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Explicit Models for Threefolds Fibred by K3 Surfaces of Degree Two

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Abstract. We consider threefolds that admit a fibration by K3 surfaces over a nonsingular curve, equipped with a divisorial sheaf that defines a polarization of degree two on the general fibre. Under certain assumptions on the threefold we show that its relative log canonical model exists and can be explicitly reconstructed from a small set of data determined by the original fibration. Finally, we prove a converse to this statement: under certain assumptions, any such set of data determines a threefold that arises as the relative log canonical model of a threefold admitting a fibration by K3 surfaces of degree two.

1 Introduction

The aim of this paper is to produce an explicit construction for the relative log canonical model of a threefold that admits a fibration by K3 surfaces of degree two.

The explicit construction of threefolds is a problem that has attracted significant interest in recent years, fuelled largely by open questions about mirror symmetry and the classification of Calabi–Yau threefolds. Motivated by general classification theory for algebraic varieties, a common method by which such threefolds are constructed is by way of a K3 fibration. Most approaches to date involve embedding such fibrations into a toric ambient space [1, 7, 8, 13], as this provides a setting under which many properties of the constructed threefolds can be easily calculated.

In this paper we find an alternative method by which K3-fibred threefolds may be constructed. We begin by restricting our attention to threefolds that admit fibrations by K3 surfaces equipped with a polarization of degree two; the standard example of such a K3 surface is a double cover of $\mathbb{P}^2$ ramified over a sextic curve. These K3 surfaces can be thought of as higher dimensional analogues of hyperelliptic curves of genus two, which can be seen as double covers of $\mathbb{P}^1$ ramified over six points. Using this analogy, in this paper we generalize to higher dimensions a construction of Catanese and Pignatelli [2], which produces an explicit model for a surface admitting a fibration by hyperelliptic curves of genus two. This construction should provide a more general way to construct K3-fibred threefolds than the toric embedding method, but remains explicit enough that many properties of the constructed threefolds can still be easily calculated (see [16, Chapter 5]).

More specifically, given a threefold X admitting a fibration by K3 surfaces $\pi\colon X \to S$ over a smooth curve S, along with a divisorial (*i.e.,* rank one reflexive)

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sheaf $\mathcal{L}$ inducing a polarization of degree two on the general fibre, our aim is to construct a birational model for $(X, \pi, \mathcal{L})$ over S. The model we choose to construct is known as the *relative log canonical model*, a birational model arising from the minimal model programme (see [10, Section 3.8]) that has a good explicit description and is unique in its birational equivalence class.

The construction proceeds as follows: starting with a threefold fibred by K3 surfaces of degree two $(X, \pi, \mathcal{L})$ over a curve S that satisfies certain assumptions (see Conditions 2.2 below), we begin by finding a 5-tuple of data $(\mathcal{E}_1, \tau, \xi, \mathcal{E}_3^+, \beta)$ on S (see Definition 4.7) that is determined by $(X, \pi, \mathcal{L})$. We then show that from this 5-tuple it is possible to explicitly reconstruct the *relative log canonical algebra* $\mathcal{R}(X, \pi, \mathcal{L})$ of $(X, \pi, \mathcal{L})$ (defined in Section 2), from which the relative log canonical model of $(X, \pi, \mathcal{L})$ is easily computed as $\mathbf{Proj}_S \mathcal{R}(X, \pi, \mathcal{L})$.

The following theorem is the main result of this paper (Theorem 5.2), which categorizes the output of the above construction. It shows that any threefold fibred by K3 surfaces of degree two $(X, \pi, \mathcal{L})$ satisfying Conditions 2.2 determines a 5-tuple that is *admissible* (see Definition 5.1), from which its relative log canonical model can be explicitly reconstructed, and, furthermore, that given an admissible 5-tuple we may always find a threefold fibred by K3 surfaces of degree two that determines that 5-tuple. Therefore, our result gives a complete description of the threefolds that can arise as the relative log canonical models of threefolds fibred by K3 surfaces of degree two, in terms of their associated admissible 5-tuples, along with an explicit method to construct them.

Theorem 1.1 *Fix a nonsingular complex curve S. Let $(X, \pi, \mathcal{L})$ be a threefold fibred by K3 surfaces of degree two over S that satisfies Conditions 2.2. Then the associated* 5*-tuple of $(X, \pi, \mathcal{L})$ over S is admissible.*

Conversely, let $\mathcal{R}$ be a sheaf of $\mathcal{O}_S$-algebras defined by an admissible 5*-tuple $(\mathcal{E}_1, \tau, \xi, \mathcal{E}_3^+, \beta)$. Let $X = \mathbf{Proj}_S(\mathcal{R})$ and $\pi \colon X \to S$ be the natural projection. Then there is a canonically defined polarization sheaf $\mathcal{L}$ on X that makes $(X, \pi, \mathcal{L})$ into a threefold fibred by K3 surfaces of degree two that satisfies Conditions 2.2, and, furthermore, $\mathbf{Proj}_S(\mathcal{R})$ is the relative log canonical algebra of $(X, \pi, \mathcal{L})$ over S and $(\mathcal{E}_1, \tau, \xi, \mathcal{E}_3^+, \beta)$ is its associated* 5*-tuple.*

In order to prove this result we rely heavily on results of Catanese and Pignatelli [2], who perform an analogous construction for the relative canonical models of surfaces fibred by hyperelliptic curves of genus two. The closeness of the analogy between K3 surfaces of degree two and hyperelliptic curves of genus two means that several of the results in Sections 3, 4, and 5 of this paper correspond directly with results in [2], although many of the proofs have been substantially modified to work in the higher dimensional case. In particular, our main result (Theorem 5.2) should be seen as a higher-dimensional analogue of [2, Theorem 4.13].

Notation We briefly mention some notation that will be used throughout this paper. Firstly, let D be a Weil divisor on a normal variety X. Then D determines a divisorial sheaf $\mathcal{O}_X(D)$ on X. The m-th reflexive power of $\mathcal{O}_X(D)$ is defined to be $\mathcal{O}_X(D)^{[m]} := (\mathcal{O}_X(D)^{\otimes m})^{\vee\vee}$, where $^\vee$ denotes the dual sheaf, and agrees with the divisorial sheaf

$\mathcal{O}_X(mD)$. A more detailed discussion of the correspondence between Weil divisors and divisorial sheaves may be found in [14, Appendix 1].

Next let $f\colon X-\to Y$ be a birational map and let D be a Weil divisor on X. Then we denote the exceptional set of f by $\mathrm{Ex}(f)$, defined to be the set of points $x\in X$ such that f^{-1} is not well defined at $f(x)$. Finally, to avoid confusion with the direct image, we denote the strict transform of D under f by the nonstandard notation f_+D.

2 The Relative Log Canonical Model of a Threefold Fibred by K3 Surfaces of Degree Two

The aim of this paper is to find an explicit method to construct the relative log canonical model of a threefold fibred by K3 surfaces of degree two. However, before doing this we should check that this model is always well defined. Begin by fixing a nonsingular complex curve S.

Definition 2.1 A *threefold fibred by K3 surfaces of degree two* over S is a triple $(X,\pi,\mathcal{L})$ consisting of

- a $\mathbb{Q}$-Gorenstein normal complex variety X of dimension 3,
- a flat, projective, surjective morphism $\pi\colon X\to S$ with connected fibres, whose general fibre is a K3 surface with at worst Du Val singularities, and
- a divisorial sheaf $\mathcal{L}$ on X with $\mathcal{L}^{[m]}$ invertible for some $m>0$, that induces a nef and big divisorial sheaf $\mathcal{L}_s$ satisfying $\mathcal{L}_s.\mathcal{L}_s=2$ on a general fibre X_s of π.

Given a threefold fibred by K3 surfaces of degree two $(X,\pi,\mathcal{L})$, the *relative log canonical algebra* of $(X,\pi,\mathcal{L})$ is defined to be the $\mathcal{O}_S$-algebra

$$\mathcal{R}(X,\pi,\mathcal{L}):=\bigoplus_{n\geq 0}\pi_*\big((\omega_X\otimes\mathcal{L})^{[n]}\big).$$

Under the assumption that the relative log canonical algebra $\mathcal{R}(X,\pi,\mathcal{L})$ is finitely generated as an $\mathcal{O}_S$-algebra, the *relative log canonical model* X^c of $(X,\pi,\mathcal{L})$ over S, defined in [10, Section 3.8], is well-defined and equal to

$$X^c:=\mathbf{Proj}_S\mathcal{R}(X,\pi,\mathcal{L}).$$

This model admits a natural morphism $\pi^c\colon X^c\to S$, and, furthermore, there is a birational map $\phi\colon X-\to X^c$ over S satisfying $\operatorname{codim}\mathrm{Ex}(\phi^{-1})\geq 2$.

The aim of this paper is to find an explicit method to construct the relative log canonical model of a threefold fibred by K3 surfaces of degree two. However, before we can do this, we must first impose some conditions on our threefold fibred by K3 surfaces of degree two.

Conditions 2.2 Let $(X,\pi,\mathcal{L})$ be a threefold fibred by K3 surfaces of degree two. We will consider the following conditions:

(i) The divisorial sheaf $\mathcal{L}$ is isomorphic to $\mathcal{O}_X(H)\otimes\pi^*\mathcal{M}$, where H is a prime divisor on X that is flat over S and $\mathcal{M}$ is an invertible sheaf on S.

(ii) The log pair (X, H) is canonical (see [10, Definition 2.34]).

(iii) The sheaf $\mathcal{L}_s$ induced on a general fibre of $\pi\colon X \to S$ by $\mathcal{L}$ is invertible and generated by its global sections.

(iv) The localization of the relative log canonical algebra $\mathcal{R}(X, \pi, \mathcal{L})_s \otimes_{\mathcal{O}_{S,s}} k(s)$ at any point $s \in S$ is isomorphic to one of the following:

- (hyperelliptic case) $\mathbb{C}[x_1, x_2, x_3, z]/(z^2 - f_6(x_i))$, where $\deg(x_i) = 1$ and $\deg(z) = 3$; or
- (unigonal case) $\mathbb{C}[x_1, x_2, x_3, y, z]/(z^2 - g_6(x_i, y),\ g_2(x_i))$, where $\deg(x_i) = 1$, $\deg(y) = 2$, $\deg(z) = 3$, and $g_6(0, 0, 0, 1) \neq 0$.

2.1 Remarks on these Conditions

We use this subsection to briefly remark upon the reasons behind these conditions and to discuss when they hold in certain special cases. However, the first thing that we should check is that, under the assumptions above, the relative log canonical model X^c of X is well defined. This will follow if we can show that the relative log canonical algebra $\mathcal{R}(X, \pi, \mathcal{L})$ is finitely generated as an $\mathcal{O}_S$-algebra.

Lemma 2.3 *Suppose that $(X, \pi, \mathcal{L})$ is a threefold fibred by K3 surfaces of degree two that satisfies Conditions 2.2*(i) *and* (ii). *Then the relative log canonical algebra $\mathcal{R}(X, \pi, \mathcal{L})$ is finitely generated as an $\mathcal{O}_S$-algebra.*

Proof We begin by noting that, as $\mathcal{M}$ is invertible, we have isomorphisms

$$(\omega_X \otimes \mathcal{L})^{[n]} \cong \big(\omega_X \otimes \mathcal{O}_X(H)\big)^{[n]} \otimes \pi^*\mathcal{M}^n$$

for all $n \geq 0$. So, by the projection formula, there are isomorphisms

$$\pi_*\big((\omega_X \otimes \mathcal{L})^{[n]}\big) \cong \pi_*\Big(\big(\omega_X \otimes \mathcal{O}_X(H)\big)^{[n]}\Big) \otimes \mathcal{M}^n$$

for all $n \geq 0$. Thus the $\mathcal{O}_S$-algebra $\mathcal{R}(X, \pi, \mathcal{L})$ is just the twist of the $\mathcal{O}_S$-algebra $\mathcal{R}(X, \pi, \mathcal{O}_X(H))$ by the invertible sheaf $\mathcal{M}$, and so $\mathcal{R}(X, \pi, \mathcal{L})$ is finitely generated as an $\mathcal{O}_S$-algebra if and only if $\mathcal{R}(X, \pi, \mathcal{O}_X(H))$ is (note that this also implies that the corresponding relative log canonical models $\mathbf{Proj}_S\mathcal{R}(X, \pi, \mathcal{L})$ and $\mathbf{Proj}_S\mathcal{R}(X, \pi, \mathcal{O}_X(H))$ are isomorphic, a fact that will be useful later).

Finally, as the log pair (X, H) is canonical, finite generation of $\mathcal{R}(X, \pi, \mathcal{O}_X(H))$ follows from results of the log minimal model program for threefolds [3, Theorem 3.14]. ∎

Remark 2.4 We note that both the proof of this lemma and the construction in this paper work just as well when Condition 2.2(ii) is replaced by the weaker assumption "the log pair (X, H) is log canonical". However, under this weaker assumption, the proof of the final Theorem 5.2, that describes the output of our construction, fails to hold.

With this in place, we will briefly discuss Condition 2.2(i). Naïvely, one might expect to define a polarization on a threefold fibred by K3 surfaces of degree two simply

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by specifying a prime divisor H on X that is flat over S and that induces a polarization of the required type on a general fibre. Indeed, Condition 2.2(i) implies that such a divisor always exists, in the form of the divisor H, and it follows from the proof of Lemma 2.3 that the relative log canonical models of $(X, \pi, \mathcal{L})$ and $(X, \pi, \mathcal{O}_X(H))$ are isomorphic over S. However, when we come to construct the relative log canonical model for a threefold fibred by K3 surfaces of degree two, we find that our construction produces both the model threefold and a polarization sheaf on it (see Theorem 5.2), and that this polarization sheaf does not necessarily admit a flat section. To account for this, Condition 2.2(i) allows the polarization to be twisted by the inverse image of a divisor on S.

Next, we prove a result that will allow Conditions 2.2(i) and 2.2(ii) to be checked locally on S.

Proposition 2.5 *Let $(X, \pi, \mathcal{L})$ be a threefold fibred by K3 surfaces of degree two. Suppose that for every closed point $s \in S$ there exists an affine open set $U_s \subset S$ containing s and a reduced, irreducible divisor H_{U_s} defined by a section in $H^0(\pi^{-1}(U_s), \mathcal{L})$, such that H_{U_s} is flat over U_s and the log pair $(\pi^{-1}(U_s), H_{U_s})$ is canonical. Then Conditions* 2.2(i) *and* (ii) *hold for $(X, \pi, \mathcal{L})$.*

Proof Note that in order to prove the proposition, it suffices to show that we may find an invertible sheaf $\mathcal{M}$ on S such that $\mathcal{L} \otimes \pi^*\mathcal{M}^{-1} \cong \mathcal{O}_X(H)$ for some prime divisor H that is flat over S and makes (X, H) canonical.

To construct $\mathcal{M}$, we begin by choosing some ample invertible sheaf $\mathcal{N}$ on S. By the ampleness property, we may find an integer $m > 0$ such that $\pi_*\mathcal{L} \otimes \mathcal{N}^m$ is generated by its global sections. Furthermore, by the projection formula and the Leray spectral sequence, we have an isomorphism

$$H^0(X, \mathcal{L} \otimes \pi^*\mathcal{N}^m) \cong H^0(S, \pi_*\mathcal{L} \otimes \mathcal{N}^m). \tag{2.1}$$

In particular, the space of sections $H^0(X, \mathcal{L} \otimes \pi^*\mathcal{N}^m)$ is nonempty. Let D be an effective divisor defined by a general section in this space.

Now, we say that an irreducible component D_i of an effective divisor D is *horizontal* if $\pi(D_i) = S$ and *vertical* if $\pi(D_i)$ is a closed point in S. Let D^h denote the sum of the horizontal components of D and let D^v denote the sum of the vertical components. As π is proper, the image of any irreducible component must be closed and connected, so any irreducible component of D is either horizontal or vertical and $D = D^h + D^v$. Furthermore, D^h and D^v must be effective because D is.

Now let $s \in S$ be a point over which the fibre X_s is reducible or non-reduced and let V be any prime divisor in $\mathrm{Supp}(X_s)$. By assumption, there exists an affine neighbourhood U_s of s and a section in $H^0(\pi^{-1}(U_s), \mathcal{L}) \cong H^0(\pi^{-1}(U_s), \mathcal{L} \otimes \pi^*\mathcal{N}^m)$ that does not vanish on V. So, since $\pi_*\mathcal{L} \otimes \mathcal{N}^m$ is generated by its global sections, using the isomorphism (2.1) we find that there exists a global section of $\mathcal{L} \otimes \pi^*\mathcal{N}^m$ that does not vanish on V. Therefore, the natural injection

$$H^0\big(X, \mathcal{L}(-V) \otimes \pi^*\mathcal{N}^m\big) \longrightarrow H^0(X, \mathcal{L} \otimes \pi^*\mathcal{N}^m)$$

cannot be surjective, so its image is Zariski closed in $H^0(X, \mathcal{L} \otimes \pi^*\mathcal{N}^m)$ and V does not appear in D. Repeating this argument for the (finitely many) other components of reducible or non-reduced fibres, we see that no components of such fibres appear in D.

Therefore, only components of reduced, irreducible fibres may appear in D^v. So D^v must be a sum of fibres and as such can be written as the inverse image of an effective divisor E on S. We have

$$\mathcal{O}_X(D^h) \cong \mathcal{L} \otimes \pi^*\big(\mathcal{N}^m \otimes \mathcal{O}_S(-E)\big).$$

Let $\mathcal{M} = \mathcal{N}^{-m} \otimes \mathcal{O}_S(E)$ and $H = D^h$. In order to complete the proof of Proposition 2.5 we just need to show that H is reduced, irreducible, and flat over S, and that (X, H) is canonical.

We begin with flatness. Let H_i denote a prime divisor in $\operatorname{Supp}(H)$. To show that H_i is flat over S (as a divisor), it suffices to show that H_i is flat when considered as a scheme over S. As S is a nonsingular curve, by [4, Proposition III.9.7] this will follow if we can show that any associated point of H_i maps to the generic point of S. But H_i is reduced and irreducible, so its only associated point is the generic point, which maps to the generic point of S as $\pi|_{H_i}$ is surjective. Thus every component H_i of H is flat over S, so H must be also.

Finally, we have to show that H is reduced and irreducible and that (X, H) is canonical. Pick a finite subcover $\mathcal{U}$ of $\{U_s | s \in S\}$. Then, since $\pi_*\mathcal{O}_X(D)$ is generated by its global sections, using the isomorphism (2.1) we see that D may be chosen so that $H|_{\pi^{-1}(U_s)}$ is a general member of the linear system $|H_{U_s}|$ for each $U_s \in \mathcal{U}$. But such a member is reduced and irreducible by Bertini's theorem, and the pair $(\pi^{-1}(U_s), H|_{\pi^{-1}(U_s)})$ is canonical by [10, Corollary 2.33]. So H must be reduced and irreducible, and, as the canonical property can be checked locally, the pair (X, H) is canonical. ∎

We will devote the remainder of this subsection to a discussion of Conditions 2.2(iii) and (iv), beginning with a lemma that will, in certain cases, allow us to check Condition 2.2(iv) as a statement on the cohomology of the fibres.

Lemma 2.6 *Suppose that $(X, \pi, \mathcal{L})$ is a threefold fibred by K3 surfaces of degree two that satisfies Conditions 2.2*(i) *and* (ii). *Let $s \in S$ be any point and let X_s denote the fibre of $\pi\colon X \to S$ over s. If $\mathcal{L}$ and $(\omega_X \otimes \mathcal{L})$ are π-nef in a neighbourhood of X_s, then there is an isomorphism*

$$\mathcal{R}(X, \pi, L)_s \otimes_{\mathcal{O}_{S,s}} k(s) \cong \bigoplus_{n \geq 0} H^0\big(X_s, (\omega_X \otimes \mathcal{L})^{[n]}\big).$$

Proof It suffices to prove that the natural maps

$$\pi_*\big((\omega_X \otimes \mathcal{L})^{[n]}\big)_s \otimes_{\mathcal{O}_{S,s}} k(s) \longrightarrow H^0\big(X_s, (\omega_X \otimes \mathcal{L})^{[n]}\big)$$

are isomorphisms for all $n > 0$. This will follow from the theorem on cohomology and base change if we can show that the higher direct images $R^i\pi_*\left((\omega_X \otimes \mathcal{L})^{[n]}\right)$ vanish in a neighbourhood of s for all $i > 0$ and all $n > 0$.

In order to show this note first that, since H is effective and (X, H) is canonical, by [10, Corollary 2.35] we have that $(X, 0)$ is also canonical. Furthermore, as $\mathcal{L}$ is π-nef in a neighbourhood of X_s and has self-intersection number two on any fibre of π, it is also π-big in a neighbourhood of X_s. Using this, the vanishing of the higher direct images follows immediately by applying [9, Theorem 1.2.5 and Remark 1.2.6] to the pair $(X, 0)$ and the divisorial sheaves $(\omega_X \otimes \mathcal{L})^{[n]}$. ■

We use this lemma to prove the next proposition, which serves to motivate Condition 2.2(iv) by showing that it holds for a smooth generic fibre in a threefold fibred by K3 surfaces of degree two.

Proposition 2.7 *Let $(X, \pi, \mathcal{L})$ be a threefold fibred by K3 surfaces of degree two that satisfies Conditions* 2.2(i) *and* (ii)*. Suppose that the generic fibre of $\pi\colon X \to S$ is smooth. Then Condition* 2.2(iv) *holds for a general $s \in S$.*

Proof Let $s \in S$ be a general point and let X_s denote the fibre over s. Then, by definition, $\mathcal{L}$ is π-nef in a neighbourhood of X_s, so we may apply Lemma 2.6 and adjunction to X_s to obtain that

$$\mathcal{R}(X, \pi, \mathcal{L})_s \otimes_{\mathcal{O}_{S,s}} k(s) \cong \bigoplus_{n \geq 0} H^0(X_s, \mathcal{L}_s^n),$$

where $\mathcal{L}_s$ denotes the invertible sheaf induced on X_s by $\mathcal{L}$.

By assumption, we see that X_s is a smooth K3 surface of degree two with polarization $\mathcal{L}_s$. Let H_s be a divisor on X_s defined by a section of $\mathcal{L}_s$. [11, Proposition 8] shows that $|H_s|$ is either base point free or has the form $|H_s| = |2E| + F$, where E is a smooth elliptic curve and F is a fixed rational (-2)-curve. In either case, the proof of [11, Corollary 5] shows that $|2H_s|$ is base point free and the morphism to projective space defined by $|3H_s|$ is birational onto its image. Using this, the algebra $\bigoplus_{n \geq 0} H^0(X_s, \mathcal{L}_s^n)$ is easily calculated using the Riemann–Roch theorem, to give the hyperelliptic case of Condition 2.2(iv) when $|H_s|$ is base point free and the unigonal case of Condition 2.2(iv) when $|H_s|$ has base points. ■

Our next result shows that, on a general fibre of a threefold fibred by K3 surfaces of degree two, Condition 2.2(iii) implies Condition 2.2(iv). In fact, we see that more than this is true: Condition 2.2(iii) implies that the *hyperelliptic case* of Condition 2.2(iv) holds on a general fibre. This condition is crucial to our construction: whilst we expect that a related construction should exist for threefolds fibred by K3 surfaces of degree two with unigonal general fibre, this is a subject for another paper.

Proposition 2.8 *Suppose that $(X, \pi, \mathcal{L})$ is a threefold fibred by K3 surfaces of degree two that satisfies Conditions* 2.2(i), (ii), *and* (iii)*. Then the hyperelliptic case of Condition* 2.2(iv) *holds at a general point $s \in S$.*

Proof Let $s \in S$ be a general point and let X_s denote the fibre over s. Then, exactly as in the proof of Proposition 2.7, we see that it is enough to study the algebra $\bigoplus_{n \geq 0} H^0(X_s, \mathcal{L}_s^n)$, where $\mathcal{L}_s$ denotes the invertible sheaf induced on X_s by $\mathcal{L}$.

By definition, X_s is a K3 surface of degree two with polarization $\mathcal{L}_s$, that may have Du Val singularities. Let $f\colon Y \to X_s$ be a minimal resolution of X_s, where Y is a smooth K3 surface. The inverse image $f^*\mathcal{L}_s$ is an invertible sheaf that defines a polarization of degree two on Y and $H^0(Y, f^*\mathcal{L}_s^n) \cong H^0(X_s, \mathcal{L}_s^n)$ for all $n \geq 0$. Given this, we argue in the same way as in the proof of Proposition 2.7 on Y, whilst noting that Condition 2.2(iii) implies that the linear system defined on Y by $f^*\mathcal{L}_s$ is base point free, so only the hyperelliptic case may occur. ■

The last result of this subsection is the strongest. It shows that if X is smooth and $\pi\colon X \to S$ is semistable, then Condition 2.2(iv) holds at *every* point $s \in S$.

Theorem 2.9 *Suppose that* $(X, \pi, \mathcal{L})$ *is a threefold fibred by K3 surfaces of degree two satisfying Conditions* 2.2(i) *and* (ii)*, and suppose further that* X *is smooth and* $\pi\colon X \to S$ *is semistable (*i.e., *all fibres of* π *are reduced and have simple normal crossings). Then Condition* 2.2(iv) *holds for* $(X, \pi, \mathcal{L})$.

Proof Applying the results of [15, Section 2] locally around the degenerate fibres of π, we see that we may find a threefold fibred by K3 surfaces of degree two $(X', \pi', \mathcal{L}')$ that is birational to $(X, \pi, \mathcal{L})$ over S and has the same relative log canonical algebra, but for which $\mathcal{L}'$ is π'-nef and $\omega_{X'}$ is trivial in a neighbourhood of any fibre (this construction is detailed in full in [16, Section 2.4]). Given this, applying [15, Theorem 3.1] in a neighbourhood of every fibre shows that Condition 2.2(iv) holds for $(X', \pi', \mathcal{L}')$, and so, as $(X', \pi', \mathcal{L}')$ and $(X, \pi, \mathcal{L})$ have the same relative log canonical algebra, it must also hold for $(X, \pi, \mathcal{L})$. ■

In light of this result, we conjecture that Condition 2.2(iv) should hold for any threefold fibred by K3 surfaces of degree two that satisfies Conditions 2.2(i) and (ii). An analogous result is known to hold for genus two curves, this was proved by Mendes-Lopes [12, Theorem 3.7].

3 Structure of the Relative Log Canonical Algebra

We now embark upon our construction of the relative log canonical model of a threefold fibred by K3 surfaces of degree two. In order to do this, we will try to emulate the construction of the relative canonical model for a fibration by genus 2 curves, given originally by Catanese and Pignatelli [2]. As such, the course of our construction will follow [2] quite closely.

We begin by recalling the set up. Fix a nonsingular complex curve S, and let $(X, \pi, \mathcal{L})$ be a threefold fibred by K3 surfaces of degree two over S satisfying Conditions 2.2. Then it follows from Lemma 2.3 that the relative log canonical model X^c of X over S is well defined, and the fibres of $\pi^c\colon X^c \to S$ are classified by Condition 2.2(iv).

By definition, the general fibre X_s of $\pi\colon X \to S$ is a (possibly singular) K3 surface of degree two with polarization $\mathcal{L}_s$ induced by $\mathcal{L}$. Furthermore, by Condition 2.2(iii), $\mathcal{L}_s$ defines a base point free linear system on X_s, so the restriction of the birational map $\phi\colon X - \to X^c$ to X_s is a birational morphism. It follows from Proposition 2.8 that the image of X_s under this morphism is a double cover of $\mathbb{P}^2$ ramified over a (possibly singular) sextic curve.

We are now ready to start our pursuit of an explicit construction for the relative log canonical model of $(X, \pi, \mathcal{L})$. Recall from Section 2 that the relative log canonical algebra of $(X, \pi, \mathcal{L})$ is defined to be the graded algebra

$$\mathcal{R}(X, \pi, \mathcal{L}) = \bigoplus_{n=0}^{\infty} \mathcal{E}_n := \bigoplus_{n=0}^{\infty} \pi_*\big((\omega_X \otimes \mathcal{L})^{[n]}\big),$$

and the relative log canonical model of $(X, \pi, \mathcal{L})$ is $X^c := \mathbf{Proj}_S(\mathcal{R}(X, \pi, \mathcal{L}))$. We will try to find a way to construct $\mathcal{R}(X, \pi, \mathcal{L})$ explicitly, which will in turn allow us to construct the relative log canonical model. First, however, we would like to know more about the structure of $\mathcal{R}(X, \pi, \mathcal{L})$.

First, by definition, the sheaves $(\omega_X \otimes \mathcal{L})^{[n]}$ are reflexive for all $n \geq 0$. So, by [5, Corollary 1.7], the sheaves $\mathcal{E}_n := \pi_*((\omega_X \otimes \mathcal{L})^{[n]})$ are also reflexive and thus, since S is a smooth curve, must be locally free $\mathcal{O}_S$-modules by [5, Corollary 1.4].

Next, since the general fibre of $\pi \colon X \to S$ admits a birational morphism to a double cover of $\mathbb{P}^2$, there exists a birational involution ι on X exchanging the sheets of this cover. We can use this involution to split the relative log canonical algebra into an invariant and an antiinvariant part. Let $U' \subset S$ be an open set. Then $U := \pi^{-1}(U')$ is ι-invariant and ι acts linearly on the space of sections $H^0(U, (\omega_X \otimes \mathcal{L})^{[n]}) = \mathcal{E}_n(U)$, which splits as the direct sum of the $(+1)$-eigenspace and the (-1)-eigenspace.

This allows us to decompose $\mathcal{E}_n$ into $\mathcal{E}_n = \mathcal{E}_n^+ \oplus \mathcal{E}_n^-$, and we can split the relative log canonical algebra as

$$\mathcal{R}(X, \pi, \mathcal{L}) = \mathcal{R}(X, \pi, \mathcal{L})^+ \oplus \mathcal{R}(X, \pi, \mathcal{L})^-.$$

Furthermore, observe that $\mathcal{R}(X, \pi, \mathcal{L})^+$ is a subalgebra of $\mathcal{R}(X, \pi, \mathcal{L})$, and that $\mathcal{R}(X, \pi, \mathcal{L})^-$ is an $\mathcal{R}(X, \pi, \mathcal{L})^+$-module.

This decomposition will prove to be invaluable when we attempt to construct $\mathcal{R}(X, \pi, \mathcal{L})$. We can calculate the ranks of the locally free sheaves $\mathcal{E}_n^+$ and $\mathcal{E}_n^-$ for $n \geq 1$ to get the following table:

n	rank $\mathcal{E}_n^+$	rank $\mathcal{E}_n^-$
even	$\frac{(n+1)(n+2)}{2}$	$\frac{(n-1)(n-2)}{2}$
odd	$\frac{(n-1)(n-2)}{2}$	$\frac{(n+1)(n+2)}{2}$

Furthermore, we know that $\mathcal{E}_0 = \mathcal{E}_0^+ = \mathcal{O}_S$ and $\mathcal{E}_1 = \mathcal{E}_1^-$.

Next, we would like to study the multiplicative structure of $\mathcal{R}(X, \pi, \mathcal{L})$, paying particular attention to how it interacts with the decomposition above. So let $\mu_{n,m} \colon \mathcal{E}_n \otimes \mathcal{E}_m \to \mathcal{E}_{n+m}$ and $\sigma_n \colon \mathrm{Sym}^n(\mathcal{E}_1) \to \mathcal{E}_n$ denote the homomorphisms induced by multiplication in $\mathcal{R}(X, \pi, \mathcal{L})$. The maps σ_n will prove to be particularly useful as, if we can determine more information about them, we should be able to use them to reconstruct the sheaves $\mathcal{E}_n$ from $\mathcal{E}_1$. We have the following lemma.

Lemma 3.1 *The maps $\sigma_n \colon \mathrm{Sym}^n(\mathcal{E}_1) \to \mathcal{E}_n$ are injective for all $n \geq 1$, and their image is contained in $\mathcal{E}_n^+$ when n is even and in $\mathcal{E}_n^-$ when n is odd.*

Proof We begin by showing injectivity. As $\mathrm{Sym}^n(\mathcal{E}_1)$ and $\mathcal{E}_n$ are locally free for all $n > 0$, it is enough to show that σ_n is injective on the fibres of the associated vector bundles. But this follows easily from the explicit description of these fibres given by Condition 2.2(iv). With this in place, the statement on the images of σ_n follows immediately from the fact that $\mathcal{E}_1 = \mathcal{E}_1^-$. ∎

Define $\mathcal{T}_n := \mathrm{coker}(\sigma_n)$ and, using Lemma 3.1, write

$$\mathcal{T}_n^+ := \mathrm{coker}(\mathrm{Sym}^n(\mathcal{E}_1) \to \mathcal{E}_n^+) \quad \text{for } n \text{ even}$$
$$\mathcal{T}_n^- := \mathrm{coker}(\mathrm{Sym}^n(\mathcal{E}_1) \to \mathcal{E}_n^-) \quad \text{for } n \text{ odd.}$$

Then, by Lemma 3.1 again, we can decompose

$$\mathcal{T}_n = \begin{cases} \mathcal{T}_n^+ \oplus \mathcal{E}_n^- & \text{for } n \text{ even,} \\ \mathcal{T}_n^- \oplus \mathcal{E}_n^+ & \text{for } n \text{ odd.} \end{cases}$$

Finally, note that the sheaves $\mathcal{T}_n^{\pm}$ are torsion sheaves.

With this in place, we are ready to begin describing how to construct $\mathcal{R}(X, \pi, \mathcal{L})$.

4 Constructing the Relative Log Canonical Algebra

In this section we detail the explicit construction of the relative log canonical algebra $\mathcal{R}(X, \pi, \mathcal{L})$. In order to do this we follow the construction given by Catanese and Pignatelli in [2]. This will involve constructing a graded subalgebra $\mathcal{A}$ of $\mathcal{R}(X, \pi, \mathcal{L})$ that is simpler to construct explicitly, and that can act as a stepping stone on the way to the construction of $\mathcal{R}(X, \pi, \mathcal{L})$.

Before we start, however, it is convenient to explain some of the geometry that motivates this algebraic approach. As we mentioned before, it follows from Proposition 2.8 that the general fibre of $\pi\colon X \to S$ admits a birational morphism to a double cover of $\mathbb{P}^2$ ramified over a sextic curve. In a similar fashion to Horikawa's [6] construction of models for surfaces fibred by genus two curves, one might consider constructing a naïve model for X as a double cover of a $\mathbb{P}^2$-bundle on S ramified over a divisor that intersects the general fibre in a sextic. However, it is shown in [16, Example 2.1.1] that if $\pi\colon X \to S$ contains any unigonal fibres, then their structure is destroyed by such a construction.

To explain how we will solve this problem, we need to be a little more precise about what is going wrong. Let S_0 be the subset of S over which the hyperelliptic case of Condition 2.2(iv) holds, which is Zariski open by Proposition 2.8. The open set $\pi^{-1}(S_0)$ in X is isomorphic to a double cover of the $\mathbb{P}^2$-bundle on S_0 given by $\mathbb{P}_{S_0}(\mathcal{E}_1)$. The branch divisor is defined using the cokernel of the map $\sigma_3\colon \mathrm{Sym}^3(\mathcal{E}_1) \to \mathcal{E}_3$, which is locally free on S_0. Unfortunately, we find that if we try to extend this definition to all of S, then we lose the local freeness of the cokernel, so the branch divisor is no longer well defined. This can be solved by performing the construction on S_0 and extending to the whole of S using the properness of the Hilbert scheme. However, this process may destroy the structure of the fibres over $S - S_0$. We refer the interested reader to [16, Sections 1.3 and 2.1] for more details.

This problem only occurs on fibres where the cokernel of the map σ_3 is not locally free. As we saw above, this cokernel can be written as $(\mathcal{T}_3^- \oplus \mathcal{E}_3^+)$, where $\mathcal{T}_3^-$ is a torsion sheaf. Furthermore, as we shall see in Lemma 4.1, $\mathcal{T}_3^-$ is supported exactly on the points of S corresponding to the unigonal fibres. This explains why this construction fails on such fibres.

To solve this problem, we will construct an algebra $\mathcal{A}$ that takes better account of the properties of the maps σ_n than $\mathrm{Sym}(\mathcal{E}_1)$ does. Instead of a $\mathbb{P}^2$-bundle, $\mathbf{Proj}_S(\mathcal{A})$ will be a fibration of S by rational surfaces. We can then try to construct $X^c = \mathbf{Proj}_S(\mathcal{R}(X,\pi,\mathcal{L}))$ as a double cover of $\mathbf{Proj}_S(\mathcal{A})$.

However, in order to do this we will need to better understand the maps σ_n. We begin by studying the structure of the cokernels $\mathcal{T}_n$. We have the following analogue of [2, Lemma 4.1].

Lemma 4.1 *Let $(X,\pi,\mathcal{L})$ be a threefold fibred by K3 surfaces of degree two over S that satisfies Conditions 2.2. Then*

(i) *$\mathcal{T}_2 = \mathcal{T}_2^+$ is isomorphic to the structure sheaf of an effective divisor τ, supported on the points of S corresponding to the unigonal fibres of π;*

(ii) *τ determines all the sheaves $\mathcal{T}_n$ as follows:*

$$\mathcal{T}_{2n}^+ \cong \bigoplus_{i=1}^{n} \mathcal{O}_{i\tau}^{\oplus(4(n-i)+1)}, \quad \mathcal{T}_{2n+1}^- \cong \bigoplus_{i=1}^{n} \mathcal{O}_{i\tau}^{\oplus(4(n-i)+3)}.$$

Proof (Following the proof of [2, Lemma 4.1]). Condition 2.2(iv) describes the two possibilities (hyperelliptic and unigonal) for the localization of the relative log canonical algebra $\mathcal{R}(X,\pi,\mathcal{L})_s \otimes_{\mathcal{O}_{S,s}} k(s)$ at any point $s \in S$. In the notation of that condition, we see that x_i are the ι-antiinvariant sections and y and z are ι-invariant. Furthermore, examination of the two cases shows that the cokernels $\mathcal{T}_n$ are locally free away from the points where the unigonal case holds, so the torsion sheaves $\mathcal{T}_{2n}^+$ and $\mathcal{T}_{2n+1}^-$ are supported on these points.

Thus, we may restrict our attention to those points where the unigonal case holds. Around such a point P, the sheaf $\mathcal{E}_2^+$ is locally generated by the sections x_1^2, x_2^2, x_3^2, x_1x_2, x_1x_3, x_2x_3, and y. Furthermore, as $g_6(0,0,0,1) \neq 0$ at such a point, we may assume that the coefficient of y^3 in g_6 is non-zero and, by completing the square in the x_i, we may also assume that

$$g_2(x_i) = x_1^2 - x_2(ax_2 + bx_3),$$

for some $a, b \in \mathbb{C}$ not both zero. Then, by flatness, if t is a uniformizing parameter for $\mathcal{O}_{S,P}$, we can lift the relation g_2 to

$$g_2(t) = x_1^2 - x_2(ax_2 + bx_3) + t\mu(t)y + t\psi(x_i, t).$$

Note that $\mu(t)$ is not identically zero, as x_1, x_2, and x_3 are algebraically independent for $t \neq 0$. Therefore, after changing coordinates in S, we may assume that $\mu(t) = t^{r-1}$ for a suitable integer $r \geq 1$. We call r the *multiplicity* of the point P. Using this and the relation above, the stalk of $\mathcal{T}_2$ at P is the $\mathcal{O}_{S,P}$-module

$$\mathcal{T}_{2,P} = \big(\mathrm{coker}(\sigma_2)\big)_P \cong \mathcal{E}_{2,P}^+/\mathrm{Im}(\sigma_{2,P}) \cong \mathcal{O}_{S,P}/(t^r),$$

generated by the class of y.

Define τ to be the divisor on S given by $\sum_i r_i P_i$, where P_i are the points in S over which the fibres are unigonal and the r_i are the corresponding multiplicities. Then the stalk of $\mathcal{O}_\tau$ at P_i is given by $\mathcal{O}_{\tau,P_i} \cong \mathcal{O}_{S,P_i}/(t^{r_i})$, and thus $\mathcal{T}_2 \cong \mathcal{O}_\tau$. This proves Lemma 4.1(i).

Next, we can also choose a lifting of g_6 of the form

$$g_6(t) = z^2 - g_6'(x_i, y, t).$$

Since g_6 is ι-invariant, $g_6(t)$ must be also, otherwise z would vanish identically on the fibre over P. By flatness, $g_2(t)$ and $g_6(t)$ are all the relations of the stalk of $\mathcal{R}(X, \pi, \mathcal{L})$ at P.

Now consider $\mathcal{T}_{2n+1}^-$. Its stalk at P is given by

$$(\mathcal{T}_{2n+1}^-)_P = (\operatorname{coker}(\sigma_{2n+1}))_P \cong \mathcal{E}_{2n+1,P}^-/\operatorname{Im}(\sigma_{2n+1,P}).$$

Then $\mathcal{E}_{2n+1,P}^-$ is generated by the $2n^2 + 5n + 3$ monomials

$$\left\{ x_1 h_{2n}(x_2, x_3, y),\ h_{2n+1}(x_2, x_3, y) \right\},$$

where $h_i(x_2, x_3, y)$ denotes any monomial of degree i in x_2, x_3, and y. Similarly, $\operatorname{Im}(\sigma_{2n+1,P})$ is generated by the $4n + 3$ monomials

$$\left\{ x_1 h_{2n}(x_2, x_3),\ h_{2n+1}(x_2, x_3) \right\}.$$

So $(\mathcal{T}_{2n+1}^-)_P$ is generated by the $2n^2 + n$ monomials

$$\left\{ x_1 y h_{2n-2}(x_2, x_3, y),\ y h_{2n-1}(x_2, x_3, y) \right\}.$$

These monomials can be listed as

$$\begin{aligned}
\{y^n x_i\} &\text{ generates } \mathcal{O}_{n\tau,P}^{\oplus 3},\\
\{x_1 y^{n-1} h_2(x_2, x_3),\ y^{n-1} h_3(x_2, x_3)\} &\text{ generates } \mathcal{O}_{(n-1)\tau,P}^{\oplus 7},\\
\{x_1 y^{n-2} h_4(x_2, x_3),\ y^{n-2} h_5(x_2, x_3)\} &\text{ generates } \mathcal{O}_{(n-2)\tau,P}^{\oplus 11},\\
&\vdots\\
\{x_1 y h_{2n-2}(x_2, x_3),\ y h_{2n-1}(x_2, x_3)\} &\text{ generates } \mathcal{O}_{\tau,P}^{\oplus(4n-1)},
\end{aligned}$$

and we see that $\mathcal{T}_{2n+1}^- \cong \bigoplus_{i=1}^n \mathcal{O}_{i\tau}^{\oplus(4(n-i)+3)}$.

Finally, a similar calculation gives $\mathcal{T}_{2n}^+ \cong \bigoplus_{i=1}^n \mathcal{O}_{i\tau}^{\oplus(4(n-i)+1)}$; full details may be found in the proof of [16, Lemma 4.2.1]. This completes the proof of Lemma 4.1. ■

Using Lemma 4.1, if we know $\mathcal{T}_2$, we can determine all of the cokernels $\mathcal{T}_n$. So it seems sensible to expect that the structure of $\mathcal{R}(X, \pi, \mathcal{L})$ might be determined by its structure in low degrees. With this in mind, we define the following.

Definition 4.2 Let $\mathcal{A}$ be the graded subalgebra of $\mathcal{R}(X,\pi,\mathcal{L})$ generated by $\mathcal{E}_1$ and $\mathcal{E}_2$. Let $\mathcal{A}_n$ denote its graded part of degree n and write

$$\mathcal{A} = \mathcal{A}_{\text{even}} \oplus \mathcal{A}_{\text{odd}} = \Big(\bigoplus_{n=0}^{\infty} \mathcal{A}_{2n}\Big) \oplus \Big(\bigoplus_{n=0}^{\infty} \mathcal{A}_{2n+1}\Big).$$

We similarly decompose $\mathcal{R}(X,\pi,\mathcal{L}) = \mathcal{R}_{\text{even}} \oplus \mathcal{R}_{\text{odd}}$. Then we have the following analogue of [2, Lemma 4.3]:

Lemma 4.3 *$\mathcal{R}(X,\pi,\mathcal{L})$ is isomorphic to $\mathcal{A} \oplus (\mathcal{A}[-3] \otimes \mathcal{E}_3^+)$ as a graded $\mathcal{A}$-module. Furthermore, $\mathcal{A}_{\text{even}}$ is the ι-invariant part of $\mathcal{R}_{\text{even}}$, and $\mathcal{A}_{\text{odd}}$ is the ι-antiinvariant part of $\mathcal{R}_{\text{odd}}$.*

Proof (Following the proof of [2, Lemma 4.3]). We can unify the hyperelliptic and unigonal cases from Condition 2.2(iv) by writing the localization of the relative log canonical algebra $\mathcal{R}(X,\pi,\mathcal{L})_s \otimes_{\mathcal{O}_{S,s}} k(s)$ to a point over which the fibre is hyperelliptic as

$$\mathbb{C}[x_1,x_2,x_3,y,z]/(y,\ z^2 - f_6(x_i)),$$

where the x_i are ι-antiinvariant of degree 1, and y and z are ι-invariant with degrees 2 and 3 respectively. Then in both cases the stalk of $\mathcal{A}$ is the subalgebra generated by x_1, x_2, x_3, and y, so $\mathcal{A}_{\text{even}}$ is ι-invariant and $\mathcal{A}_{\text{odd}}$ is ι-antiinvariant.

In both cases, locally on S we may write

$$\mathcal{R}(X,\pi,\mathcal{L}) \cong \mathcal{O}_S[x_1,x_2,x_3,y,z]/\big(f_2(t), f_6(t)\big)$$

with $f_6(t) = z^2 - f_6'(x_i,y,t)$, so locally we have

(i) $\mathcal{A} \cong \mathcal{O}_S[x_1,x_2,x_3,y]/(f_2(t))$, and
(ii) $\mathcal{R}(X,\pi,\mathcal{L}) \cong \mathcal{A} \oplus z\mathcal{A}$.

As z is a local generator of $\mathcal{E}_3^+$, this gives $\mathcal{R}(X,\pi,\mathcal{L}) \cong \mathcal{A} \oplus (\mathcal{A}[-3] \otimes \mathcal{E}_3^+)$.

Finally, the statement on the ι-invariant and ι-antiinvariant parts follows from the fact that $\mathcal{A}_{\text{even}}$ is ι-invariant and $\mathcal{R}_{\text{even}} \cong \mathcal{A}_{\text{even}} \oplus z\mathcal{A}_{\text{odd}}$, and $\mathcal{A}_{\text{odd}}$ is ι-antiinvariant and $\mathcal{R}_{\text{odd}} \cong \mathcal{A}_{\text{odd}} \oplus z\mathcal{A}_{\text{even}}$. ∎

We have that $\mathbf{Proj}_S(\mathcal{A})$ is a fibration of S by rational surfaces with natural projection map $\pi_{\mathcal{A}}\colon \mathbf{Proj}_S(\mathcal{A}) \to S$. The inclusion $\mathcal{A} \subset \mathcal{R}(X,\pi,\mathcal{L})$ yields a factorization of the fibration $\pi\colon X \to S$ as

$$X \overset{\phi}{\dashrightarrow} X^c = \mathbf{Proj}_S(\mathcal{R}(X,\pi,\mathcal{L})) \xrightarrow{\psi} \mathbf{Proj}_S(\mathcal{A}) \xrightarrow{\pi_{\mathcal{A}}} S.$$

We will attempt to construct $\mathcal{A}$ first, then use the properties of the map ψ to reconstruct $\mathcal{R}(X,\pi,\mathcal{L})$.

As $\mathcal{A}$ is generated by $\mathcal{E}_1$ and $\mathcal{E}_2$, we might expect that $\mathcal{A}$ can be reconstructed from the locally free sheaves $\mathcal{E}_1$ and $\mathcal{E}_2$ and the map σ_2 that relates them. The next proposition, our analogue of [2, Lemma 4.4], gives us a way to do this.

Proposition 4.4 *With notation as above, there are exact sequences*

$$\mathrm{Sym}^2(\mathcal{E}_1 \wedge \mathcal{E}_1) \otimes \mathrm{Sym}^{n-2}(\mathcal{E}_2) \xrightarrow{i_n} \mathrm{Sym}^n(\mathcal{E}_2) \longrightarrow \mathcal{A}_{2n} \longrightarrow 0 \quad (n \geq 2) \tag{4.1}$$

$$\mathcal{E}_1 \otimes (\mathcal{E}_1 \wedge \mathcal{E}_1) \otimes \mathcal{A}_{2n-2} \xrightarrow{j_n} \mathcal{E}_1 \otimes \mathcal{A}_{2n} \longrightarrow \mathcal{A}_{2n+1} \longrightarrow 0 \quad (n \geq 1), \tag{4.2}$$

where

$$i_n\big((x_i \wedge x_j)(x_k \wedge x_l) \otimes r\big) := \big(\sigma_2(x_ix_k)\sigma_2(x_jx_l) - \sigma_2(x_ix_l)\sigma_2(x_jx_k)\big)\, r,$$
$$j_n\big(l \otimes (x_i \wedge x_j) \otimes r\big) := x_i \otimes \big(\sigma_2(x_jl)r\big) - x_j \otimes (\sigma_2(x_il)r\big).$$

Furthermore, if $n = 2$, then sequence (4.1) is also exact on the left.

Proof (Based upon the proof of [2, Lemma 4.4]). The maps $\mathrm{Sym}^n(\mathcal{E}_2) \to \mathcal{A}_{2n}$ and $\mathcal{E}_1 \otimes \mathcal{A}_{2n} \to \mathcal{A}_{2n+1}$, induced by the ring structure of $\mathcal{A}$, are surjective, because $\mathcal{A}$ is generated in degrees ≤ 2 by definition. Since $\mathcal{E}_n$ and $\mathcal{A}_n$ are locally free, the respective kernels are locally free also. Furthermore, both sequences are complexes, by virtue of associativity and commutativity in $\mathcal{R}(X, \pi, \mathcal{L})$.

It remains to show that (4.1) and (4.2) are exact in the middle. Since the kernels of the maps to $\mathcal{A}_n$ are locally free, it is enough to prove this on the fibres of the associated vector bundles.

We begin with sequence (4.1). Suppose that f is contained in the kernel of the map to $\mathcal{A}_{2n}$. We wish to show that f is also in the image of i_n.

If the fibre of $\pi\colon X \to S$ over the point under consideration is hyperelliptic, then $\mathcal{E}_2$ is generated by the images $\sigma_2(x_ix_j)$ for all $i, j \in \{1, 2, 3\}$. Express f in terms of these generators. Then perform the following algorithm on f:

(i) If any monomial of f contains a factor of $\sigma_2(x_1x_i)\sigma_2(x_1x_j)$, with $i, j \in \{2, 3\}$, replace this factor with $\sigma_2(x_1^2)\sigma_2(x_ix_j)$. Repeat this step until it terminates.
(ii) If any monomial of f contains a factor of $\sigma_2(x_1x_3)\sigma_2(x_2x_i)$, with $i \in \{2, 3\}$, replace this factor with $\sigma_2(x_1x_2)\sigma_2(x_3x_i)$.
(iii) If any monomial of f contains a factor of $\sigma_2(x_2x_3)\sigma_2(x_2x_3)$, replace this factor with $\sigma_2(x_2^2)\sigma_2(x_3^2)$. Repeat this step until it terminates.
(iv) Collect like terms in f and simplify.

Call the result f'. Note that the kernel of the map to $\mathcal{A}_{2n}$ is closed under these operations, so f' is in this kernel. Furthermore, $\mathrm{Im}(i_n)$ is also closed under these operations and their inverses, so $f \in \mathrm{Im}(i_n)$ if and only if $f' \in \mathrm{Im}(i_n)$.

Now, any monomial in f' must have the form

$$\sigma_2(x_1^2)^{n_{1,1}}\sigma_2(x_1x_2)^{n_{1,2}}\sigma_2(x_2^2)^{n_{2,2}}\sigma_2(x_2x_3)^{n_{2,3}}\sigma_2(x_3^2)^{n_{3,3}}\sigma_2(x_1x_3)^{n_{1,3}},$$

with $n_{1,2}, n_{2,3}, n_{1,3} \in \{0, 1\}$ and $n_{1,3} = 1$ only if $n_{1,2} = n_{2,2} = n_{2,3} = 0$. However, under the map to $\mathcal{A}_{2n}$ there are no relations between monomials of this form, so, since f' is in the kernel of this map, f' must be the zero polynomial. But $0 \in \mathrm{Im}(i_n)$, so $f \in \mathrm{Im}(i_n)$ also.

The proof for points corresponding to unigonal fibres is very similar. This time, $\mathcal{E}_2$ is generated by y and the images $\sigma_2(x_ix_j)$ for all $i, j \in \{1, 2, 3\}$ with the exception of $(i, j) = (1, 1)$. We perform the same set of operations on f, but with step (i) replaced by the following.

(i′) If any monomial of f contains a factor of $\sigma_2(x_1x_i)\sigma_2(x_1x_j)$, with $i, j \in \{2, 3\}$, replace this factor with $(a\sigma_2(x_2^2) + b\sigma_2(x_2x_3))\sigma_2(x_ix_j)$, where the degree 2 relation in the unigonal fibre is given by $q(x_1, x_2, x_3) = x_1^2 - x_2(ax_2 + bx_3) = 0$ for some $a, b \in \mathbb{C}$. Repeat this step until it terminates.

Any monomial in the resulting f' must have the form

$$y^{n_0}\sigma_2(x_1x_2)^{n_{1,2}}\sigma_2(x_2^2)^{n_{2,2}}\sigma_2(x_2x_3)^{n_{2,3}}\sigma_2(x_3^2)^{n_{3,3}}\sigma_2(x_1x_3)^{n_{1,3}},$$

with $n_{1,2}, n_{2,3}, n_{1,3} \in \{0, 1\}$ and $n_{1,3} = 1$ only if $n_{1,2} = n_{2,2} = n_{2,3} = 0$. With this, the remainder of the proof proceeds exactly as in the hyperelliptic case.

It remains to show that this sequence is exact on the left when $n = 2$. This will again follow from the corresponding statement on the fibres of the associated vector bundles. As the map induced by i_2 on the fibres of the associated vector bundles is linear, in order to prove that it is injective, we need only show that the dimension (as a complex vector space) of its domain is equal to that of its image. A simple calculation yields that the dimension of a fibre of $\mathcal{A}_4$ is 21, and the dimension of a fibre of $\mathrm{Sym}^2(\mathcal{E}_2)$ is 15. So, as sequence (4.1) is exact in the middle, the image of i_2 has dimension 6. But a fibre of $\mathrm{Sym}^2(\mathcal{E}_1 \wedge \mathcal{E}_1)$ also has dimension 6. Hence, i_2 is injective, and sequence (4.1) is exact on the left when $n = 2$.

Next we consider sequence (4.2). Given f contained in the kernel of the map to $\mathcal{A}_{2n+1}$, we wish to show that f is contained in the image of j_n.

First consider the case where the fibre of $\pi\colon X \to S$ over the point under consideration is hyperelliptic. Then the fibre of the $\mathcal{O}_S$-algebra $\mathcal{A}$ over this point is isomorphic to $\mathbb{C}[x_1, x_2, x_3]$. Since the x_i form a basis for the fibre of $\mathcal{E}_1$, we may write f as

$$f = x_1 \otimes f_1 + x_2 \otimes f_2 + x_3 \otimes f_3$$

for some $f_1, f_2, f_3 \in \mathbb{C}[x_1, x_2, x_3]$ of degree $2n$. This maps to $x_1f_1 + x_2f_2 + x_3f_3$ under the map to $\mathcal{A}_{2n+1}$, so the condition that f is in the kernel of this map is equivalent to $x_1f_1 + x_2f_2 + x_3f_3 = 0$.

Using this equation, we have $x_1 | (x_2f_2 + x_3f_3)$. This implies that f_2 and f_3 have the form

$$f_2 = x_1\, r_2(x_1, x_2, x_3) + x_3\, s_{23}(x_2, x_3),$$
$$f_3 = x_1\, r_3(x_1, x_2, x_3) - x_2\, s_{23}(x_2, x_3),$$

for $r_i, s_{ij} \in \mathbb{C}[x_1, x_2, x_3]$ of degree $(2n - 1)$. Repeating this process for x_2 and x_3, we get

$$f_1 = x_2x_3\, r_1(x_1, x_2, x_3) + x_2\, s_{12}(x_1, x_2) + x_3\, s_{13}(x_1, x_3),$$
$$f_2 = x_1x_3\, r_2(x_1, x_2, x_3) - x_1\, s_{12}(x_1, x_2) + x_3\, s_{23}(x_2, x_3),$$
$$f_3 = x_1x_2\, r_3(x_1, x_2, x_3) - x_1\, s_{13}(x_1, x_3) - x_2\, s_{23}(x_2, x_3),$$

for $r_i, s_{ij} \in \mathbb{C}[x_1, x_2, x_3]$ of degrees $(2n - 2)$ and $(2n - 1)$ respectively. Furthermore, as f is in the kernel of the map to $\mathcal{A}_{2n+1}$, we must have $r_1 + r_2 + r_3 = 0$.

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Let $l_{ij}(x_i, x_j)$ be any linear factor of $s_{ij}(x_i, x_j)$. Using this, we can express $s_{ij}(x_i, x_j) = l_{ij}(x_i, x_j)\, s'_{ij}(x_i, x_j)$ for $s'_{ij} \in \mathbb{C}[x_1, x_2, x_3]$ of degree $(2n-2)$. Then we have

$$\begin{aligned} f &= x_1 \otimes (x_2x_3\, r_1 + x_2\, l_{12}\, s'_{12} + x_3\, l_{13}\, s'_{13}) + x_2 \otimes (x_1x_3\, r_2 - x_1\, l_{12}\, s'_{12} + x_3\, l_{23}\, s'_{23}) \\ &\quad + x_3 \otimes (-x_1x_2\, r_1 - x_1x_2\, r_2 - x_1\, l_{13}\, s'_{13} - x_2\, l_{23}\, s'_{23}) \\ &= j_n\big(x_2 \otimes (x_1 \wedge x_3) \otimes r_1 + x_1 \otimes (x_2 \wedge x_3) \otimes r_2 + l_{12} \otimes (x_1 \wedge x_2) \otimes s'_{12} \\ &\quad + l_{13} \otimes (x_1 \wedge x_3) \otimes s'_{13} + l_{23} \otimes (x_2 \wedge x_3) \otimes s'_{23}\big). \end{aligned}$$

Hence $f \in \operatorname{Im}(j_n)$, and sequence (4.2) is exact in the middle.

Finally, we have to show that sequence (4.2) is exact in the middle when the fibre of $\pi\colon X \to S$ over the point under consideration is unigonal. In this case, the fibre of the $\mathcal{O}_S$-algebra $\mathcal{A}$ over this point is isomorphic to

$$\frac{\mathbb{C}[x_1, x_2, x_3, y]}{(x_1^2 - x_2(ax_2 + bx_3))} = \left(\frac{\mathbb{C}[x_1, x_2, x_3]}{(x_1^2 - x_2(ax_2 + bx_3))}\right)[y]$$

for some $a, b \in \mathbb{C}$.

Once again, let f denote an element of the kernel of the map to $\mathcal{A}_{2n+1}$. Then, since the x_i form a basis for $\mathcal{E}_i$, we may write $f = x_1 \otimes f_1 + x_2 \otimes f_2 + x_3 \otimes f_3$, and, using the above characterization of the fibres of $\mathcal{A}$, without loss of generality we can replace f_1, f_2, and f_3 with their coefficients in $\mathbb{C}[x_1, x_2, x_3]/(x_1^2 - x_2(ax_2 + bx_3))$. Then the remainder of the proof proceeds much as in the hyperelliptic case. Full details may be found in the proof of [16, Proposition 4.2.4]. ■

The exact sequences (4.1) and (4.2) in Proposition 4.4 allow us to describe $\mathcal{A}_{\text{even}}$ as a quotient algebra of $\operatorname{Sym}(\mathcal{E}_2)$ and $\mathcal{A}_{\text{odd}}$ as an $\mathcal{A}_{\text{even}}$-module. The multiplication map $\mathcal{A}_{\text{odd}} \times \mathcal{A}_{\text{odd}} \to \mathcal{A}_{\text{even}}$ is induced by the composition

$$\mathcal{E}_1 \otimes \mathcal{E}_1 \xrightarrow{\mu_{1,1}} \operatorname{Sym}^2(\mathcal{E}_1) \xrightarrow{\sigma_2} \mathcal{E}_2.$$

Thus, $\mathcal{A}$ is completely determined as an $\mathcal{O}_S$-algebra by the locally free sheaves $\mathcal{E}_1$ and $\mathcal{E}_2$ and the map $\sigma_2\colon \operatorname{Sym}^2(\mathcal{E}_1) \to \mathcal{E}_2$.

The structure of $\mathcal{A}_{\text{even}}$ as a quotient algebra of $\operatorname{Sym}(\mathcal{E}_2)$ gives a Veronese embedding of $\mathbf{Proj}_S(\mathcal{A})$ into $\mathbb{P}_S(\mathcal{E}_2)$ that commutes with the projection to S. The projective space bundle $\mathbb{P}_S(\mathcal{E}_2)$ comes equipped with natural invertible sheaves $\mathcal{O}(n)$ for all $n \in \mathbb{Z}$, which induce invertible sheaves $\mathcal{O}_{\mathbf{Proj}_S(\mathcal{A})}(2n)$ on $\mathbf{Proj}_S(\mathcal{A})$.

Now that we have a way to construct $\mathcal{A}$, we would like to find a way to reconstruct $\mathcal{R}(X, \pi, \mathcal{L})$ from it. By Lemma 4.3, we can already construct $\mathcal{R}(X, \pi, \mathcal{L})$ as an $\mathcal{A}$-module. However, we need to give $\mathcal{R}(X, \pi, \mathcal{L})$ a multiplicative structure to make it into an $\mathcal{A}$-algebra. In order to do this, we need to determine the multiplication map from $\mathcal{E}_3^+ \otimes \mathcal{E}_3^+$ to $\mathcal{E}_6$. By Lemma 4.3, this multiplication map has image contained in $\mathcal{A}_6$. So the ring structure on $\mathcal{R}(X, \pi, \mathcal{L})$ induces a map $\beta\colon (\mathcal{E}_3^+)^2 \to \mathcal{A}_6$.

To determine β, we will study the map $\psi\colon X^c \to \mathbf{Proj}_S(\mathcal{A})$. First, however, we need a definition.

Definition 4.5 Let P be a point in the support of τ. The fibre of $\mathbf{Proj}_S(\mathcal{A})$ over P is of the form

$$\{x_1^2 - x_2(ax_2 + bx_3) = 0\} \subset \mathbb{P}_{(1,1,1,2)}[x_1, x_2, x_3, y].$$

This is a cone over the rational normal curve of degree 4 and is singular at the point $(0:0:0:1)$.

Taking all such singular points associated with the points of $\operatorname{Supp}(\tau)$, we get a subset of $\mathbf{Proj}_S(\mathcal{A})$ that we will denote by $\mathcal{P}$. Note that the projection onto S maps $\mathcal{P}$ bijectively onto $\operatorname{Supp}(\tau)$.

Then we have the following analogue of [2, Theorem 4.7].

Proposition 4.6 *$X^c = \mathbf{Proj}_S(\mathcal{R}(X,\pi,\mathcal{L}))$ is a double cover of $\mathbf{Proj}_S(\mathcal{A})$, with branch locus consisting of the set of isolated points $\mathcal{P}$ together with the divisor $B_{\mathcal{A}}$ in the linear system $|\mathcal{O}_{\mathbf{Proj}_S(\mathcal{A})}(6) \otimes \pi_{\mathcal{A}}^*(\mathcal{E}_3^+)^{-2}|$ determined by β ($B_{\mathcal{A}}$ is thus disjoint from $\mathcal{P}$).*

Proof (Following the proof of [2, Theorem 4.7]). Note first that $\psi\colon X^c \to \mathbf{Proj}_S(\mathcal{A})$ is a double cover by Lemma 4.3. It just remains to calculate the branch locus of ψ.

Since the question is local on S, we may use the same method as in the proof of Lemma 4.3 and restrict our attention to an affine open set U over which X^c is isomorphic to the subscheme of $\mathbb{P}_{(1,1,1,2,3)}[x_1, x_2, x_3, y, z] \times U$ defined by the equations

$$f_2(x_1, x_2, x_3, y; t) = 0, \qquad z^2 = f_6(x_1, x_2, x_3, y; t),$$

where t is a parameter on U. Furthermore, we note that if the x_i's and y simultaneously vanish, then $z = 0$ also, which is impossible.

At a point where $x_i \neq 0$ for some i, we can localize both equations by dividing by x_i^2, respectively by x_i^6. Then $z = 0$ is the ramification divisor, and $f_6 = 0$ is the branch locus. This equation defines exactly the divisor $B_{\mathcal{A}} \subset \mathbf{Proj}_S(\mathcal{A})$.

At a point where $x_1 = x_2 = x_3 = 0$, we may assume that $y = 1$, and we have a point of $\mathcal{P}$. Note that, since the points $(0:0:0:1:a)$ and $(0:0:0:1:-a)$ are identified in $\mathbb{P}(1,1,1,2,3)$ for any $a \in \mathbb{C}$, this point must be a branch point of ψ. Furthermore, by Condition 2.2(iv), f_6 cannot vanish at such a point, so $B_{\mathcal{A}}$ is disjoint from $\mathcal{P}$. ■

Putting the results of this section together, we can list the data required to construct the relative log canonical model of a threefold fibred by K3 surfaces of degree two.

Definition 4.7 Let $(X,\pi,\mathcal{L})$ be a threefold fibred by K3 surfaces of degree two over a nonsingular curve S that satisfies Conditions 2.2. Then define the *associated* 5-*tuple of* $(X,\pi,\mathcal{L})$ over S, denoted $(\mathcal{E}_1, \tau, \xi, \mathcal{E}_3^+, \beta)$, as follows:

- $\mathcal{E}_1 = \pi_*(\omega_X \otimes \mathcal{L})$;
- τ is the effective divisor on S whose structure sheaf is isomorphic to $\mathcal{T}_2$;
- $\xi \in \operatorname{Ext}^1_{\mathcal{O}_S}(\mathcal{O}_\tau, \operatorname{Sym}^2(\mathcal{E}_1))/\operatorname{Aut}_{\mathcal{O}_S}(\mathcal{O}_\tau)$ is the isomorphism class of the pair $(\mathcal{E}_2, \sigma_2)$ in the sequence

$$0 \longrightarrow \operatorname{Sym}^2(\mathcal{E}_1) \xrightarrow{\sigma_2} \mathcal{E}_2 \longrightarrow \mathcal{O}_\tau \longrightarrow 0;$$

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- $\mathcal{E}_3^+$ is the ι-invariant part of $\pi_*((\omega_X \otimes \mathcal{L})^{[3]})$;
- $\beta \in \mathbb{P}(H^0(S, \mathcal{A}_6 \otimes (\mathcal{E}_3^+)^{-2})) \cong |\mathcal{O}_{\mathbf{Proj}_S(\mathcal{A})}(6) \otimes \pi_{\mathcal{A}}^*(\mathcal{E}_3^+)^{-2}|$ is the class of a section with associated divisor $B_{\mathcal{A}}$.

Remark 4.8 We need one more piece of data than Catanese and Pignatelli [2]: the line bundle $\mathcal{E}_3^+$. This is because a surface fibred by genus two curves is naturally polarized by its canonical divisor, but we require a separate polarization sheaf $\mathcal{L}$. The extra piece of data in our case is needed to determine this polarization sheaf.

5 A Generality Result

In this section we will give a method, based upon the results of Section 4, to construct relative log canonical models of threefolds fibred by K3 surfaces of degree two, and prove a result about the generality of this construction.

Fix a nonsingular complex curve S. We begin with a 5-tuple $(\mathcal{E}_1, \tau, \xi, \mathcal{E}_3^+, \beta)$ of data on S, defined by:

- $\mathcal{E}_1$ is a rank 3 vector bundle on S;
- τ is an effective divisor on S;
- $\xi \in \mathrm{Ext}^1_{\mathcal{O}_S}(\mathcal{O}_\tau, \mathrm{Sym}^2(\mathcal{E}_1))/\mathrm{Aut}_{\mathcal{O}_S}(\mathcal{O}_\tau)$ yields a pair $(\mathcal{E}_2, \sigma_2)$ consisting of a vector bundle $\mathcal{E}_2$ on S and a map $\sigma_2 \colon \mathrm{Sym}^2(\mathcal{E}_1) \to \mathcal{E}_2$;
- $\mathcal{E}_3^+$ is a line bundle on S;
- $\beta \in \mathbb{P}(H^0(S, \mathcal{A}_6 \otimes (\mathcal{E}_3^+)^{-2}))$, where $\mathcal{A}_6$ is defined using $\mathcal{E}_1$, $\mathcal{E}_2$, σ_2 and the exact sequences of Proposition 4.4.

Given this data, we begin by constructing a sheaf of $\mathcal{O}_S$-algebras $\mathcal{A}$ using the exact sequences of Proposition 4.4. Then we may define a second sheaf of $\mathcal{O}_S$-algebras

$$\mathcal{R} := \mathcal{A} \oplus \left(\mathcal{A}[-3] \otimes \mathcal{E}_3^+\right),$$

with multiplicative structure induced by $\mathcal{A}$ and the map $(\mathcal{E}_3^+)^2 \to \mathcal{A}_6$ defined by β.

Definition 5.1 We say that a 5-tuple $(\mathcal{E}_1, \tau, \xi, \mathcal{E}_3^+, \beta)$ is *admissible* if the sheaf of algebras $\mathcal{R}$ constructed from it satisfies the following conditions:

(i) Let $B_{\mathcal{A}}$ be the divisor of β on $\mathbf{Proj}_S(\mathcal{A})$; then $B_{\mathcal{A}}$ does not contain any point of the set $\mathcal{P}$ from Definition 4.5.

(ii) $\mathbf{Proj}_S(\mathcal{R})$ has at worst canonical singularities.

We have the following generality result, an analogue of [2, Theorem 4.13].

Theorem 5.2 *Fix a nonsingular complex curve S. Let $(X, \pi, \mathcal{L})$ be a threefold fibred by K3 surfaces of degree two over S that satisfies Conditions 2.2. Then the associated 5-tuple of $(X, \pi, \mathcal{L})$ over S is admissible.*

Conversely, let $\mathcal{R}$ be a sheaf of $\mathcal{O}_S$-algebras defined by an admissible 5-tuple $(\mathcal{E}_1, \tau, \xi, \mathcal{E}_3^+, \beta)$. Let $X = \mathbf{Proj}_S(\mathcal{R})$ and $\pi \colon X \to S$ be the natural projection. Then there is a canonically defined polarization sheaf $\mathcal{L}$ on X that makes $(X, \pi, \mathcal{L})$ into a threefold fibred by K3 surfaces of degree two that satisfies Conditions 2.2, and, furthermore, $\mathbf{Proj}_S(\mathcal{R})$ is the relative log canonical algebra of $(X, \pi, \mathcal{L})$ over S, and $(\mathcal{E}_1, \tau, \xi, \mathcal{E}_3^+, \beta)$ is its associated 5-tuple.

Proof We begin by letting $(X, \pi, \mathcal{L})$ be a threefold fibred by K3 surfaces of degree two over S that satisfies Conditions 2.2. We want to show that the associated 5-tuple of $(X, \pi, \mathcal{L})$ is admissible. Note that condition (i) in the definition of admissible follows immediately from Proposition 4.6.

It remains to show that $X^c := \mathbf{Proj}_S(\mathcal{R})$ has at worst canonical singularities. This will follow from Condition 2.2(ii), that the pair (X, H) is canonical. By the proof of Lemma 2.3, the relative log canonical models of $(X, \pi, \mathcal{L})$ and $(X, \pi, \mathcal{O}_X(H))$ are isomorphic, so it is enough to prove that the relative log canonical model $\widehat{X}^c$ of $(X, \pi, \mathcal{O}_X(H))$ has at worst canonical singularities.

Let $\widehat{\phi}\colon X \dashrightarrow \widehat{X}^c$ denote the natural birational map. By Condition 2.2(i) H is irreducible, so no components of H can be contracted by $\widehat{\phi}$. Thus, by [10, Proposition 3.51] we see that $\operatorname{discrep}(\widehat{X}^c, \widehat{\phi}_* H) \geq \operatorname{discrep}(X, H) \geq 0$, so the log pair $(\widehat{X}^c, \widehat{\phi}_* H)$ is canonical. But $\widehat{\phi}_* H$ is effective on $\widehat{X}^c$, so, by [10, Corollary 2.35], the log pair $(\widehat{X}^c, 0)$ is also canonical and $\widehat{X}^c$ has at worst canonical singularities. This proves condition (ii) in the definition of admissible.

Next we prove the converse statement. Let $\mathcal{R}$ be a sheaf of $\mathcal{O}_S$-algebras defined by an admissible 5-tuple $(\mathcal{E}_1, \tau, \xi, \mathcal{E}_3^+, \beta)$. Define $X := \mathbf{Proj}_S(\mathcal{R})$ and let $\pi\colon X \to S$ denote the natural projection. As in the proof of Lemma 4.3, over an affine open set $U \subset S$ we can view X as a normal subvariety in $\mathbb{P}_{(1,1,1,2,3)}[x_1, x_2, x_3, y, z] \times U$ defined by equations

$$f_2(x_1, x_2, x_3, y; t) = 0, \qquad z^2 = f_6(x_1, x_2, x_3, y; t),$$

where t is a parameter on U.

By this local description, it is clear that X does not intersect the singular curve $(0:0:0:0:1) \times U \subset \mathbb{P}(1,1,1,2,3) \times U$. Furthermore, the divisor $B_{\mathcal{A}}$ defining the locus $f_6(x_1, x_2, x_3, y; t) = 0$ on $\mathbf{Proj}_S(\mathcal{A})$ does not intersect the set $\mathcal{P}$ by condition (ii) in the definition of admissible, so X does not intersect the singular curve $(0:0:0:1:0) \times U \subset \mathbb{P}(1,1,1,2,3) \times U$ either.

Thus, we see that X does not intersect the singular locus in $\mathbb{P}(1,1,1,2,3) \times U$ and that $\omega_X|_{\pi^{-1}(U)}$ is trivial. Therefore, X is Gorenstein, and over each such open set the sheaf $\mathcal{O}(1)$ induced on X by the weighted projective space structure is invertible. These invertible sheaves glue to give an invertible sheaf $\mathcal{O}_X(1)$ on X.

Next consider the morphism $\pi\colon X \to S$. By construction π is flat, projective and surjective. Furthermore, by the local description above, the general fibre of π is a complete intersection of type $(2,6)$ in $\mathbb{P}(1,1,1,2,3)$, which is a K3 surface by adjunction. As X has at worst canonical singularities, by [10, Lemma 5.17] we see that this K3 surface has at worst Du Val singularities.

Now define $\mathcal{L} := \mathcal{O}_X(1) \otimes \omega_X^{-1}$. Then $\mathcal{L}$ is invertible, and the local description above shows that $\mathcal{L}$ induces an ample invertible sheaf with self-intersection number two on a general fibre of π. Therefore $(X, \pi, \mathcal{L})$ is a threefold fibred by K3 surfaces of degree two.

We next show that Conditions 2.2 hold for $(X, \pi, \mathcal{L})$.

Lemma 5.3 *Define a threefold fibred by K3 surfaces of degree two $(X, \pi, \mathcal{L})$ as above. Then $(X, \pi, \mathcal{L})$ satisfies Conditions 2.2.*

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Proof We begin by proving Conditions 2.2(iii) and (iv). Let $U \subset S$ be an affine open set. Then as in the proof of Lemma 4.3, we can view $\pi^{-1}(U)$ as a complete intersection

$$\pi^{-1}(U) \cong \left\{ f_2(t) = f_6(t) = 0 \right\} \subset \mathbb{P}(1,1,1,2,3) \times U.$$

As $\omega_X|_{\pi^{-1}(U)}$ is trivial, the restriction of $\mathcal{L}$ to $\pi^{-1}(U)$ is just the sheaf induced from $\mathcal{O}(1)$ on $\mathbb{P}(1,1,1,2,3) \times U$. From this, it easily follows that Conditions 2.2(iii) and (iv) hold for $(X, \pi, \mathcal{L})$.

It remains to prove Conditions 2.2(i) and 2.2(ii). In order to do this we start by defining the divisor H. Choose an ample invertible sheaf $\mathcal{N}$ on S. Then for some $m > 0$, the sheaf $\pi_*\mathcal{L} \otimes \mathcal{N}^m$ is generated by its global sections. Let H denote a general member of the linear system $|\mathcal{L} \otimes \pi^*\mathcal{N}^m|$.

We wish to show that H is a prime divisor that is flat over S, and that (X, H) is canonical. In order to do this, we show that H may be chosen to avoid the worst singularities of Y. Specifically, we want to avoid singularities that are not compound Du Val [10, Definition 5.32].

We start by examining the linear system $|\mathcal{L} \otimes \pi^*\mathcal{N}^m|$ in which H moves. Note that as $\pi_*\mathcal{L} \otimes \mathcal{N}^m$ is generated by its global sections, for any affine open set $U \subset S$ the sections in $H^0(X, \mathcal{L} \otimes \pi^*\mathcal{N}^m)$ generate $H^0(\pi^{-1}(U), \mathcal{L} \otimes \pi^*\mathcal{N}^m)$ as an $\mathcal{O}_{\pi^{-1}(U)}$-module, so we may study this linear system locally over S.

So let $U \subset S$ be an affine open set. As $\mathcal{N}|_U \cong \mathcal{O}_U$, we have the inverse image $\pi^*\mathcal{N}^m|_{\pi^{-1}(U)} \cong \mathcal{O}_{\pi^{-1}(U)}$ and, since $\omega_X|_{\pi^{-1}(U)}$ is trivial, the restriction of $\mathcal{L} \otimes \pi^*\mathcal{N}^m$ to $\pi^{-1}(U)$ is just the sheaf induced from $\mathcal{O}(1)$ on $\mathbb{P}(1,1,1,2,3) \times U$.

The sheaf $\mathcal{O}(1)$ defines a linear system on $\mathbb{P}(1,1,1,2,3) \times U$ that is base point free outside of the locus $(0:0:0:y:z) \times U$, so the induced linear system on X is base point free outside of its intersection with this locus, consisting of precisely the points over the set $\mathcal{P}$ defined in 4.5. Thus, as $H^0(\pi^{-1}(U), \mathcal{L} \otimes \pi^*\mathcal{N}^m)$ is generated as an $\mathcal{O}_{\pi^{-1}(U)}$-module by the sections in $H^0(X, \mathcal{L} \otimes \pi^*\mathcal{N}^m)$, we see that the linear system $|\mathcal{L} \otimes \pi^*\mathcal{N}^m|$ has no base points or fixed components on X outside of the points over the set $\mathcal{P}$. In particular, we see that the linear system induced by $|H|$ on a general fibre of π is base point free.

As X has at worst canonical singularities, by [10, Corollary 5.40] all but finitely many of the singular points of X are compound Du Val. So, apart from the points lying over the set $\mathcal{P}$, we may assume that the only singularities of X lying on H are compound Du Val. Furthermore, by Bertini's theorem we may assume that H is reduced, irreducible, and nonsingular outside of the singular points of X and the points lying over $\mathcal{P}$. In particular H cannot contain any components of fibres, so it is horizontal and thus flat over S. This proves that H is flat over S and that $(X, \pi, \mathcal{L})$ satisfies Condition 2.2(i) (with $\mathcal{M} = \mathcal{N}^{-m}$).

The last step in the proof of Lemma 5.3 is to show that, with H chosen as above, the log pair (X, H) is canonical. This will follow from [10, Theorem 5.34] if we can show that all of the singularities in H are rational double points. By the argument above, these singularities arise from compound Du Val points and points lying over $\mathcal{P}$. At a compound Du Val point, the singularity in H is a rational double point by definition. So it just remains to classify the singularities lying over the points of $\mathcal{P}$.

By the proof of Lemma 4.1, after a change of coordinates locally we can write $\mathbf{Proj}_S(\mathcal{A})$ as

$$\{f_2(t) = 0\} \subset \mathbb{P}_{(1,1,1,2)}[x_1, x_2, x_3, y] \times U,$$

where

$$f_2(t) = x_1^2 - x_2(ax_2 + bx_3) + t^r y + t\psi(x_i, t)$$

for some $a, b \in \mathbb{C}$ that are not both zero and t a local parameter on the affine open set $U \subset S$. The weighted projective space structure induces a divisorial sheaf $\mathcal{O}_{\mathbf{Proj}_S(\mathcal{A})}(1)$ locally on $\mathbf{Proj}_S(\mathcal{A})$, a general section of which defines a Weil divisor that has a rational double point singularity of type A_{2r+1} at the point $(0:0:0:1;0)$. As $B_{\mathcal{A}}$ does not contain any point of $\mathcal{P}$, around the point $(0:0:0:1;0)$ we have that X is a cyclic double cover of $\mathbf{Proj}_S(\mathcal{A})$ ramified over the point $(0:0:0:1;0)$. Thus a divisor defined by a general section of $\mathcal{O}_X(1)$ is a cyclic double cover of a divisor defined by a general section of $\mathcal{O}_{\mathbf{Proj}_S(\mathcal{A})}(1)$ ramified over the singularity. Therefore, by [10, Theorem 5.43], the general section of $\mathcal{O}_X(1)$ has a rational double point singularity of type A_r.

Thus, we may assume that the only singularities occurring in H are rational double points, so by [10, Theorem 5.34] the log pair (X, H) is canonical. Therefore Condition 2.2(ii) holds for $(X, \pi, \mathcal{L})$. This completes the proof of Lemma 5.3. ∎

To complete the proof of Theorem 5.2, we just have to show that $\mathbf{Proj}_S(\mathcal{R})$ is the relative log canonical model of $(X, \pi, \mathcal{L})$ over S and $(\mathcal{E}_1, \tau, \xi, \mathcal{E}_3^+, \beta)$ is its associated 5-tuple. This will follow if we can show that $\mathcal{R}$ is the relative log canonical algebra of $(X, \pi, \mathcal{L})$.

Note that

$$\pi_*((\omega_X \otimes \mathcal{L})^{[n]}) \cong \pi_*((\omega_X \otimes \mathcal{O}_X(1) \otimes \omega_X^{-1})^{[n]}) \cong \pi_*(\mathcal{O}_X(1)^n)$$

for all $n > 0$. But this implies that the relative log canonical algebra of $(X, \pi, \mathcal{L})$ is $\mathcal{R}$, as required. This completes the proof of Theorem 5.2. ∎

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