention to the problem of determining the unknown constants in the Goldie rank polynomials in Cartan types different from A. This led Joseph to conjecture in [Jos88, Conjecture 8.4(i)] that Goldie rank polynomials always take the value 1 on some integral weight. Our final result verifies this conjecture in Cartan type A. The proof is a surprisingly simple computation from (1.13).

Theorem 1.7. *Every Goldie rank polynomial takes the value 1 on some element of P. More precisely, if $w \in W$ is minimal in its left cell and C is the unique tableau of the same shape as $Q(w)$ that has all 1s on its bottom row, all 2s on the next row up and so on, then $p_w(\alpha) = 1$, where $\alpha := w^{-1}\gamma(C)$.*

The remainder of the article is organized as follows. In § 2, we recall the highest weight classification of finite-dimensional irreducible representations of the finite W-algebra $U(\mathfrak{g}, e)$

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from [BK08a, Theorem 7.9]. Then we compare this with [Pre10, Theorem 3.3] to determine the highest weights of all the one-dimensional $U(\mathfrak{g}, e)$-modules explicitly. In particular, we see from this that every one-dimensional representation of a finite W-algebra in Cartan type A can be obtained as the restriction of a one-dimensional representation of a parabolic subalgebra of $\mathfrak{g}$, a statement which is closely related to Mœglin's theorem.

Then, in § 3, we gather together various existing results about Whittaker functors and primitive ideals in Cartan type A. In fact, we need to exploit both sorts of Whittaker functor (invariants and coinvariants) to deduce our main results. We point out in particular Remark 3.7, in which we formulate a conjecture which would imply a classification of primitive ideals in $U(\mathfrak{g}, e)$ exactly in the spirit of the Joseph–Duflo classification of Prim $U(\mathfrak{g})$.

In § 4, we use the criterion for irreducibility of standard modules from [BK08a, Theorem 8.25] to establish the first equality in Theorem 1.3.

In § 5, we review the Whittaker coinvariants construction of finite-dimensional irreducible $U(\mathfrak{g}, e)$-modules from [BK08a, Theorem 8.21].

In § 6, we explain the method from [BK08a, § 8.5] for computing dimensions of finite-dimensional irreducible $U(\mathfrak{g}, e)$-modules, and extract the polynomial on the right-hand side of the formula (1.13) from this.

Finally, we explain the alternative proof of Theorem 1.1 and derive all the other new results formulated in this introduction in § 7.

2. One-dimensional representations

In this section, we recall some basic facts about the representation theory of finite W-algebras in Cartan type A from [BK08a] and then deduce a classification of one-dimensional representations of these algebras. We continue with the basic Lie theoretic notation from the introduction; in particular, $\mathfrak{g} = \mathfrak{gl}_N(\mathbb{C})$, and $\mathfrak{t}$ and $\mathfrak{b}$ are the usual choices of Cartan and Borel subalgebras.

Let $\lambda = (p_n \geqslant \cdots \geqslant p_1)$ be a fixed partition of N. For each $i = 1, \ldots, n-1$, pick non-negative integers $s_{i,i+1}$ and $s_{i+1,i}$ such that $s_{i,i+1} + s_{i+1,i} = p_{i+1} - p_i$. Then set $s_{i,j} := s_{i,i+1} + s_{i+1,i+2} + \cdots + s_{j-1,j}$ and $s_{j,i} := s_{j,j-1} + \cdots + s_{i+2,i+1} + s_{i+1,i}$ for $1 \leqslant i \leqslant j \leqslant n$. This defines a *shift matrix* $\sigma = (s_{i,j})_{1 \leqslant i,j \leqslant n}$ in the sense of [BK06, (2.1)]. Let $l := p_n$ for short, which is called the *level* in [BK06].

We visualize this data by means of a *pyramid* π of boxes drawn in an $n \times l$ rectangle, so that there is a box in row i and column j for each $1 \leqslant i \leqslant n$ and $1 + s_{n,i} \leqslant j \leqslant l - s_{i,n}$ (where rows and columns are indexed as in a matrix). Note that there are p_i boxes in the ith row for each $i = 1, \ldots, n$. Let q_j be the number of boxes in the jth column for $j = 1, \ldots, l$. Also, number the boxes of π by $1, \ldots, N$ working in order down columns starting from the leftmost column, and write row(k) and col(k) for the row and column numbers of the kth box. For example, for $\lambda = (3, 2, 1)$ there are four possible choices for σ with corresponding pyramids

$$\sigma = \begin{pmatrix} 0 & 1 & 2 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \leftrightarrow \pi = \begin{array}{ccc} 1 & & \\ 2 & 4 & \\ 3 & 5 & 6 \end{array}, \quad \sigma = \begin{pmatrix} 0 & 1 & 1 \\ 0 & 0 & 0 \\ 1 & 1 & 0 \end{pmatrix} \leftrightarrow \pi = \begin{array}{ccc} & 2 & \\ & 3 & 5 \\ 1 & 4 & 6 \end{array},$$

$$\sigma = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix} \leftrightarrow \pi = \begin{array}{ccc} & 3 & \\ 1 & 4 & \\ 2 & 5 & 6 \end{array}, \quad \sigma = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 2 & 1 & 0 \end{pmatrix} \leftrightarrow \pi = \begin{array}{ccc} & & 4 \\ & 2 & 5 \\ 1 & 3 & 6 \end{array}.$$

If σ is upper-triangular, then π coincides with the usual Young diagram of the partition λ; we refer to this as the *left-justified* case.

By a π-*tableau*, we mean a filling of the boxes of the pyramid π by complex numbers; the left-justified tableaux from the introduction are a special case. The definitions of *column-strict*, *column-connected* and *row-equivalence* formulated in the introduction in the left-justified case extend without change to π-tableaux. Also, we say that a π-tableau A is *row-standard* if its entries are non-decreasing along rows from left to right, meaning $a \not> b$ whenever a and b are two entries from the same row with a located to the left of b.

We next define two essential maps from π-tableaux to $\mathfrak{t}^*$, denoted γ and ρ and called *column reading* and *row reading*, respectively. First, for a π-tableau A, we let

$$\gamma(A) := \sum_{i=1}^{n} a_i \varepsilon_i, \tag{2.1}$$

where $(a_1, \dots, a_N)$ is the sequence of complex numbers obtained by reading the entries of A in order down columns starting with the leftmost column; so, a_i is the entry in the ith box of A. For $\rho(A)$, we first need to convert A into a row-standard π-tableau, which we do by repeatedly transposing pairs of entries $a > b$ in the same row with a located to the left of b until we get to a (uniquely determined) row-standard tableau A'. Then let

$$\rho(A) := \sum_{i=1}^{n} a'_i \varepsilon_i, \tag{2.2}$$

where $(a'_1, \dots, a'_n)$ is the sequence obtained by reading the entries of A' in order along rows starting with the top row. Note that the map γ is obviously bijective, but ρ is definitely not.

Let $e \in \mathfrak{g}$ be the nilpotent matrix

$$e := \sum_{\substack{1 \leqslant i,j \leqslant N \\ \mathrm{row}(i)=\mathrm{row}(j) \\ \mathrm{col}(i)=\mathrm{col}(j)-1}} e_{i,j}$$

of Jordan type λ. Here $e_{i,j}$ denotes the ij-matrix unit. Introduce a $\mathbb{Z}$-grading $\mathfrak{g} = \bigoplus_{d \in \mathbb{Z}} \mathfrak{g}(d)$ by declaring that $e_{i,j}$ is of degree $2(\mathrm{col}(j) - \mathrm{col}(i))$; in particular, e is homogeneous of degree two. Let $\mathfrak{m} := \bigoplus_{d<0} \mathfrak{g}(d)$, $\mathfrak{h} := \mathfrak{g}(0)$ and $\mathfrak{p} := \bigoplus_{d \geqslant 0} \mathfrak{g}(d)$. So, $\mathfrak{p}$ is the standard parabolic subalgebra with Levi factor $\mathfrak{h}$, and $\mathfrak{h}$ is just the diagonally embedded subalgebra $\mathfrak{gl}_{q_1}(\mathbb{C}) \oplus \cdots \oplus \mathfrak{gl}_{q_l}(\mathbb{C})$. Let $\mathfrak{g}^e$ (respectively $\mathfrak{t}^e$) be the centralizer of e in $\mathfrak{g}$ (respectively $\mathfrak{t}$). It is important that $\mathfrak{g}^e \subseteq \mathfrak{p}$.

Let $\chi : \mathfrak{m} \to \mathbb{C}$ be the Lie algebra homomorphism $x \mapsto (x, e)$, where $(\cdot, \cdot)$ is the trace form. Let $\mathfrak{m}_\chi := \{x - \chi(x) \mid x \in \mathfrak{m}\} \subseteq U(\mathfrak{m})$. The *finite W-algebra* is the following subalgebra of $U(\mathfrak{p})$:

$$U(\mathfrak{g}, e) := \{u \in U(\mathfrak{p}) \mid \mathfrak{m}_\chi u \subseteq U(\mathfrak{g})\mathfrak{m}_\chi\}. \tag{2.3}$$

This definition originates in work of Kostant [Kos78], Lynch [Lyn79] and Mœglin [Mœ88] and is a special case of the construction due to Premet [Pre02] and then Gan and Ginzburg [GG02] of non-commutative filtered deformations of the coordinate algebra of the Slodowy slice associated to the nilpotent orbit $G \cdot e$; the terminology 'finite W-algebra' has emerged because they are the finite-dimensional analogues of the vertex W-algebras constructed in [KRW03]. Of course, the definition depends implicitly on the choice of grading (hence on π), but up to isomorphism the algebra $U(\mathfrak{g}, e)$ is independent of this choice; see [BK06, Corollary 10.3]. More conceptual proofs of this independence (valid in all Cartan types) were given subsequently in [BG07, Theorem 1] and [Los10a, Proposition 3.1.2].

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A special feature of the Cartan type A case is that a complete set of generators and relations for $U(\mathfrak{g},e)$ is known; see [BK06, Theorem 10.1]. The generators are certain explicit elements

$$\{D_i^{(r)} \mid 1 \leqslant i \leqslant n, r > 0\},$$
$$\{E_i^{(r)} \mid 1 \leqslant i < n, r > s_{i,i+1}\},$$
$$\{F_i^{(r)} \mid 1 \leqslant i < n, r > s_{i+1,i}\}$$

of $U(\mathfrak{p})$ defined in [BK06, § 9], and the relations are the defining relations for the shifted Yangian $Y_n(\sigma)$ recorded in [BK06, (2.4)–(2.15)], together with the relations $D_1^{(r)} = 0$ for $r > p_1$. These generators and relations were exploited in [BK08a] to classify the finite-dimensional irreducible $U(\mathfrak{g},e)$-modules.

To recall this classification in more detail, by a *highest weight vector* in a $U(\mathfrak{g},e)$-module, we mean a common eigenvector for all $D_i^{(r)}$ which is annihilated by all $E_j^{(s)}$. Assume that v_+ is a non-zero highest weight vector in a left module. Let $a_i^{(r)} \in \mathbb{C}$ be defined from $D_i^{(r)} v_+ = a_i^{(r)} v_+$ and define $a_{i,1}, \dots, a_{i,p_i} \in \mathbb{C}$ by factoring

$$u^{p_i} + a_i^{(1)} u^{p_i - 1} + \cdots + a_i^{(p_i)} = (u + a_{i,1}) \cdots (u + a_{i,p_i}). \tag{2.4}$$

Combining [BK08a, Theorem 3.5] for $j = i$ with the definition [BK08a, (2.34)], it follows that the elements $D_i^{(r)}$ for $r > p_i$ lie in the left ideal of $U(\mathfrak{g},e)$ generated by all $E_j^{(s)}$; hence, $a_i^{(r)} = 0$ for $r > p_i$. So, we have for all $r > 0$ that

$$D_i^{(r)} v_+ = e_r(a_{i,1}, \dots, a_{i,p_i}) v_+, \tag{2.5}$$

where $e_r(a_{i,1}, \dots, a_{i,p_i})$ is the rth elementary symmetric polynomial in the complex numbers $a_{i,1}, \dots, a_{i,p_i}$. We record this by writing the complex numbers $a_{i,1} - i, \dots, a_{i,p_i} - i$ into the boxes on the ith row of the pyramid π to obtain a π-tableau A, which we refer to as the *type* of the original highest weight vector v_+. Of course, A here is defined only up to row-equivalence.

Conversely, given a π-tableau A, there is a unique (up to isomorphism) irreducible left $U(\mathfrak{g},e)$-module $L(A,e)$ generated by a highest weight vector of type A, with $L(A,e) \cong L(B,e)$ if and only if $A \sim B$. The module $L(A,e)$ is constructed in [BK08a, § 6.1] as the unique irreducible quotient of the *Verma module* $M(A,e)$, which is the universal highest weight module of type A; see also [BGK08, § 4.2] for a different construction of Verma modules which avoids the explicit use of generators and relations (and so makes sense in other Cartan types).

Remark 2.1. A basic question is to compute the composition multiplicities $[M(A,e) : L(B,e)]$. In [BK08a, Conjecture 7.17], we conjectured for any π-tableaux A and B with integer entries that

$$[M(A,e) : L(B,e)] = [M(\rho(A)) : L(\rho(B))], \tag{2.6}$$

the numbers on the right-hand side being known by (1.9) and (1.11). Although not needed in the present article, we want to point out that this conjecture is now a theorem of Losev; see [Los10b, Theorems 4.1 and 4.3]. Strictly speaking, to get from Losev's result to (2.6) one needs to identify the Verma modules $M(A,e)$ defined here with the ones in [Los10b], but this has now been checked thanks to some recent work of Brown and Goodwin [BG10]; see the proof of Theorem 3.2 below for a fuller discussion. In arbitrary standard Levi type, there is an analogous conjecture formulated roughly in [dVvD95], which can also be proved using Losev's work.

The highest weight classification of finite-dimensional irreducible $U(\mathfrak{g},e)$-modules is as follows.

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Theorem 2.2 [BK08a, Theorem 7.9]. *For a π-tableau A, $L(A,e)$ is finite dimensional if and only if A is row-equivalent to a column-strict tableau. Hence, as A runs over a set of representatives for the row-equivalence classes of column-strict π-tableaux, the modules $\{L(A,e)\}$ give a complete set of pairwise-inequivalent finite-dimensional irreducible left $U(\mathfrak{g},e)$-modules.*

The proof of the 'if' part of Theorem 2.2 given in [BK08a] is quite straightforward, and is based on the construction of another family of $U(\mathfrak{g},e)$-modules called standard modules indexed by column-strict tableaux. To define these, recall the weight ρ from (1.2), and also introduce the special weight

$$\beta := \sum_{\substack{1\leqslant i,j\leqslant N\\ \mathrm{col}(i)>\mathrm{col}(j)}} (\varepsilon_i - \varepsilon_j) = \sum_{i=1}^{N} ((q_1+\cdots+q_{\mathrm{col}(i)-1}) - (q_{\mathrm{col}(i)+1}+\cdots+q_l))\varepsilon_i \in \mathfrak{t}^*. \tag{2.7}$$

This is the same as the weight β defined in [BGK08], which is important because of [BGK08, Corollary 2.9] (reproduced in Theorem 5.1 below). Notice that A is column-strict if and only if $\gamma(A)-\beta-\rho$ is a dominant weight for the Lie algebra $\mathfrak{h}=\mathfrak{g}(0)$ with respect to the Borel subalgebra $\mathfrak{b}\cap\mathfrak{h}$. Assuming that is the case, there is a finite-dimensional irreducible $\mathfrak{p}$-module $V(A)$ generated by a $\mathfrak{b}$-highest weight vector of this weight. Then we restrict the left $U(\mathfrak{p})$-module $V(A)$ to the subalgebra $U(\mathfrak{g},e)$ to obtain the *standard module* denoted $V(A,e)$. Thus, $V(A,e)=V(A)$ as vector spaces, but we use different notation since one is a $U(\mathfrak{g},e)$-module and the other is a $U(\mathfrak{p})$-module. As observed in the last paragraph of the proof of [BK08a, Theorem 7.9], the original $\mathfrak{b}$-highest weight vector in $V(A)$ is a highest weight vector of type A in $V(A,e)$; this can also be checked directly by arguing as in the proof of [BGK08, Lemma 5.4]. It follows that $L(A,e)$ is a composition factor of the finite-dimensional module $V(A,e)$; hence, $L(A,e)$ is indeed finite dimensional when A is column-strict.

We are interested next in one-dimensional modules. It is obvious from the definitions that $V(A)$ is one dimensional if and only if A is column-connected. Since $L(A,e)$ is a subquotient of $V(A,e)$, it follows that $L(A,e)$ is one dimensional if A is row-equivalent to a column-connected tableau. We are going to prove the converse of this statement to obtain the following classification of one-dimensional $U(\mathfrak{g},e)$-modules. The possibility of doing this was suggested already by Losev in the discussion in the paragraph after [Los11a, Theorem 5.2.1].

Theorem 2.3. *For a π-tableau A, $L(A,e)$ is one dimensional if and only if A is row-equivalent to a column-connected tableau. Hence, as A runs over a set of representatives for the row-equivalence classes of column-connected π-tableaux, the modules $\{L(A,e)\}$ give a complete set of pairwise-inequivalent one-dimensional left $U(\mathfrak{g},e)$-modules.*

Corollary 2.4. *Every one-dimensional left $U(\mathfrak{g},e)$-module is isomorphic to a standard module $V(A,e)$ for some column-connected π-tableau A, and so arises as the restriction of a one-dimensional $U(\mathfrak{p})$-module.*

The rest of the section is devoted to proving Theorem 2.3 and its corollary. To do this, we need to review the following theorem of Premet describing the algebra $U(\mathfrak{g},e)^{\mathrm{ab}}$, that is, the quotient of $U(\mathfrak{g},e)$ by the two-sided ideal generated by all commutators $[x,y]$ for $x,y\in U(\mathfrak{g},e)$. Of course, one-dimensional $U(\mathfrak{g},e)$-modules are identified with one-dimensional $U(\mathfrak{g},e)^{\mathrm{ab}}$-modules. It is convenient at this point to set $p_0:=0$.

Theorem 2.5 [Pre10, Theorem 3.3]. *The algebra $U(\mathfrak{g},e)^{\mathrm{ab}}$ is a free polynomial algebra of rank l generated by the images of the elements*

$$\{D_i^{(r)} \mid 1\leqslant i\leqslant n, 1\leqslant r\leqslant p_i-p_{i-1}\}. \tag{2.8}$$

Premet's proof of Theorem 2.5 is in two parts. The first step is to show that $U(\mathfrak{g}, e)^{\mathrm{ab}}$ is generated by the images of the commuting elements listed in (2.8). This is a straightforward consequence of the defining relations for $U(\mathfrak{g}, e)$ from [BK06], and is explained in the first two paragraphs of the proof of [Pre10, Theorem 3.3]. Thus, letting $X \cong \mathbb{A}^l$ be the affine space with algebraically independent coordinate functions $\{T_i^{(r)} \mid 1 \leqslant i \leqslant n, 1 \leqslant r \leqslant p_i - p_{i-1}\}$, there is a surjective map

$$\mathbb{C}[X] \twoheadrightarrow U(\mathfrak{g}, e)^{\mathrm{ab}}, \quad T_i^{(r)} \mapsto D_i^{(r)}. \tag{2.9}$$

This map identifies $\operatorname{Specm} U(\mathfrak{g}, e)^{\mathrm{ab}}$ with a closed subvariety of X. Then, to complete the proof, Premet showed quite indirectly that $\dim \operatorname{Specm} U(\mathfrak{g}, e)^{\mathrm{ab}} \geqslant l$, hence $\operatorname{Specm} U(\mathfrak{g}, e)^{\mathrm{ab}} = X$ and the surjective map is an isomorphism. In the next paragraph, we will explain an alternative argument for this second step using the following elementary lemma.

Lemma 2.6. *Given complex numbers $a_i^{(r)}$ for $1 \leqslant i \leqslant n$ and $1 \leqslant r \leqslant p_i - p_{i-1}$, there are complex numbers $a_{i,j}$ for $1 \leqslant i \leqslant n$ and $1 \leqslant j \leqslant p_i$ such that*

$$a_{i,p_i - p_{i-1}+r} = a_{i-1,r} \quad \text{for } 1 \leqslant r \leqslant p_{i-1}, \tag{2.10}$$

$$e_r(a_{i,1}, \ldots, a_{i,p_i}) = a_i^{(r)} \quad \text{for } 1 \leqslant r \leqslant p_i - p_{i-1}. \tag{2.11}$$

Proof. We prove existence of numbers $a_{i,j}$ for $1 \leqslant j \leqslant p_i$ satisfying (2.10)–(2.11) by induction on $i = 1, \ldots, n$. For the base case $i = 1$, we define $a_{1,1}, \ldots, a_{1,p_1}$ from the factorization (2.4), and (2.10)–(2.11) are clear. For the induction step, suppose that we have already found $a_{i-1,1}, \ldots, a_{i-1,p_{i-1}}$. Define $a_{i,p_i-p_{i-1}+1}, \ldots, a_{i,p_i}$ so that (2.10) holds. Then we need to find complex numbers $a_{i,1}, \ldots, a_{i,p_i-p_{i-1}}$ satisfying (2.11). The equations (2.11) are equivalent to the equations

$$b_i^{(r)} = a_i^{(r)} - \sum_{s=0}^{r-1} b_i^{(s)} e_{r-s}(a_{i-1,1}, \ldots, a_{i-1,p_{i-1}})$$

for $1 \leqslant r \leqslant p_i - p_{i-1}$, where $b_i^{(r)}$ denotes $e_r(a_{i,1}, \ldots, a_{i,p_i-p_{i-1}})$. Proceeding by induction on $r = 1, \ldots, p_i - p_{i-1}$, we solve these equations uniquely for $b_i^{(r)}$ and then define $a_{i,1}, \ldots, a_{i,p_i-p_{i-1}}$ by factoring

$$u^{p_i-p_{i-1}} + b_i^{(1)} u^{p_i-p_{i-1}-1} + \cdots + b_i^{(p_i-p_{i-1})} = (u + a_{i,1}) \cdots (u + a_{i,p_i-p_{i-1}}).$$

This does the job. □

Now take any point $x \in X$, set $a_i^{(r)} := T_i^{(r)}(x)$ and then define $a_{i,j}$ according to Lemma 2.6. Because of (2.10), there is a column-connected π-tableau A having entries $a_{i,1} - i, \ldots, a_{i,p_i} - i$ in its ith row for each $i = 1, \ldots, n$. This tableau A is unique up to row-equivalence; indeed, any two choices for A agree up to reordering columns of the same height. As we have already observed, the assumption that A is column-connected means that the standard module $V(A, e)$ is one dimensional; hence, so is $L(A, e) \cong V(A, e)$. By (2.5) and (2.11), we see that $D_i^{(r)}$ acts on $L(A, e)$ by the scalar $a_i^{(r)}$, showing that the point x lies in $\operatorname{Specm} U(\mathfrak{g}, e)^{\mathrm{ab}}$. Thus, we have established that $\operatorname{Specm} U(\mathfrak{g}, e)^{\mathrm{ab}} = X$, so the map (2.9) is indeed an isomorphism as required for the alternative proof of the second part of Theorem 2.5 promised above.

This argument shows moreover that every one-dimensional left $U(\mathfrak{g}, e)$-module is isomorphic to $L(A, e) \cong V(A, e)$ for some column-connected π-tableau A, which is enough to complete the proofs of Theorem 2.3 and Corollary 2.4.

3. Whittaker functors and Duflo–Joseph classification

In this section, we review the definitions of the two sorts of Whittaker functors and explain some of the results of Premet and Losev linking finite-dimensional $U(\mathfrak{g},e)$-modules to $\operatorname{Prim} U(\mathfrak{g})$.

For any associative algebra A, we denote the category of all left (respectively right) A-modules by A-mod (respectively mod-A). If M is a left $U(\mathfrak{g})$-module, it is clear from (2.3) that the space $H^0(\mathfrak{m}_\chi, M) := \{v \in M \mid \mathfrak{m}_\chi v = \mathbf{0}\}$ of *Whittaker invariants* is stable under left multiplication by elements of $U(\mathfrak{g},e)$; hence, it is a left $U(\mathfrak{g},e)$-module. So, we get the functor

$$H^0(\mathfrak{m}_\chi, ?) : U(\mathfrak{g})\text{-mod} \to U(\mathfrak{g},e)\text{-mod}. \tag{3.1}$$

Instead, suppose that M is a right $U(\mathfrak{g})$-module. Then, by (2.3) again, the space $H_0(\mathfrak{m}_\chi, M) := M/M\mathfrak{m}_\chi$ of *Whittaker coinvariants* is naturally a right $U(\mathfrak{g},e)$-module. So, we have the functor

$$H_0(\mathfrak{m}_\chi, ?) : \text{mod-}U(\mathfrak{g}) \to \text{mod-}U(\mathfrak{g},e). \tag{3.2}$$

In the remainder of the section, we review some of the basic properties of these two Whittaker functors. Although not used here, we remark that one can also combine these functors to obtain a remarkable functor $H_0^0(\mathfrak{m}_\chi, ?)$ on bimodules introduced originally by Ginzburg; see [Gin09, § 3.3] and [Los11b, § 3.5].

We begin with the functor $H^0(\mathfrak{m}_\chi, ?)$. Let $(U(\mathfrak{g}), \mathfrak{m}_\chi)$-mod be the full subcategory of $U(\mathfrak{g})$-mod consisting of all modules on which $\mathfrak{m}_\chi$ acts locally nilpotently. By Skryabin's theorem [Skr02] (see also [GG02, § 6]), the functor $H^0(\mathfrak{m}_\chi, ?)$ restricts to an equivalence of categories

$$H^0(\mathfrak{m}_\chi, ?) : (U(\mathfrak{g}), \mathfrak{m}_\chi)\text{-mod} \to U(\mathfrak{g},e)\text{-mod}.$$

The quasi-inverse equivalence is the *Skryabin functor*

$$S_\chi : U(\mathfrak{g},e)\text{-mod} \to (U(\mathfrak{g}), \mathfrak{m}_\chi)\text{-mod} \tag{3.3}$$

defined by tensoring with the $(U(\mathfrak{g}), U(\mathfrak{g},e))$-bimodule $U(\mathfrak{g})/U(\mathfrak{g})\mathfrak{m}_\chi$.

This equivalence has proved useful for the study of primitive ideals in $U(\mathfrak{g})$. For a two-sided ideal I of $U(\mathfrak{g})$, we define its associated variety $\mathcal{VA}(I)$ as in [Jan04, § 9.3], viewing it as a closed subvariety of $\mathfrak{g}$ via the trace form. Let $\mathcal{VA}'(I)$ denote the image of $\mathcal{VA}(I)$ under the natural projection $\mathfrak{g} \twoheadrightarrow [\mathfrak{g},\mathfrak{g}] = \mathfrak{sl}_N(\mathbb{C})$. By Joseph's irreducibility theorem, it is known that $\mathcal{VA}'(I)$ is the closure of a single nilpotent orbit for every $I \in \operatorname{Prim} U(\mathfrak{g})$. This follows in Cartan type A from [Jos81a, § 3.3]; for other Cartan types see [Jos85, § 3.10] as well as [Vog91, Corollary 4.7] and [Los11b, Remark 3.4.4] for alternative proofs (the second of which goes via finite W-algebras in the spirit of the present article). Let $\operatorname{Prim}_\lambda U(\mathfrak{g})$ denote the set of $I \in \operatorname{Prim} U(\mathfrak{g})$ such that $\mathcal{VA}'(I)$ is the closure of the orbit $G \cdot e$ of all nilpotent matrices of Jordan type λ.

Given any non-zero left $U(\mathfrak{g},e)$-module L, we get a two-sided ideal

$$I(L) := \operatorname{Ann}_{U(\mathfrak{g})} S_\chi(L) \tag{3.4}$$

of $U(\mathfrak{g})$ by applying Skryabin's functor (3.3) and then taking the annihilator. If L is irreducible, then Skryabin's theorem implies $I(L) \in \operatorname{Prim} U(\mathfrak{g})$. The following fundamental theorem of Premet implies moreover that $I(L) \in \operatorname{Prim}_\lambda U(\mathfrak{g})$ if L is finite dimensional and irreducible. Premet's proof of this result also uses Joseph's irreducibility theorem. Although not needed here, we remark that the converse of the second statement of the theorem is also true by [Los10a, Theorem 1.2.2(ii), (ix)].

Theorem 3.1 [Pre07a, Theorem 3.1]. *For any non-zero left $U(\mathfrak{g}, e)$-module L, we have that*

$$\mathcal{VA}'(I(L)) \supseteq \overline{G \cdot e}.$$

Moreover, equality holds if L is finite dimensional.

Recalling Theorem 2.2, this gives us an ideal $I(L(A, e)) \in \mathrm{Prim}_\lambda\, U(\mathfrak{g})$ for each column-strict π-tableau A. The next theorem explains how to identify this primitive ideal in the Duflo labelling from the introduction. It is a special case of a general result of Losev [Los11a, Theorem 5.1.1] (a closely related statement was conjectured in [BGK08, § 5.1]).

Theorem 3.2. *For any π-tableau A, we have that*

$$I(L(A, e)) = I(\rho(A)),$$

where $\rho(A) \in \mathfrak{t}^*$ is defined by (2.2).

Proof. Recall that we have labelled the boxes of π in order down columns starting from the leftmost column. Let $1', 2', \dots, N'$ be the sequence of integers obtained by reading these labels from left to right along rows starting from the top row. There is a unique permutation $w \in W$ such that $w(i) = i'$ for each $i = 1, \dots, N$. Let $\mathfrak{b}' := w \cdot \mathfrak{b} = \langle e_{i',j'} \mid 1 \leqslant i \leqslant j \leqslant N\rangle$ and $\rho' := w\rho = -\sum_{i=1}^N i\varepsilon_{i'}$. For any $\alpha' \in \mathfrak{t}^*$, let $L'(\alpha')$ be the irreducible $\mathfrak{g}$-module generated by a $\mathfrak{b}'$-highest weight vector of weight $\alpha' - \rho'$. Now take a π-tableau A and let $\rho'(A) := w\rho(A)$. An easy argument involving twisting the action by w shows that $\mathrm{Ann}_{U(\mathfrak{g})} L'(\rho'(A)) = \mathrm{Ann}_{U(\mathfrak{g})} L(\rho(A)) \stackrel{\text{def}}{=} I(\rho(\alpha))$. Thus, to complete the proof of the theorem, it suffices to show that

$$I(L(A, e)) \stackrel{\text{def}}{=} \mathrm{Ann}_{U(\mathfrak{g})} S_\chi(L(A, e)) = \mathrm{Ann}_{U(\mathfrak{g})} L'(\rho'(A)). \tag{3.5}$$

We will ultimately deduce this from [Los11a, Theorem 5.1.1], which is in phrased in terms of highest weight theory [BGK08].

To recall a little of this theory, for $\mathfrak{a} \in \{\mathfrak{g}, \mathfrak{p}, \mathfrak{h}, \mathfrak{m}, \mathfrak{b}, \mathfrak{b}'\}$, let $\mathfrak{a}_0$ be the zero weight space of $\mathfrak{a}$ for the adjoint action of the torus $\mathfrak{t}^e$. In particular, we have that $\mathfrak{g}_0 = \langle e_{i,j} \mid \mathrm{row}(i) = \mathrm{row}(j)\rangle \cong \mathfrak{gl}_{p_1}(\mathbb{C}) \oplus \cdots \oplus \mathfrak{gl}_{p_n}(\mathbb{C})$, while $\mathfrak{p}_0 = \mathfrak{b}_0 = \mathfrak{b}'_0$ and $\mathfrak{h}_0 = \mathfrak{t}$. We have in front of us the necessary data to define another finite W-algebra $U(\mathfrak{g}_0, e) \subseteq U(\mathfrak{p}_0)$, which plays the role of 'Cartan subalgebra'. Choose a parabolic subalgebra $\mathfrak{q}$ of $\mathfrak{g}$ with Levi factor $\mathfrak{g}_0$ by setting $\mathfrak{q} := \mathfrak{g}_0 + \mathfrak{b}' = \langle e_{i,j} \mid \mathrm{row}(i) \leqslant \mathrm{row}(j)\rangle$. This choice determines a certain $(U(\mathfrak{g}, e), U(\mathfrak{g}_0, e))$-bimodule denoted $U(\mathfrak{g}, e)/U(\mathfrak{g}, e)_\sharp$ in [BGK08, § 4.1]; the right $U(\mathfrak{g}_0, e)$-module structure here is defined using a homomorphism defined in [BGK08, Theorem 4.3]. Then, given any finite-dimensional irreducible left $U(\mathfrak{g}_0, e)$-module Λ, we can form the Verma module

$$M(\Lambda, e) := U(\mathfrak{g}, e)/U(\mathfrak{g}, e)_\sharp \otimes_{U(\mathfrak{g}_0, e)} \Lambda \tag{3.6}$$

as in [BGK08, § 5.2]. As usual, it has a unique irreducible quotient denoted $L(\Lambda, e)$; see [BGK08, Theorem 4.5(4)]. On the other hand, in [Los11a, § 4.3], Losev made a very similar construction of Verma modules, but replaced the homomorphism from [BGK08, Theorem 4.3] with a map constructed in a completely different way in [Los10b, (5.6)]. It is far from clear that Losev's map is the same as the one in [BGK08], but fortunately this has recently been checked by Brown and Goodwin (in standard Levi type); see [BG10, Proposition 3.12]. Hence, as noted in [BG10, § 3.5], the Verma modules constructed in [Los11a] are the same as the Verma modules $M(\Lambda, e)$ above coming from [BGK08]. This is a crucial point.

As we are in standard Levi type, i.e. e is regular in $\mathfrak{g}_0$, we have simply that $U(\mathfrak{g}_0, e) \cong Z(\mathfrak{g}_0)$, the center of $U(\mathfrak{g}_0)$, as goes back to Kostant [Kos78, § 2]. More precisely, there is a canonical algebra isomorphism

$$\mathrm{Pr}_0 : Z(\mathfrak{g}_0) \xrightarrow{\sim} U(\mathfrak{g}_0, e) \tag{3.7}$$

induced by the unique linear projection $\mathrm{Pr}_0 : U(\mathfrak{g}_0) \twoheadrightarrow U(\mathfrak{b}_0)$ that sends $u(x - \chi(x))$ to zero for each $u \in U(\mathfrak{g}_0)$ and $x \in \mathfrak{m}_0$. For $\alpha' \in \mathfrak{t}^*$, let $L'_0(\alpha')$ denote the irreducible $U(\mathfrak{g}_0)$-module generated by a $\mathfrak{b}'_0$-highest weight vector of weight $\alpha' - \rho'$. Let W_0 be the subgroup of W consisting of all permutations such that $\mathrm{row}(i) = \mathrm{row}(w(i))$ for each $1 \leqslant i \leqslant N$, which is the Weyl group of $\mathfrak{g}_0$. Then we have the Harish-Chandra isomorphism

$$\Psi_0 : Z(\mathfrak{g}_0) \xrightarrow{\sim} S(\mathfrak{t})^{W_0}, \tag{3.8}$$

which we normalize so that $z \in Z(\mathfrak{g}_0)$ acts on $L'_0(\alpha')$ by the scalar $\alpha'(\Psi_0(z))$ for each $\alpha' \in \mathfrak{t}^*$. Let Λ be the one-dimensional left $U(\mathfrak{g}_0, e)$-module corresponding under the isomorphisms (3.7) and (3.8) to the $S(\mathfrak{t})^{W_0}$-module $\mathbb{C}_{\rho'(A)}$ of weight $\rho'(A)$. By the proof of [BGK08, Theorem 5.5] and [BGK08, Lemma 5.1], we have that $M(\Lambda, e) \cong M(A, e)$ as left $U(\mathfrak{g}, e)$-modules; hence, $L(\Lambda, e) \cong L(A, e)$. So, we have identified $L(A, e)$ with a highest weight module exactly as in [Los11a], and our problem (3.5) now reduces to showing that

$$\mathrm{Ann}_{U(\mathfrak{g})} S_\chi(L(\Lambda, e)) = \mathrm{Ann}_{U(\mathfrak{g})} L'(\rho'(A)). \tag{3.9}$$

By the definition of Λ and (3.8), the character of $Z(\mathfrak{g}_0)$ arising from Λ via (3.7) is the same as the central character of $L'_0(\rho'(A))$. Moreover, by the definition of $\rho'(A)$, $L'_0(\rho'(A))$ is an 'anti-dominant' irreducible Verma module, so by [Dix96, Theorem 8.4.3] its annihilator in $U(\mathfrak{g}_0)$ is the minimal primitive ideal generated by the kernel of this central character. By [Kos78, Theorem 3.9], this minimal primitive ideal is also the annihilator of the $U(\mathfrak{g}_0)$-module obtained from Λ by applying the $\mathfrak{g}_0$-version of Skryabin's equivalence. Now apply [Los11a, Theorem 5.1.1] to deduce (3.9). □

Theorem 3.2 has a number of important consequences. Recalling the definition of the left-justified tableau $Q(\alpha)$ from the introduction, let

$$\mathfrak{t}^*_\lambda := \{\alpha \in \mathfrak{t}^* \mid Q(\alpha) \text{ has shape } \lambda\}. \tag{3.10}$$

For $\alpha \in \mathfrak{t}^*_\lambda$, we define a π-tableau $Q_\pi(\alpha)$ by taking $Q(\alpha)$ and sliding the boxes to the right as necessary in order to convert it to a π-tableau. Note that $Q_\pi(\alpha)$ is row-equivalent to a column-strict π-tableau.

Lemma 3.3. *For any column-strict π-tableau A, we have that $\rho(A) \in \mathfrak{t}^*_\lambda$ and $A \sim Q_\pi(\rho(A))$.*

Proof. This follows easily from the algorithm to compute $Q(\rho(A))$.

Theorem 3.4. *For $\alpha \in \mathfrak{t}^*_\lambda$, we have that $I(\alpha) = I(L(A, e))$, where A is any column-strict π-tableau with $A \sim Q_\pi(\alpha)$.*

Proof. Lemma 3.3 implies $Q_\pi(\rho(A)) \sim A \sim Q_\pi(\alpha)$. Hence, $Q(\rho(A)) \sim Q(\alpha)$, and we get that $I(\rho(A)) = I(\alpha)$ by (1.3). Also, by Theorem 3.2, we have that $I(L(A, e)) = I(\rho(A))$. Hence, $I(\alpha) = I(L(A, e))$. □

The next two corollaries are certainly not new, but still we have included self-contained proofs in order to illustrate the usefulness of Theorems 3.2 and 3.4. The first recovers fully the result of Joseph from [Jos81a, § 3.3].

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Corollary 3.5 (Joseph). $\mathrm{Prim}_\lambda U(\mathfrak{g}) = \{I(\alpha) \mid \alpha \in \mathfrak{t}^*_\lambda\}$.

Proof. This follows from Theorems 3.4 and 3.1, since we know already by Duflo's theorem and Joseph's irreducibility theorem that $\mathrm{Prim}\, U(\mathfrak{g}) = \{I(\alpha) \mid \alpha \in \mathfrak{t}^*\}$ is the disjoint union of the sets $\mathrm{Prim}_\lambda U(\mathfrak{g})$ for all λ. □

The next corollary is a special case of a result proved in arbitrary Cartan type by Losev; see [Los10a, Theorem 1.2.2(viii)] for the surjectivity of the map in the statement of the corollary, and Premet's conjecture formulated in [Los11b, Conjecture 1.2.1] and proved in [Los11b, § 4.2] for the injectivity (which simplifies in Cartan type A because centralizers are connected).

Corollary 3.6 (Losev). *The map*

$$\left\{\begin{matrix}\text{isomorphism classes of finite-dimensional}\\ \text{irreducible left } U(\mathfrak{g},e)\text{-modules}\end{matrix}\right\} \to \mathrm{Prim}_\lambda U(\mathfrak{g}) \quad [L] \mapsto I(L)$$

is a bijection.

Proof. By Corollary 3.5, any $I \in \mathrm{Prim}_\lambda U(\mathfrak{g})$ can be represented as $I(\alpha)$ for some $\alpha \in \mathfrak{t}^*_\lambda$. By Theorem 3.4, we see that $I(\alpha) = I(L)$ for some finite-dimensional irreducible left $U(\mathfrak{g}, e)$-module; hence, the map is surjective. For injectivity, by Theorem 2.2, it suffices to show that $I(L(A, e)) = I(L(B, e))$ implies $A \sim B$ for any column-strict π-tableaux A and B. To prove this, use Theorem 3.2 and (1.3) to see that $I(L(A, e)) = I(L(B, e))$ implies $Q(\rho(A)) \sim Q(\rho(B))$; hence, $A \sim B$ by Lemma 3.3. □

Remark 3.7. Let $\mathrm{Prim}\, U(\mathfrak{g}, e)$ denote the space of all primitive ideals in $U(\mathfrak{g}, e)$. In [Los10a], Losev showed that there is a well-defined map

$$?^\dagger : \mathrm{Prim}\, U(\mathfrak{g}, e) \to \bigcup_{\mu \geqslant \lambda} \mathrm{Prim}_\mu U(\mathfrak{g})$$

such that $(\mathrm{Ann}_{U(\mathfrak{g},e)} M)^\dagger = I(M)$ for any irreducible left $U(\mathfrak{g}, e)$-module M; here $\geqslant$ is the usual dominance ordering on partitions. Using Theorem 3.2, Corollary 3.5 and (1.3), it is a purely combinatorial exercise to check that this map sends the subset

$$\mathrm{Prim}_{hw} U(\mathfrak{g}, e) := \{\mathrm{Ann}_{U(\mathfrak{g},e)} L(A, e) \mid \text{for all } \pi\text{-tableaux } A\} \subseteq \mathrm{Prim}\, U(\mathfrak{g}, e)$$

of *highest weight* primitive ideals surjectively onto $\bigcup_{\mu \geqslant \lambda} \mathrm{Prim}_\mu U(\mathfrak{g})$; hence, Losev's map $?^\dagger$ is surjective. We conjecture that it is also injective (in Cartan type A). Combined with the preceding observations, this conjecture would imply that $\mathrm{Prim}\, U(\mathfrak{g}, e) = \mathrm{Prim}_{hw} U(\mathfrak{g}, e)$ and moreover

$$\mathrm{Ann}_{U(\mathfrak{g},e)} L(A, e) = \mathrm{Ann}_{U(\mathfrak{g},e)} L(B, e) \Leftrightarrow Q(\rho(A)) \sim Q(\rho(B)). \tag{3.11}$$

This would give a classification of $\mathrm{Prim}\, U(\mathfrak{g}, e)$ exactly in the spirit of the Duflo–Joseph classification of $\mathrm{Prim}\, U(\mathfrak{g})$ from (1.3).

Now we turn our attention to deriving some basic properties of the coinvariant Whittaker functor from (3.2). This functor has its origins in the work of Kostant and Lynch (see e.g. [Kos78, § 3.8] and [Lyn79, ch. 4]), though we give a self-contained treatment here.

Lemma 3.8. *The functor $H_0(\mathfrak{m}_\chi, ?)$ sends right $U(\mathfrak{g})$-modules that are finitely generated over $\mathfrak{m}$ to finite-dimensional right $U(\mathfrak{g}, e)$-modules.*

Proof. This is obvious from the definition (3.2). □

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Lemma 3.9. *For any right* $U(\mathfrak{p})$*-module* V*,* $H_0(\mathfrak{m}_\chi, V \otimes_{U(\mathfrak{p})} U(\mathfrak{g}))$ *is isomorphic to the restriction of* V *to* $U(\mathfrak{g}, e)$*.*

Proof. By the PBW theorem, $V \otimes_{U(\mathfrak{p})} U(\mathfrak{g}) \cong V \otimes U(\mathfrak{m})$ as a right $U(\mathfrak{m})$-module. It follows easily that the map $V \to H_0(\mathfrak{m}_\chi, V \otimes_{U(\mathfrak{p})} U(\mathfrak{g}))$ sending v to the image of $v \otimes 1$ is a vector space isomorphism. For $u \in U(\mathfrak{g}, e)$, this map sends vu to the image of $vu \otimes 1$, which is the same as the image of $(v \otimes 1)u$. Hence, our map is a homomorphism of right $U(\mathfrak{g}, e)$-modules. □

Given a vector space M, let M^* be the full linear dual $\mathrm{Hom}_{\mathbb{C}}(M, \mathbb{C})$, and denote the annihilator in M^* of a subspace $N \leqslant M$ by N° (which is of course canonically isomorphic to $(M/N)^*$). If M is a left module over an associative algebra A, then M^* is naturally a right module with action $(fa)(v) := f(av)$ for $f \in M^*$, $a \in A$ and $v \in M$. Similarly, if M is a right module, then M^* is a left module with action $(af)(v) = f(va)$.

For a right $U(\mathfrak{m})$-module M, its $\mathfrak{m}_\chi$*-restricted dual* $M^\#$ is defined from

$$M^\# := \bigcup_{i \geqslant 0} (M\mathfrak{m}_\chi^i)^\circ \subseteq M^*. \tag{3.12}$$

This gives a functor $?^\#$ from mod-$U(\mathfrak{m})$ to vector spaces.

Lemma 3.10. *The functor* $?^\#$ *is exact.*

Proof. Let I_χ be the two-sided ideal of $U(\mathfrak{m})$ generated by $\mathfrak{m}_\chi$. The subspace $(I_\chi^i)^\circ$ of $U(\mathfrak{m})^*$ is naturally a right $U(\mathfrak{m})$-module with action $(fx)(y) = f(xy)$. For any right $U(\mathfrak{m})$-module M, we claim that the linear map

$$\theta : \mathrm{Hom}_{\mathfrak{m}}(M, (I_\chi^i)^\circ) \to (M\mathfrak{m}_\chi^i)^\circ, \quad f \mapsto \mathrm{ev} \circ f$$

is an isomorphism, where $\mathrm{ev} : U(\mathfrak{m})^* \to \mathbb{C}$ is evaluation at 1. To see this, take $f \in \mathrm{Hom}_{\mathfrak{m}}(M, (I_\chi^i)^\circ)$ and observe that $\theta(f)$ annihilates $M\mathfrak{m}_\chi^i$; indeed,

$$(\mathrm{ev} \circ f)(vx) = f(vx)(1) = (f(v)x)(1) = f(v)(x) = 0$$

for $v \in M$ and $x \in \mathfrak{m}_\chi^i$. Hence, the map makes sense. To prove that it is an isomorphism, construct a two-sided inverse $\varphi : (M\mathfrak{m}_\chi^i)^\circ \to \mathrm{Hom}_{\mathfrak{m}}(M, (I_\chi^i)^\circ)$ by defining $\varphi(g) \in \mathrm{Hom}_{\mathfrak{m}}(M, (I_\chi^i)^\circ)$ for $g \in (M\mathfrak{m}_\chi^i)^\circ$ from $\varphi(g)(v)(u) := g(vu)$ for $v \in M$ and $u \in U(\mathfrak{m})$.

Now let $E_\chi := \bigcup_{i \geqslant 0} (I_\chi^i)^\circ$, the space of all $f : U(\mathfrak{m}) \to \mathbb{C}$ which annihilate I_χ^i for sufficiently large i. The result from the previous paragraph taken for all i gives us a natural isomorphism

$$\mathrm{Hom}_{\mathfrak{m}}(M, E_\chi) = \bigcup_{i \geqslant 0} \mathrm{Hom}_{\mathfrak{m}}(M, (I_\chi^i)^\circ) \xrightarrow{\sim} \bigcup_{i \geqslant 0} (M\mathfrak{m}_\chi^i)^\circ = M^\#, \quad f \mapsto \mathrm{ev} \circ f$$

for every right $U(\mathfrak{m})$-module M. Hence, the functors $?^\#$ and $\mathrm{Hom}_{\mathfrak{m}}(?, E_\chi)$ are isomorphic. The latter functor is exact because E_χ is an injective right $U(\mathfrak{m})$-module; see [Skr02, Assertion 2]. □

Now suppose that M is a right $U(\mathfrak{g})$-module. We observe that the subspace $M^\#$ of M^* from (3.12) is actually a left $U(\mathfrak{g})$-submodule belonging to the category $(U(\mathfrak{g}), \mathfrak{m}_\chi)$-mod. So, we can view $?^\#$ as an exact functor from mod-$U(\mathfrak{g})$ to $(U(\mathfrak{g}), \mathfrak{m}_\chi)$-mod.

Lemma 3.11. *For any right* $U(\mathfrak{g})$*-module* M*, we have that*

$$H^0(\mathfrak{m}_\chi, M^\#) = H^0(\mathfrak{m}_\chi, M^*) = (M\mathfrak{m}_\chi)^\circ$$

as subspaces of M^**. Moreover, there is a natural isomorphism of left* $U(\mathfrak{g}, e)$*-modules* $(M\mathfrak{m}_\chi)^\circ \cong H_0(\mathfrak{m}_\chi, M)^*$*.*

Proof. For the first statement, we observe that

$$\begin{aligned} H^0(\mathfrak{m}_\chi, M^*) &= \{f \in M^* \mid xf = 0 \text{ for all } x \in \mathfrak{m}_\chi\} \\ &= \{f \in M^* \mid (xf)(v) = 0 \text{ for all } x \in \mathfrak{m}_\chi, v \in M\} \\ &= \{f \in M^* \mid f(vx) = 0 \text{ for all } v \in M, x \in \mathfrak{m}_\chi\} \\ &= \{f \in M^* \mid f(v) = 0 \text{ for all } v \in M\mathfrak{m}_\chi\} = (M\mathfrak{m}_\chi)^\circ. \end{aligned}$$

We get that $(M\mathfrak{m}_\chi)^\circ = H^0(\mathfrak{m}_\chi, M^\#)$ too since there are obviously inclusions

$$(M\mathfrak{m}_\chi)^\circ \subseteq H^0(\mathfrak{m}_\chi, M^\#) \subseteq H^0(\mathfrak{m}_\chi, M^*).$$

Then, for the second isomorphism, just use the usual natural isomorphism $(M\mathfrak{m}_\chi)^\circ \cong (M/M\mathfrak{m}_\chi)^*$. □

Theorem 3.12. *There are natural isomorphisms of right $U(\mathfrak{g}, e)$-modules*

$$H^0(\mathfrak{m}_\chi, M^\#)^* \cong H_0(\mathfrak{m}_\chi, M) \cong H^0(\mathfrak{m}_\chi, M^*)^*$$

for any right $U(\mathfrak{g})$-module that is finitely generated over $\mathfrak{m}$.

Proof. Take the duals of the isomorphisms

$$H^0(\mathfrak{m}_\chi, M^\#) \cong H_0(\mathfrak{m}_\chi, M)^* \cong H^0(\mathfrak{m}_\chi, M^\#)$$

from Lemma 3.11 and note that $(H_0(\mathfrak{m}_\chi, M)^*)^* \cong H_0(\mathfrak{m}_\chi, M)$ by Lemma 3.8. □

The following corollary is equivalent to [Lyn79, Lemma 4.6] (attributed there to N. Wallach).

Corollary 3.13. *The functor $H_0(\mathfrak{m}_\chi, ?)$ sends short exact sequences of right $U(\mathfrak{g})$-modules that are finitely generated over $\mathfrak{m}$ to short exact sequences of finite-dimensional right $U(\mathfrak{g}, e)$-modules.*

Proof. In view of Theorem 3.12, it suffices to show that the functor $H^0(\mathfrak{m}_\chi, ?^\#)^*$ is exact. This is clear, as it is a composition of three exact functors: the functor $?^\# : U(\mathfrak{g})\text{-mod} \to (U(\mathfrak{g}), \mathfrak{m}_\chi)\text{-mod}$, which is exact by Lemma 3.10, then the functor $H^0(\mathfrak{m}_\chi, ?) : (U(\mathfrak{g}), \mathfrak{m}_\chi)\text{-mod} \to U(\mathfrak{g}, e)\text{-mod}$, which is exact as it is an equivalence of categories by Skryabin's theorem, and then the duality $?^*$. □

4. Irreducible standard modules and induced primitive ideals

Continuing with our fixed pyramid π, we define *column-separated* π-tableaux in exactly the same way as was done in the introduction in the left-justified case. The following theorem explains the significance of this notion from a representation theoretic perspective. (We point out that there is a typo in the definition of 'separated' in [BK08a], in which the inequalities $r < s$ and $r > s$ are the wrong way round.)

Theorem 4.1 [BK08a, Theorem 8.25]. *For a column-strict π-tableau A, the standard module $V(A, e)$ is irreducible if and only if A is column-separated, in which case $V(A, e) \cong L(A, e)$.*

In the rest of the section, we are going to apply this to deduce (a slight generalization of) the first equality in Theorem 1.3; see Theorem 4.6 below.

Lemma 4.2. *Let M be a right $U(\mathfrak{g})$-module that is free as a $U(\mathfrak{m})$-module. Then $\mathrm{Ann}_{U(\mathfrak{g})} M = \mathrm{Ann}_{U(\mathfrak{g})}(M^\#)$, where $M^\#$ is the left $U(\mathfrak{g})$-module defined in the previous section.*

Proof. Take $u \in \mathrm{Ann}_{U(\mathfrak{g})} M$ and $f \in M^{\#}$. Then $(uf)(v) = f(vu) = 0$ for every $v \in M$, so $uf = 0$. This shows that $\mathrm{Ann}_{U(\mathfrak{g})} M \subseteq \mathrm{Ann}_{U(\mathfrak{g})}(M^{\#})$. Conversely, by the definition (3.12), we have that

$$\mathrm{Ann}_{U(\mathfrak{g})}(M^{\#}) = \bigcap_{i \geqslant 0} \mathrm{Ann}_{U(\mathfrak{g})}(M\mathfrak{m}_\chi^i)^\circ.$$

So, any $u \in \mathrm{Ann}_{U(\mathfrak{g})}(M^{\#})$ satisfies $f(vu) = (uf)(v) = 0$ for all $i \geqslant 0$, $f \in (M\mathfrak{m}_\chi^i)^\circ$ and $v \in M$. This implies for any $v \in M$ that $vu \in M\mathfrak{m}_\chi^i$. It remains to observe that $\bigcap_{i \geqslant 0} M\mathfrak{m}_\chi^i = \mathbf{0}$. To see this, it suffices in view of the assumption that M is a free $U(\mathfrak{m})$-module to check that $\bigcap_{i \geqslant 0} U(\mathfrak{m})\mathfrak{m}_\chi^i = \mathbf{0}$. Twisting by the automorphism of $U(\mathfrak{m})$ sending $x \in \mathfrak{m}$ to $x + \chi(x)$, this is equivalent to the statement $\bigcap_{i \geqslant 0} U(\mathfrak{m})\mathfrak{m}^i = \mathbf{0}$, which is easy to see by considering the (strictly negative) grading on $\mathfrak{m}$. $\square$

Lemma 4.3. *Let V be a finite-dimensional left $U(\mathfrak{p})$-module and V^* be the dual right $U(\mathfrak{p})$-module as in the previous section. Then*

$$(V^* \otimes_{U(\mathfrak{p})} U(\mathfrak{g}))^{\#} \cong S_\chi(V)$$

as left $U(\mathfrak{g})$-modules. (On the right-hand side, we are viewing V as a left $U(\mathfrak{g}, e)$-module by the natural restriction.)

Proof. Both modules belong to the category $(U(\mathfrak{g}), \mathfrak{m}_\chi)$-mod. So, by Skryabin's equivalence of categories, it suffices to show that

$$H^0(\mathfrak{m}_\chi, (V^* \otimes_{U(\mathfrak{p})} U(\mathfrak{g}))^{\#}) \cong V$$

as left $U(\mathfrak{g}, e)$-modules. By Lemma 3.11, we have that

$$H^0(\mathfrak{m}_\chi, (V^* \otimes_{U(\mathfrak{p})} U(\mathfrak{g}))^{\#}) \cong H_0(\mathfrak{m}_\chi, V^* \otimes_{U(\mathfrak{p})} U(\mathfrak{g}))^*.$$

It remains to observe by Lemma 3.9 that $H_0(\mathfrak{m}_\chi, V^* \otimes_{U(\mathfrak{p})} U(\mathfrak{g})) \cong V^*$; hence, $H_0(\mathfrak{m}_\chi, V^* \otimes_{U(\mathfrak{p})} U(\mathfrak{g}))^* \cong V$ as V is finite dimensional. $\square$

Let A be a column-strict π-tableau. Recall the weight $\gamma(A)$ from (2.1) and the subsequent definition of the standard module $V(A, e)$; it is the restriction of the left $U(\mathfrak{p})$-module $V(A)$ to the subalgebra $U(\mathfrak{g}, e)$.

Lemma 4.4. *For any column-strict π-tableau A, we have that*

$$\mathrm{Ann}_{U(\mathfrak{g})}(V(A)^* \otimes_{U(\mathfrak{p})} U(\mathfrak{g})) = I(V(A, e)). \tag{4.1}$$

Proof. This is a consequence of the previous two lemmas and the definition (3.4). $\square$

It is a bit awkward at this point that the module on the left-hand side of (4.1) is a right module. We will get around this by twisting with a suitable anti-automorphism, at the price of a shift by the special weight β from (2.7) (and some temporary notational issues). Observe that β extends uniquely to a character of $\mathfrak{p}$. Let $\mathbb{C}_\beta$ be the corresponding one-dimensional left $U(\mathfrak{p})$-module.

We need to work momentarily with a different pyramid π^t associated to the transpose σ^t of the shift matrix σ; in other words, π^t is obtained from π by reversing the order of the columns. For example, if

$$\pi = \begin{array}{|c|c|c|} \cline{2-2} \multicolumn{1}{c|}{} & 3 & \multicolumn{1}{c}{} \\ \cline{1-2} 1 & 4 & \multicolumn{1}{c}{} \\ \hline 2 & 5 & 6 \\ \hline \end{array}\,, \quad \text{then } \pi^t = \begin{array}{|c|c|c|} \cline{2-2} \multicolumn{1}{c|}{} & 2 & \multicolumn{1}{c}{} \\ \cline{2-3} \multicolumn{1}{c|}{} & 3 & 5 \\ \hline 1 & 4 & 6 \\ \hline \end{array}\,. \tag{4.2}$$

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Let $\mathfrak{p}^t$ (respectively e^t, respectively $U(\mathfrak{g}, e^t)$) be defined in the same way as $\mathfrak{p}$ (respectively e, respectively $U(\mathfrak{g}, e)$) but starting from the pyramid π^t instead of π. If A is any π-tableau, we obtain a π^t-tableau A^t by reversing the order of the columns again. It makes sense to talk about $V(A^t)$, $V(A^t, e^t)$ and $L(A^t, e^t)$, which are $U(\mathfrak{p}^t)$- and $U(\mathfrak{g}, e^t)$-modules.

Now we define the appropriate anti-automorphism. As usual, label the boxes of π in order down columns starting from the leftmost column. Let i' be the entry in the ith box of the tableau obtained by writing the numbers $1, \dots, N$ into the boxes of π working in order down columns starting from the rightmost column; for example, in the situation of (4.2), we have that $1' = 5, 2' = 6, 3' = 2, 4' = 3, 5' = 4, 6' = 1$. Let $t : U(\mathfrak{g}) \to U(\mathfrak{g})$ be the anti-automorphism with $t(e_{i,j}) = e_{j',i'}$. Then we have that $t(e) = e^t$ and $t(\mathfrak{p}) = \mathfrak{p}^t$, so t restricts to an anti-isomorphism $t : U(\mathfrak{p}) \xrightarrow{\sim} U(\mathfrak{p}^t)$.

LEMMA 4.5. *Suppose that A is a column-strict π-tableau, so that A^t is a column-strict π^t-tableau. The pull-back $t^*(V(A^t)^*)$ of the right $U(\mathfrak{p}^t)$-module $V(A^t)^*$ is a left $U(\mathfrak{p})$-module isomorphic to $\mathbb{C}_\beta \otimes V(A)$. Hence, we have that*

$$t^*(V(A^t)^* \otimes_{U(\mathfrak{p}^t)} U(\mathfrak{g})) \cong U(\mathfrak{g}) \otimes_{U(\mathfrak{p})} (\mathbb{C}_\beta \otimes V(A)) \tag{4.3}$$

as left $U(\mathfrak{g})$-modules.

Proof. Suppose that M is a finite-dimensional left $U(\mathfrak{p}^t)$-module M and we are given an isomorphism of left $U(\mathfrak{p})$-modules $\theta : K \to t^*(M^*)$. Then it is clear that the map $U(\mathfrak{g}) \otimes_{U(\mathfrak{p})} K \to t^*(M^* \otimes_{U(\mathfrak{p}^t)} U(\mathfrak{g})), u \otimes v \mapsto \theta(v) \otimes t(u)$ is an isomorphism. So, the second part of the lemma follows from the first part. The first part is a routine exercise in highest weight theory. □

The module on the right-hand side of (4.3) is a parabolic Verma module attached to the parabolic $\mathfrak{p}$ in the usual sense. Let us give it a special name: for a column-strict π-tableau A, we set

$$M(A) := U(\mathfrak{g}) \otimes_{U(\mathfrak{p})} (\mathbb{C}_\beta \otimes V(A)). \tag{4.4}$$

This module has irreducible head

$$L(A) := M(A)/\mathrm{rad}\, M(A). \tag{4.5}$$

As $V(A)$ has highest weight $\gamma(A) - \beta - \rho$, $L(A)$ is the usual irreducible highest weight module $L(\gamma(A))$ of highest weight $\gamma(A) - \rho$.

THEOREM 4.6. *If A is a column-separated π-tableau, then*

$$I(L(A, e)) = \mathrm{Ann}_{U(\mathfrak{g})} M(A).$$

Proof. We need to work with the finite W-algebra $U(\mathfrak{g}, e^t)$, notation as introduced just before Lemma 4.5. Let A be a column-separated π-tableau. Then A^t is a column-connected π^t-tableau, so $V(A^t, e^t) \cong L(A^t, e^t)$ by Theorem 4.1. By Lemma 4.4 (for π^t rather than π), we get that

$$I(L(A^t, e^t)) = \mathrm{Ann}_{U(\mathfrak{g})}(V(A^t)^* \otimes_{U(\mathfrak{p}^t)} U(\mathfrak{g})).$$

Note that $Q(\rho(A)) \sim Q(\rho(A^t))$ by Lemma 3.3; hence, $I(L(A, e)) = I(L(A^t, e^t))$ by Theorem 3.2 and (1.3). Also, Lemma 4.5 implies that

$$\mathrm{Ann}_{U(\mathfrak{g})}(V(A^t)^* \otimes_{U(\mathfrak{p}^t)} U(\mathfrak{g})) = t(\mathrm{Ann}_{U(\mathfrak{g})} M(A)).$$

So, we have established that $I(L(A, e)) = t(\mathrm{Ann}_{U(\mathfrak{g})} M(A))$ or equivalently

$$t^{-1}(I(L(A, e))) = \mathrm{Ann}_{U(\mathfrak{g})} M(A).$$

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It remains to observe for any $I \in \operatorname{Prim} U(\mathfrak{g})$ that $t^{-1}(I) = I$; this follows from [Jan83, 5.2(2)] on noting that t^{-1} is equal to the usual Chevalley anti-automorphism up to composing with an inner automorphism. □

5. Irreducible modules and Whittaker coinvariants

In this section, we recall the construction of the finite-dimensional irreducible left $U(\mathfrak{g}, e)$-modules from [BK08a, § 8.5] by taking Whittaker coinvariants in certain irreducible highest weight modules for $\mathfrak{g}$. Before we can begin, we need to modify the definition (3.2), since we want now to use the coinvariant Whittaker functor in the context of left modules. Actually, both of the definitions (3.1)–(3.2) are rather asymmetric with respect to left and right modules. The reason for this goes back to the original definition of the finite W-algebra from (2.3): one could just as naturally consider

$$\overline{U}(\mathfrak{g}, e) := \{u \in U(\mathfrak{p}) \mid u\mathfrak{m}_\chi \subseteq \mathfrak{m}_\chi U(\mathfrak{g})\}. \tag{5.1}$$

We call this the *opposite finite W-algebra*, since there is an *anti-isomorphism* between $\overline{U}(\mathfrak{g}, e)$ and $U(\mathfrak{g}, e)$. More precisely, let $U(\mathfrak{g}, -e)$ be defined exactly as in (2.3) but with e replaced by $-e$ (hence χ replaced by $-\chi$). The antipode $S : U(\mathfrak{g}) \to U(\mathfrak{g})$ sending $x \mapsto -x$ for each $x \in \mathfrak{g}$ obviously sends $\overline{U}(\mathfrak{g}, e)$ to $U(\mathfrak{g}, -e)$, and then $U(\mathfrak{g}, -e)$ is isomorphic to $U(\mathfrak{g}, e)$, since $-e$ is conjugate to e. Composing, we get an anti-isomorphism $\overline{U}(\mathfrak{g}, e) \xrightarrow{\sim} U(\mathfrak{g}, e)$.

Using this anti-isomorphism, it is rather routine to deduce opposite versions of most of the results in § 3 with $U(\mathfrak{g}, e)$ replaced by $\overline{U}(\mathfrak{g}, e)$. For example, the opposite versions of the functors (3.1)–(3.2) are functors

$$\overline{H}^0(\mathfrak{m}_\chi, ?) : \text{mod-}U(\mathfrak{g}) \to \text{mod-}\overline{U}(\mathfrak{g}, e), \quad M \mapsto \{v \in M \mid v\mathfrak{m}_\chi = \mathbf{0}\}, \tag{5.2}$$

$$\overline{H}_0(\mathfrak{m}_\chi, ?) : U(\mathfrak{g})\text{-mod} \to \overline{U}(\mathfrak{g}, e)\text{-mod}, \quad M \mapsto M/\mathfrak{m}_\chi M. \tag{5.3}$$

The first of these functors gives an equivalence between mod-$(U(\mathfrak{g}), \mathfrak{m}_\chi)$ and mod-$\overline{U}(\mathfrak{g}, e)$, where mod-$(U(\mathfrak{g}), \mathfrak{m}_\chi)$ is the full subcategory of mod-$U(\mathfrak{g})$ consisting of all modules that are locally nilpotent over $\mathfrak{m}_\chi$ (the opposite version of Skryabin's theorem). Defining $\# : U(\mathfrak{g})\text{-mod} \to$ mod-$(U(\mathfrak{g}), \mathfrak{m}_\chi)$ in the opposite way to in § 3, the second of these functors satisfies

$$\overline{H}_0(\mathfrak{m}_\chi, M) \cong \overline{H}^0(\mathfrak{m}_\chi, M^\#)^* \tag{5.4}$$

for any left $U(\mathfrak{g})$-module M that is finitely generated over $\mathfrak{m}$ (the opposite version of Theorem 3.12).

Less obviously, there is also a canonical *isomorphism* between $U(\mathfrak{g}, e)$ and $\overline{U}(\mathfrak{g}, e)$. To record this, recall that the weight β from (2.7) extends uniquely to a character of $\mathfrak{p}$. The following theorem was proved originally (in Cartan type A only) by explicit computation in [BK08a, Lemma 3.1], but we cite instead a more conceptual proof found subsequently (which is valid in all Cartan types).

Theorem 5.1 [BGK08, Corollary 2.9]. *The automorphisms $S_{\pm\beta} : U(\mathfrak{p}) \to U(\mathfrak{p})$ sending $x \in \mathfrak{p}$ to $x \pm \beta(x)$ restrict to mutually inverse isomorphisms*

$$S_\beta : \overline{U}(\mathfrak{g}, e) \xrightarrow{\sim} U(\mathfrak{g}, e), \quad S_{-\beta} : U(\mathfrak{g}, e) \xrightarrow{\sim} \overline{U}(\mathfrak{g}, e).$$

We get an isomorphism of categories $S_{-\beta}^* : \overline{U}(\mathfrak{g}, e)\text{-mod} \to U(\mathfrak{g}, e)\text{-mod}$ by pulling back the action through $S_{-\beta}$. Composing with $S_{-\beta}^*$, we will always from now on view the functors (5.2)–(5.3) as functors

$$\overline{H}^0(\mathfrak{m}_\chi, ?) : \text{mod-}U(\mathfrak{g}) \to \text{mod-}U(\mathfrak{g}, e), \tag{5.5}$$

$$\overline{H}_0(\mathfrak{m}_\chi, ?) : U(\mathfrak{g})\text{-mod} \to U(\mathfrak{g}, e)\text{-mod}. \tag{5.6}$$

Of course, we are abusing notation here, but we will not mention $\overline{U}(\mathfrak{g}, e)$ again so there should be no confusion.

Now let $\mathcal{O}_\pi$ be the parabolic category $\mathcal{O}$ consisting of finitely generated $\mathfrak{g}$-modules that are locally finite over $\mathfrak{p}$ and semi-simple over $\mathfrak{h}$. The basic objects in $\mathcal{O}_\pi$ are the parabolic Verma modules $M(A)$ and their irreducible quotients $L(A)$ from (4.4)–(4.5). Recall that both of these modules are of highest weight $\gamma(A) - \rho$.

Lemma 5.2. *The restriction of the functor $\overline{H}(\mathfrak{m}_\chi, ?)$ to $\mathcal{O}_\pi$ is exact and it sends modules in $\mathcal{O}_\pi$ to finite-dimensional left $U(\mathfrak{g}, e)$-modules.*

Proof. Every module in $\mathcal{O}_\pi$ has a composition series with composition factors of the form $L(A)$ for various column-strict π-tableaux A. Since $L(A)$ is a quotient of $M(A)$, it is clearly finitely generated as an $\mathfrak{m}$-module. Hence, every object in $\mathcal{O}_\pi$ is finitely generated over $\mathfrak{m}$ and we are done by the opposite version of Corollary 3.13. □

Lemma 5.3. *For a column-strict π-tableau A, we have that*

$$\overline{H}_0(\mathfrak{m}_\chi, M(A)) \cong V(A, e)$$

as left $U(\mathfrak{g}, e)$-modules.

Proof. We have that $\overline{H}_0(\mathfrak{m}_\chi, M(A)) \cong S^*_{-\beta}(\mathbb{C}_\beta \otimes V(A, e)) \cong V(A, e)$ by the definition of $M(A)$ and the opposite version of Lemma 3.9. □

Call a π-tableau A *semi-standard* if it is column-strict and $\gamma(A) \in \mathfrak{t}^*_\lambda$, i.e. $Q(\gamma(A))$ has shape λ. In the left-justified case, it is an easy exercise to check that A is semi-standard if and only if A is both column-strict and row-standard, which hopefully justifies our choice of language. In other cases, the semi-standard π-tableaux are harder to characterize from a combinatorial point of view. For example, here are all the semi-standard π-tableaux for one particular π with entries 1, 2, 3, 3, 4, 4:

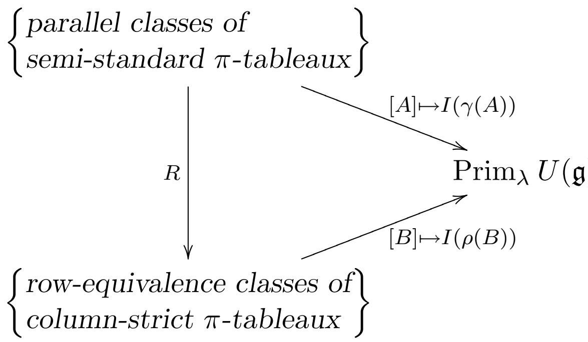
$$A = \begin{array}{|c|c|c|}\cline{2-2} \multicolumn{1}{c|}{} & 4 & \multicolumn{1}{c}{} \\ \cline{1-2} 3 & 3 & \multicolumn{1}{c}{} \\ \hline 2 & 1 & 4 \\ \hline \end{array}, \quad B = \begin{array}{|c|c|c|}\cline{2-2} \multicolumn{1}{c|}{} & 4 & \multicolumn{1}{c}{} \\ \cline{1-2} 4 & 3 & \multicolumn{1}{c}{} \\ \hline 2 & 1 & 3 \\ \hline \end{array}, \quad C = \begin{array}{|c|c|c|}\cline{2-2} \multicolumn{1}{c|}{} & 4 & \multicolumn{1}{c}{} \\ \cline{1-2} 3 & 3 & \multicolumn{1}{c}{} \\ \hline 1 & 2 & 4 \\ \hline \end{array}.$$

To illustrate the next lemma, we note for these that

$$Q_\pi(\gamma(A)) \sim \begin{array}{|c|c|c|}\cline{2-2} \multicolumn{1}{c|}{} & 3 & \multicolumn{1}{c}{} \\ \cline{1-2} 4 & 2 & \multicolumn{1}{c}{} \\ \hline 3 & 1 & 4 \\ \hline \end{array}, \quad Q_\pi(\gamma(B)) \sim \begin{array}{|c|c|c|}\cline{2-2} \multicolumn{1}{c|}{} & 4 & \multicolumn{1}{c}{} \\ \cline{1-2} 4 & 2 & \multicolumn{1}{c}{} \\ \hline 3 & 1 & 3 \\ \hline \end{array}, \quad Q_\pi(\gamma(C)) \sim \begin{array}{|c|c|c|}\cline{2-2} \multicolumn{1}{c|}{} & 4 & \multicolumn{1}{c}{} \\ \cline{1-2} 3 & 3 & \multicolumn{1}{c}{} \\ \hline 1 & 2 & 4 \\ \hline \end{array}.$$

Two semi-standard π-tableaux A and B are *parallel*, denoted $A \| B$, if one is obtained from the other by a sequence of transpositions of pairs of columns of the same height whose entries lie in different cosets of $\mathbb{C}$ modulo $\mathbb{Z}$.

Lemma 5.4. *There is a unique map R making the following into a commuting diagram of bijections.*



More explicitly, R maps $[A]$ to $[B]$, where B is any column-strict π-tableau such that $B \sim Q_\pi(\gamma(A))$. In the special case that π is left-justified (when a π-tableau is semi-standard if and only if it is both column-strict and row-standard), the map R is induced by the natural inclusion of semi-standard π-tableaux into column-strict π-tableaux.

Proof. In [BK08a, § 4.1], the following purely combinatorial statement was established: there is a well-defined bijection R from parallel classes of semi-standard π-tableaux to row-equivalence classes of column-strict π-tableaux sending $[A]$ to $[B]$, where $B \sim Q_\pi(\gamma(A))$. To deduce the first part of the lemma from this, note for such A and B that $B \sim Q_\pi(\rho(B))$ by Lemma 3.3; hence, our bijection R sends $[A]$ to $[B]$, where $Q(\gamma(A)) \sim Q(\rho(B))$. In view of (1.3), we deduce that the diagram in the statement of the lemma commutes. It remains to observe that the top right map in the diagram is already known to be a bijection, thanks to Corollary 3.6 and Theorems 2.2 and 3.2. The last statement of the lemma is clear, as $Q_\pi(\gamma(A)) \sim A$ in the case π is left-justified and A is semi-standard. □

Now we can state (and slightly extend) the main result from [BK08a, § 8.5], which identifies some of the $\overline{H}_0(\mathfrak{m}_\chi, L(A))$ with the $L(B, e)$. The equivalences in this theorem originate in work of Irving [Irv85] and proofs in varying degrees of generality can be found in several places in the literature.

Theorem 5.5. *Let A be a column-strict π-tableau. The following conditions are equivalent:*

1. *A is semi-standard;*
2. *the projective cover of $L(A)$ in $\mathcal{O}_\pi$ is self-dual;*
3. *$L(A)$ is isomorphic to a submodule of a parabolic Verma module in $\mathcal{O}_\pi$;*
4. $\operatorname{gkdim} L(A) = \dim \mathfrak{m}$, *which is the maximum Gelfand–Kirillov dimension of any module in $\mathcal{O}_\pi$;*
5. $\operatorname{gkdim} (U(\mathfrak{g})/\operatorname{Ann}_{U(\mathfrak{g})} L(A)) = \dim G \cdot e = 2 \dim \mathfrak{m}$;
6. *the associated variety $\mathcal{VA}'(\operatorname{Ann}_{U(\mathfrak{g})} L(A))$ is the closure of $G \cdot e$;*
7. *the module $\overline{H}_0(\mathfrak{m}_\chi, L(A))$ is non-zero.*

Assuming that these conditions hold, we have that

$$\overline{H}_0(\mathfrak{m}_\chi, L(A)) \cong L(B, e),$$

where B is a column-strict π-tableau with $B \sim Q_\pi(\gamma(A))$, i.e. $[B]$ is the image of $[A]$ under the bijection from Lemma 5.4.

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Proof. By (5.4) and the first paragraph of the proof of [BK08a, Lemma 8.20], the restriction of the functor $\overline{H}_0(\mathfrak{m}_\chi, ?)$ to $\mathcal{O}_\pi$ is isomorphic to the restriction of the functor $\mathbb{V}$ defined in [BK08a, § 8.5]. Given this and assuming just that (1) holds, the existence of an isomorphism $\overline{H}_0(\mathfrak{m}_\chi, L(A)) \cong L(B, e)$ follows from [BK08a, Corollary 8.24]. In particular, $\overline{H}_0(\mathfrak{m}_\chi, L(A)) \neq \mathbf{0}$, establishing that (1) $\Rightarrow$ (7). (In fact, [BK08a, Corollary 8.24] also proved (7) $\Rightarrow$ (1) but via an argument that uses the Kazhdan–Lusztig conjecture; we will give an alternative argument shortly avoiding that.)

The equivalence (1) $\Leftrightarrow$ (6) follows from Corollary 3.6, since $L(A) \cong L(\gamma(A))$ and, by definition, A is semi-standard if and only if $Q(\gamma(A))$ is of shape λ. The equivalence (4) $\Leftrightarrow$ (5) follows by standard properties of Gelfand–Kirillov dimension; see [Jos78b, Proposition 2.7]. We refer to [BK08b, Theorem 4.8] for (1) $\Leftrightarrow$ (2) $\Leftrightarrow$ (3) and postpone (4) until the next paragraph. Note that [BK08b] proved a slightly weaker result (integral weights, left-justified π) but the argument there extends.

It remains to check (5) $\Leftrightarrow$ (6) $\Leftarrow$ (7). We have that

$$\mathrm{Ann}_{U(\mathfrak{g})} L(A) \supseteq \mathrm{Ann}_{U(\mathfrak{g})} M(A) = \mathrm{Ann}_{U(\mathfrak{g})}(M(A)^{\#}),$$

using the opposite version of Lemma 4.2. Hence,

$$\mathcal{VA}'(\mathrm{Ann}_{U(\mathfrak{g})} L(A)) \subseteq \mathcal{VA}'(\mathrm{Ann}_{U(\mathfrak{g})}(M(A)^{\#})).$$

Since $\overline{H}_0(\mathfrak{m}_\chi, M(A))^* \cong \overline{H}^0(\mathfrak{m}_\chi, M(A)^{\#})$ by (5.4), we see using Lemma 5.3 that $\overline{H}^0(\mathfrak{m}_\chi, M(A)^{\#})$ is finite dimensional and non-zero. Hence, we can invoke the opposite version of Theorem 3.1 to deduce $\mathcal{VA}'(\mathrm{Ann}_{U(\mathfrak{g})}(M(A)^{\#})) = \overline{G \cdot e}$. Hence, $\mathcal{VA}'(\mathrm{Ann}_{U(\mathfrak{g})} L(A)) \subseteq \overline{G \cdot e}$ and the equivalence of (5) and (6) follows by standard dimension theory. Also, it is obvious that

$$\mathrm{Ann}_{U(\mathfrak{g})} L(A) \subseteq \mathrm{Ann}_{U(\mathfrak{g})}(L(A)^{\#}),$$

so

$$\overline{G \cdot e} \supseteq \mathcal{VA}'(\mathrm{Ann}_{U(\mathfrak{g})} L(A)) \supseteq \mathcal{VA}'(\mathrm{Ann}_{U(\mathfrak{g})}(L(A)^{\#})).$$

Finally, we repeat the earlier argument with (5.4) and the opposite version of Theorem 3.1 to see that $\mathcal{VA}'(\mathrm{Ann}_{U(\mathfrak{g})}(L(A)^{\#})) = \overline{G \cdot e}$ assuming (7) holds. Hence, (7) $\Rightarrow$ (6). ☐

From this, we obtain the following alternative classification of the finite-dimensional irreducible left $U(\mathfrak{g}, e)$-modules; cf. Theorem 2.2.

COROLLARY 5.6. *As A runs over a set of representatives for the parallel classes of semi-standard π-tableaux, the modules $\{\overline{H}_0(\mathfrak{m}_\chi, L(A))\}$ give a complete set of pairwise non-isomorphic irreducible $U(\mathfrak{g}, e)$-modules.*

Proof. Combine Theorems 2.2 and 5.5 and the bijection in Lemma 5.4. ☐

6. Dimension formulae

Now we are ready to look more closely at the dimensions of the finite-dimensional irreducible $U(\mathfrak{g}, e)$-modules. We note for column-strict π-tableaux A and B that the composition multiplicity $[M(A) : L(B)]$ is zero unless A and B have the same *content* (multiset of entries), as follows by central character considerations. Define $(L(A) : M(B)) \in \mathbb{Z}$ from the expansion

$$[L(A)] = \sum_B (L(A) : M(B))[M(B)], \tag{6.1}$$

equality in the Grothendieck group of $\mathcal{O}_\pi$, where we adopt the convention here and for the rest of the section that summation over B always means summation over all column-strict π-tableaux B having the same content as A. Also, define

$$h_\pi := \prod_{\substack{1 \leqslant i < j \leqslant N \\ \mathrm{col}(i) = \mathrm{col}(j)}} \frac{x_i - x_j}{j - i} \in \mathbb{C}[\mathfrak{t}^*],$$

which is relevant because the Weyl dimension formula tells us that

$$\dim V(A, e) = \dim V(A) = \dim(\mathbb{C}_\beta \otimes V(A)) = h_\pi(\gamma(A)) \tag{6.2}$$

for any column-strict π-tableau A.

Theorem 6.1. *For any column-strict π-tableau A, we have that*

$$\dim \overline{H}_0(\mathfrak{m}_\chi, L(A)) = \sum_B (L(A) : M(B)) h_\pi(\gamma(B)).$$

Moreover, $\dim \overline{H}_0(\mathfrak{m}_\chi, L(A)) = 0$ unless A is semi-standard, when it is equal to $\dim L(B, e)$, where B is any column-strict π-tableau with $B \sim Q_\pi(\gamma(A))$.

Proof. The final statement of the theorem is clear from Theorem 5.5. For the first statement, we know by Lemma 5.2 that the functor $\overline{H}_0(\mathfrak{m}_\chi, ?)$ induces a linear map between the Grothendieck group of $\mathcal{O}_\pi$ and the Grothendieck group of the category of finite-dimensional left $U(\mathfrak{g}, e)$-modules. Applying this map to (6.1) and using Lemma 5.3 gives the identity

$$[\overline{H}_0(\mathfrak{m}_\chi, L(A))] = \sum_B (L(A) : M(B))[V(B, e)].$$

The dimension formula follows immediately from this and (6.2). □

In the rest of the section, we explain how to rewrite the sum appearing in Theorem 6.1 in terms of the Kazhdan–Lusztig polynomials from (1.12). Actually, for simplicity, we will restrict attention from now on to integral weights, an assumption which can be justified in several different ways, one being the following result from [BK08a].

Theorem 6.2 [BK08a, Theorem 7.14]. *Suppose that A is a column-strict π-tableau. Partition the set $\{1, \ldots, l\}$ into subsets $\{i_1 < \cdots < i_k\}$ and $\{j_1 < \cdots < j_{l-k}\}$ in such a way that no entry in any of the columns $i_1, \ldots, i_k$ of A is in the same coset of $\mathbb{C}$ modulo $\mathbb{Z}$ as any of the entries in the columns $j_1, \ldots, j_{l-k}$. Let A' (respectively A'') be the column-strict tableau consisting just of columns $i_1, \ldots, i_k$ (respectively $j_1, \ldots, j_{l-k}$) of A arranged in order from left to right. Then*

$$\dim L(A, e) = \dim L(A', e') \times \dim L(A'', e''),$$

where e' and e'' are the nilpotent elements associated to the pyramids of shapes A' and A'', respectively.

For an anti-dominant weight $\delta \in P$, recall from the introduction that W_δ denotes its stabilizer and D_δ is the set of minimal length W/W_δ-coset representatives. Also, let

$$W^\pi := \{w \in W \mid \mathrm{col}(w(i)) = \mathrm{col}(i) \text{ for all } i = 1, \ldots, N\}, \tag{6.3}$$

the *column stabilizer* of our pyramid π, and D^π denote the set of all maximal length $W^\pi \backslash W$-coset representatives.

Lemma 6.3. *For column-strict π-tableaux A and B, we have that*

$$(L(A) : M(B)) = (L(\gamma(A)) : M(\gamma(B))).$$

If A and B have integer entries, these numbers can be expressed in terms of Kazhdan–Lusztig polynomials using (1.12) and (1.10).

Proof. We will work in the Grothendieck group $[\mathcal{O}]$ of the full Bernstein–Gelfand–Gelfand category $\mathcal{O}$. By the Weyl character formula, we have that

$$[M(B)] = \sum_{x \in W^\pi} (-1)^{\ell(x)} [M(x\gamma(B))].$$

Substituting this into (6.1) and comparing with the identity (1.8) for $\alpha = \gamma(A)$, we get that

$$\sum_B \sum_{x \in W^\pi} (-1)^{\ell(x)} (L(A) : M(B))[M(x\gamma(B))] = \sum_\beta (L(\gamma(A)) : M(\beta))[M(\beta)].$$

Equating coefficients of $[M(\gamma(B))]$ on both sides gives the conclusion. □

Finally, for each $w \in W$ we introduce the polynomial

$$p_w^\pi := \sum_{z \in D^\pi} (L(w) : M(z)) z^{-1}(h_\pi) \in \mathbb{C}[\mathfrak{t}^*]. \tag{6.4}$$

Comparing the following with Theorem 6.1 and recalling Corollary 5.6, these can be viewed as *dimension polynomials* computing the dimensions of finite-dimensional irreducible $U(\mathfrak{g}, e)$-modules in families.

Theorem 6.4. *Let A be a column-strict π-tableau such that $\gamma(A) \in W\delta$ for some anti-dominant $\delta \in P$. Then*

$$p_w^\pi(\delta) = \sum_B (L(A) : M(B)) h_\pi(\gamma(B)),$$

where $w = d(\gamma(A))$ and the sum is over all column-strict π-tableaux B having the same content as A.

Proof. Let A and δ be fixed as in the statement of the theorem. Let $\mathscr{T}$ be the set of all π-tableaux having the same content as A. Notice that γ restricts to a bijection $\gamma : \mathscr{T} \to W\delta$. Using this bijection, we lift the action of W on $\mathfrak{t}^*$ to an action on $\mathscr{T}$, which is just the natural left action of the symmetric group S_N on tableaux given by place permutation of entries, indexing entries in order down columns starting from the leftmost column as usual. Similarly, we view functions in $\mathbb{C}[\mathfrak{t}^*]$ now as functions on $\mathscr{T}$, so $x_i(B)$ is just the ith entry of B. Let $S \in \mathscr{T}$ be the special tableau with $\gamma(S) = \delta$ and write simply $d(B)$ for $d(\gamma(B))$ for $B \in \mathscr{T}$. We make several routine observations.

(1) The map $\mathscr{T} \to D_\delta, B \mapsto d(B)$ is a bijection with inverse $x \mapsto xS$.

(2) For any $x \in W$, we have that $h_\pi(xS) \neq 0$ if and only if xS has no repeated entries in any column.

(3) The set $D_\delta^\pi := D^\pi \cap D_\delta$ is a set of (W^π, W_δ)-coset representatives.

(4) Assume $x \in W$ is such that $h_\pi(xS) \neq 0$. Then we have that $x \in D^\pi$ if and only if xS is column-strict.

(5) The restriction of the bijection from (1) is a bijection between the set of all column-strict $B \in \mathscr{T}$ and the set $\{x \in D_\delta^\pi \mid h_\pi(x\delta) \neq 0\}$.

(6) For $x \in D_\delta^\pi$ with $h_\pi(x\delta) \neq 0$, we have that $D^\pi \cap (W^\pi x W_\delta) = xW_\delta$.

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By Lemma 6.3 and (1.10), then (5), (3) and (6), we get that

$$\begin{aligned}\sum_B (L(A):M(B))h_\pi(\gamma(B)) &= \sum_B \sum_{y\in W_\delta} (L(d(A)):M(d(B)y))h_\pi(B)\\ &= \sum_{x\in D_\delta^\pi} \sum_{y\in W_\delta} (L(d(A)):M(xy))h_\pi(x\delta)\\ &= \sum_{z\in D^\pi} (L(d(A)):M(z))z^{-1}(h_\pi)(\delta).\end{aligned}$$

Comparing with (6.4), this proves the theorem. □

7. Main results

In this section, we prove Theorems 1.1–1.7 exactly as formulated in the introduction. We begin with the promised reproof of Premet's theorem.

Proof of Premet's theorem 1.1. We recall Joseph's algorithm for computing Goldie ranks of primitive quotients of $U(\mathfrak{g})$ mentioned already in the introduction. Let $\mathscr{L}(M,M)$ denote the space of all ad $\mathfrak{g}$-locally finite maps from a left $U(\mathfrak{g})$-module M to itself. Joseph established the following statements.

(1) [Jos80a, § 5.10] For any column-strict π-tableau A, we have that

$$\operatorname{rk}\mathscr{L}(M(A),M(A)) = h_\pi(\gamma(A)).$$

(To state Joseph's result in this way, we have used (4.4) and (6.2).)

(2) [Jos80b, § 8.1] The following additivity principle holds:

$$\operatorname{rk}\mathscr{L}(M(A),M(A)) = \sum_B [M(A):L(B)]\operatorname{rk}(B),$$

where $\operatorname{rk}(B) := \operatorname{rk}\mathscr{L}(L(B),L(B))$ if $L(B)$ is a module of maximal Gelfand–Kirillov dimension in $\mathcal{O}_\pi$, and $\operatorname{rk}(B) := 0$ otherwise. (Again we are using the convention that summation over B means summation over all column-strict π-tableaux B having the same content as A.)

(3) [Jos80b, § 9.1] For any $\alpha\in\mathfrak{t}^*$, $\operatorname{rk}\mathscr{L}(L(\alpha),L(\alpha)) = \operatorname{rk} U(\mathfrak{g})/I(\alpha)$.

By (1)–(2), we get that $h_\pi(\gamma(A)) = \sum_B [M(A):L(B)]\operatorname{rk}(B)$. Inverting this gives that $\operatorname{rk}(A) = \sum_B (L(A):M(B))h_\pi(\gamma(B))$. Recall also from (4.5) that $L(A)\cong L(\gamma(A))$. So, using (3) and the implication (1) ⇒ (4) from Theorem 5.5, we have established that

$$\operatorname{rk} U(\mathfrak{g})/I(\gamma(A)) = \sum_B (L(A):M(B))h_\pi(\gamma(B)) \tag{7.1}$$

for any semi-standard π-tableau A. Now take any finite-dimensional irreducible left $U(\mathfrak{g},e)$-module L. By Corollary 5.6, we may assume $L = \overline{H}_0(\mathfrak{m}_\chi, L(A))$ for a semi-standard π-tableau A. Comparing Theorem 6.1 with Joseph's formula (7.1), we see that $\dim L = \operatorname{rk} U(\mathfrak{g})/I(\gamma(A))$. Finally, observe that $I(\gamma(A)) = I(L)$ by Lemma 5.4 and Theorems 3.2 and 5.5. □

For the rest of the section, we assume that the pyramid π is left-justified, keeping λ fixed as before.

Proof of Theorem 1.2. It suffices to show for $\alpha \in \mathfrak{t}_\lambda^*$ that $\operatorname{rk} U(\mathfrak{g})/I(\alpha) = 1$ if and only if $Q(\alpha)$ is row-equivalent to a column-connected tableau. By Theorem 3.4, we have that $I(\alpha) = I(L(A, e))$, where A is any column-strict tableau that is row-equivalent to $Q(\alpha)$. Hence, by Theorem 1.1, we see that $\operatorname{rk} U(\mathfrak{g})/I(\alpha) = \dim L(A, e)$. Now apply Theorem 2.3. □

Proof of Theorem 1.3. We may assume $\alpha \in \mathfrak{t}_\lambda^*$ and $Q(\alpha) \sim A$ for some column-separated tableau A. By Theorems 3.4 and 4.6, we deduce that $I(\alpha) = I(L(A, e)) = \operatorname{ann}_{U(\mathfrak{g})} M(A)$. Moreover, by Theorems 1.1 and 4.1, we have that

$$\operatorname{rk} U(\mathfrak{g})/I(\alpha) = \operatorname{rk} U(\mathfrak{g})/I(L(A, e)) = \dim L(A, e) = \dim V(A, e) = \dim V(A).$$

It remains to observe from the definition (4.4) that $M(A) \cong U(\mathfrak{g}) \otimes_{U(\mathfrak{p})} F$, where F is as in the statement of Theorem 1.3, and also $\dim V(A) = \dim F$, since they are equal up to tensoring by a one-dimensional representation. □

Proof of Theorem 1.4. Take any $\alpha \in \mathfrak{t}_\lambda^*$ and set $A := Q(\alpha)$. Then, for each $z \in \mathbb{C}$, let A_z be the tableau obtained by erasing all entries of A that are not in $z + \mathbb{Z}$, subtracting z from all remaining entries and then sliding all boxes to the left to get a left-justified tableau with integer entries. It is clear from the definition of $Q(\alpha)$ that each A_z is a column-strict tableau; indeed, $A_z = Q(\alpha_z)$ for α_z as in the statement of Theorem 1.4. Finally, let e_z be the nilpotent in $\mathfrak{g}_z$ associated to the pyramid of the same shape as A_z. Applying Theorem 6.2 (perhaps several times), we get that

$$\dim L(A, e) = \prod_z \dim L(A_z, e_z),$$

where the product is over a set of coset representatives for $\mathbb{C}$ modulo $\mathbb{Z}$. This implies Theorem 1.4 thanks to Theorems 3.4 and 1.1. □

Proof of Theorem 1.5. We may assume that w is minimal in its left cell and that $Q(w)$ is of shape λ. Take any regular anti-dominant δ and set $\alpha := w\delta \in \widehat{C}_w$. Since the entries of $Q(\alpha)$ satisfy the same system of inequalities as the entries of $Q(w)$, we see that $Q(\alpha) \sim B$ for a column-separated tableau B which is obtained from $Q(\alpha)$ by permuting entries within rows in exactly the same way as A is obtained from $Q(w)$. Theorem 1.3 tells us that $\operatorname{rk} U(\mathfrak{g})/I(\alpha)$ is the dimension of the irreducible $\mathfrak{h}$-module of highest weight $\gamma(B) - \rho$, where $\mathfrak{h}$ is the standard Levi subalgebra $\mathfrak{gl}_{\lambda'_1}(\mathbb{C}) \oplus \mathfrak{gl}_{\lambda'_2}(\mathbb{C}) \oplus \cdots$ and $\lambda' = (\lambda'_1 \geqslant \lambda'_2 \geqslant \cdots)$ is the transpose of λ. Using the Weyl dimension formula for $\mathfrak{h}$, we deduce that

$$\operatorname{rk} U(\mathfrak{g})/I(\alpha) = h_\lambda(\gamma(B)).$$

Using (1.6), the definition of h_λ from the statement of Theorem 1.6 and the assumption that w is minimal in its left cell, the right-hand side here is the same as

$$\left(\prod_{(i,j)} \frac{x_{w(i)} - x_{w(j)}}{d(i, j)}\right)(\gamma(Q(\alpha))) = \left(\prod_{(i,j)} \frac{x_i - x_j}{d(i, j)}\right)(\delta),$$

where the product is over pairs (i, j) as in the statement of the theorem. By the definition (1.7), this establishes that p_w and $\prod_{(i,j)}(x_i - x_j)/d(i, j)$ take the same values on all regular anti-dominant γ. The theorem follows by density. □

Proof of Joseph's theorem 1.6. Take any $w \in W$ that is minimal in its left cell and assume that $Q(w)$ has shape λ. Take any regular anti-dominant δ. Set $\alpha := w\delta \in \widehat{C}_w$ and $A := Q(\alpha)$, which is a semi-standard tableau of shape λ. By (1.4) and (1.6), we have that $d(\alpha) = w$ and $\gamma(A) = \alpha$. So, Theorems 6.1 and 6.4 give that $\dim \overline{H}_0(\mathfrak{m}_\chi, L(A)) = p_w^\pi(\delta)$. By Lemma 5.4 and Theorems 3.2

and 5.5, we know that $I(\overline{H}_0(\mathfrak{m}_\chi, L(A))) = I(\alpha)$. Hence, by Theorem 1.1, we deduce that

$$\operatorname{rk} U(\mathfrak{g})/I(\alpha) = p_w^\pi(\delta).$$

(This equality can also be deduced without finite W-algebras using Theorem 6.4 and Joseph's (7.1) directly.) Comparing with (1.7), we have therefore shown that $p_w(\delta) = p_w^\pi(\delta)$ for all δ in a Zariski dense subset of $\mathfrak{t}^*$, so $p_w = p_w^\pi$. It remains to observe that the polynomial p_w^π from (6.4) is the same as the one on the right-hand side of (1.13) in the left-justified case. □

Proof of Theorem 1.7. Let $w \in W$ be minimal in its left cell and assume that $Q(w)$ is of shape λ. Like in the proof of Theorem 6.4, we use the map γ from (1.1) to lift the action of W on $\mathfrak{t}^*$ to an action on tableaux of shape λ by place permutation. Let $\mathscr{T}$ be the set of all tableaux of shape λ with entries $\{1, \ldots, N\}$ and $S \in \mathscr{T}$ be the unique tableau with $\gamma(S) = -\rho$. We obviously get a bijection $W \to \mathscr{T}, w \mapsto wS$. For any $x \in W$, we have that $x \in D^\lambda$ if and only if xS is column-strict, so our bijection identifies D^λ with the column-strict tableaux in $\mathscr{T}$. Under this identification, it is well known that the usual Bruhat order $\geqslant$ on D^λ corresponds to the partial order $\geqslant$ on column-strict tableaux such that $A \geqslant B$ if and only if we can pass from column-strict tableau A to column-strict tableau B by repeatedly applying the following basic move:

(1) find entries $i > j$ in A such that the column containing i is strictly to the left of the column containing j;

(2) interchange these entries and then reorder entries within columns to obtain another column-strict tableau.

Now, to prove the result, let C be the tableau from the statement of Theorem 1.7. Using the explicit formula for p_w from Theorem 1.6, we need to show that

$$\sum_{z \in D^\lambda} (L(w) : M(z)) h_\lambda(zw^{-1}\gamma(C)) = 1.$$

By (1.5) and (1.6), we know that $wS = Q(w)$, which is standard and so certainly column-strict; hence, $w \in D^\lambda$. So, there is a term in the above sum with $z = w$ and for this z it is obvious that $(L(w) : M(z)) h_\lambda(zw^{-1}\gamma(C)) = h_\lambda(\gamma(C)) = 1$. Since $(L(w) : M(z)) = 0$ unless $z \leqslant w$ in the Bruhat order on W, it remains to show that $h_\lambda(zw^{-1}\gamma(C)) = 0$ for any $z \in D^\lambda$ such that $z < w$. To see this, take such an element z and let $A := wS$ and $B := zS$, so A is standard, B is column-strict and $A > B$ (in the partial order on column-strict tableaux defined in the first paragraph of the proof). In the next paragraph, we show that there exist $1 \leqslant i < j \leqslant N$ such that the numbers i and j appear in the same row of A and in the same column of B. We deduce in the notation from § 2 that $\operatorname{row}(w(i)) = \operatorname{row}(w(j))$ and $\operatorname{col}(z(i)) = \operatorname{col}(z(j))$. Hence,

$$(x_{z(i)} - x_{z(j)})(zw^{-1}\gamma(C)) = (x_i - x_j)(w^{-1}\gamma(C)) = (x_{w(i)} - x_{w(j)})(\gamma(C)) = 0$$

and $x_{z(i)} - x_{z(j)}$ is a linear factor of h_λ. This implies that $h_\lambda(zw^{-1}\gamma(C)) = 0$, as required.

It remains to prove the following claim: given tableaux $A > B$ of shape λ with A standard and B column-strict, there exist $1 \leqslant i < j \leqslant N$ such that i and j appear in the same row of A and in the same column of B. To see this, let $A_{\leqslant j}$ (respectively $B_{\leqslant j}$) denote the diagram obtained from A (respectively B) by removing all boxes containing entries $> j$. Choose $1 \leqslant j \leqslant N$ so that $A_{\leqslant (j-1)} = B_{\leqslant (j-1)}$ but $A_{\leqslant j} \neq B_{\leqslant j}$. Suppose that j appears in column c of B, and observe since $A > B$ that this column is strictly to the left of the column of A containing j. Suppose also that j appears in row r of A, and observe since A is standard that this row is strictly below the row of B containing j. As $A_{\leqslant (j-1)} = B_{\leqslant (j-1)}$ and B is column-strict, A and B have the same entry $i \leqslant j - 1$ in row r and column c. Thus, the entries i and j lie in the same row r of A, and in the same column c of B.

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Jonathan Brundan brundan@uoregon.edu
Department of Mathematics, University of Oregon, Eugene, OR 97403, USA

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