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OPTIMAL PATHS ON THE SPACE–TIME SINR RANDOM GRAPH

FRANÇOIS BACCELLI,* ** *INRIA/ENS*
BARTŁOMIEJ BŁASZCZYSZYN,* *** *INRIA/ENS and University of Wrocław*
MIR-OMID HAJI-MIRSADEGHI,**** *Sharif University of Technology and INRIA/ENS*

Abstract

We analyze a class of signal-to-interference-and-noise-ratio (SINR) random graphs. These random graphs arise in the modeling packet transmissions in wireless networks. In contrast to previous studies on SINR graphs, we consider both a space and a time dimension. The spatial aspect originates from the random locations of the network nodes in the Euclidean plane. The time aspect stems from the random transmission policy followed by each network node and from the time variations of the wireless channel characteristics. The combination of these random space and time aspects leads to fluctuations of the SINR experienced by the wireless channels, which in turn determine the progression of packets in space and time in such a network. In this paper we study optimal paths in such wireless networks in terms of first passage percolation on this random graph. We establish both 'positive' and 'negative' results on the associated time constant. The latter determines the asymptotics of the minimum delay required by a packet to progress from a source node to a destination node when the Euclidean distance between the two tends to ∞. The main negative result states that this time constant is infinite on the random graph associated with a Poisson point process under natural assumptions on the wireless channels. The main positive result states that, when adding a periodic node infrastructure of arbitrarily small intensity to the Poisson point process, the time constant is positive and finite.

Keywords: Poisson point process; random graph; first passage percolation; shot noise process; SINR

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1. Introduction

There is a rich literature on random graphs generated over a random point process. These graphs are often motivated by physical, biological, or social networks. Many interesting large-scale properties of these networks related to connectivity have been studied in terms of the percolation of the associated graphs. An early example of such a study can be found in [12], where the connectivity of large networks was defined as the supercritical phase in what is today

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* Postal address: INRIA, 23 Avenue d'Italie, CS 81321, 75214 Paris Cedex 13, France.
** Email address: francois.baccelli@ens.fr
*** Email address: bartek.blaszczyszyn@ens.fr
**** Postal address: Department of Mathematical Sciences, Sharif University of Technology, Azadi Av., Tehran, Iran.
Email address: mirsadeghi@mehr.sharif.edu
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called the continuum (Boolean) percolation model. More recently, a random SINR graph model for wireless networks was studied with the same perspective in [9] and [10].

The routeing, and, more precisely, the speed of delivery of information in networks are further problems which motivated the study of random graphs. The main object in this context is the evaluation of the so-called *time constant*, which gives the asymptotic behavior of the number of edges (hops) in the paths (optimal or produced by some particular routeing protocol) joining two given nodes as a function of the (Euclidean) distance between these nodes, when this distance tends to ∞. In the case of a shortest (in terms of the number of hops) path, this problem is usually called the *first passage percolation problem* and was originally stated by Broadbent and Hammersley in [7] to study the spread of fluid in a porous medium. More recently, in [6] and [19], such time constants were studied on so-called *small-word graphs*, motivated by routeing in certain social networks, where any two given nodes are joined by an edge independently with a probability that decays as some power function with the Euclidean distance between them. The complete graph on a Poisson point process with 'nearest neighbor' routeing policy was studied in this context in [2]. The first passage percolation problem on the Poisson–Delaunay graph was considered in [20] and [23]. In the case of graphs whose edges are marked by some weights, we can extend the notion of the time constant by studying the sum of the edge weights. First passage percolation on the complete Poisson point process graph, with weights proportional to some power of the distance between the nodes, was studied in [13].

The present paper focuses on the speed of delivery of information in SINR graphs. In contrast to previous studies of this subject, in particular to [9] and [10], we consider graphs with *space–time* vertices. This new model is motivated by multihop routeing protocols used in wireless ad-hoc networks. In this framework, the random point process on the plane describes the locations of the users of an ad-hoc network and the discrete-time dimension corresponds to successive time slots in which these nodes exchange information (here packets). As in [3], we assume the spatial Aloha policy to decide which node transmits at a given time slot. We also assume some space–time fading model (already used in, e.g. [5]) to describe the variability of the wireless channel conditions (see, e.g. [22, Chapter 2]). In this *space–time SINR graph*, a directed edge represents the feasibility of the wireless transmission between two given network nodes at a given time. More precisely, the direct transmission of a packet succeeds between two nodes in a given time slot if the ratio of the power of the signal between these nodes to the interference and noise at the receiver is larger than a threshold at this time slot. This definition has an information theoretic basis (see, e.g. [22, Chapter 4]). It is rigorously defined below using some power path-loss model and an associated shot noise model representing the interference.

We study various problems on this random graph, including the law of its in- and out-degrees, the number of paths originating from (or terminating at) a typical node or its connectedness. The most important results bear on the first passage percolation problem in this graph. In the case of a Poisson point process for the node locations, we show that the time constant is infinite. We then show that, when adding a periodic node infrastructure of arbitrarily small intensity to the Poisson point process, the time constant is positive and finite. These results lead to bounds on the delays in ad-hoc networks which hold for all routeing algorithms. This subject or, more generally, the question of the speed of the delivery of information in large wireless ad-hoc networks currently receives a lot of attention in the engineering literature; see, e.g. [11] and [14].

The paper is organized as follows. In Section 2 we introduce the space–time SINR graph model. The results are presented in Section 3. Most of the proofs are deferred to Section 4. Some implications on routeing in ad-hoc networks are presented in Section 5.

2. The model

2.1. Probabilistic assumptions

Throughout the paper we consider a simple, stationary, independently marked (i.m.) point process (PP) $\widetilde{\Phi} = \{(X_i, \boldsymbol{e}_i, \boldsymbol{F}_i, \boldsymbol{W}_i)\}$ with finite, positive intensity λ on $\mathbb{R}^2$. In this model, we use the following notation.

- $\Phi = \{X_i\}$ denotes the locations of the network nodes in the $\mathbb{R}^2$ plane. The following three cases regarding the distribution of Φ will be considered:

  General PP: Φ is a general (stationary, nonnull, with finite intensity) PP,

  Poisson PP: Φ is a Poisson PP,

  Poisson+grid PP: $\Phi = \Phi_M + \Phi_G$ is the superposition of two independent PPs, where Φ_M denotes a stationary Poisson PP with finite, nonnull intensity λ_M and $\Phi_G = s\mathbb{Z}^2 + U_G$ is a stationary, periodic PP whose nodes constitute a square grid with edge length s, randomly shifted by the vector U_G that is uniformly distributed in $[0, s]^2$ (this makes Φ_G stationary). Note that the intensity of Φ_G is $\lambda_G = 1/s^2$.

- $\boldsymbol{e}_i = \{e_i(n)\}_{\in n \in \mathbb{Z}}$, where $\mathbb{Z} = \{\dots, -1, 0, 1, \dots\}$ denotes the set of integers; the variables $\{e_i(n) \colon i, n\}$ are independent and identically distributed (i.i.d.) (in n and i) Bernoulli random variables (RVs) with $\mathrm{P}\{e = 1\} = 1 - \mathrm{P}\{e = 0\} = p$, where e denotes the generic RV for this family. We always assume that $0 < p < 1$. The variable $e_i(n)$ represents the *medium access indicator* of node X_i at time n; it says whether the node transmits ($e = 1$ case) or not at time n.

- $\boldsymbol{W}_i = \{W_i(n)\}_n$; $\{W_i(n) \colon i, n\}$ is a family of nonnegative i.i.d. RVs with some arbitrary distribution. The variable $W_i(n)$ represents the power of the *thermal noise* at node X_i at time n. Let W denote the generic RV for this family.

- $\boldsymbol{F}_i = \{F_{i,j}(n)\}_{j,n}$; $\{F_{i,j}(n) \colon i, j, n\}$ is a family of nonnegative i.i.d. RVs. The variable $F_{i,j}(n)$ represents the quality of the radio channel (also called *fading*) from node $X_i \in \Phi$ to node $X_j \in \Phi$ at time n. The following two cases regarding the distribution of F (denoting the generic RV for this family) will be considered:

  General fading: when F has some arbitrary distribution with finite mean,

  Exponential fading: when F has *exponential* distribution with mean $1/\mu$.

In wireless signal propagation models, the exponential distribution appears naturally as the square power of the norm of a complex RV, whose real and imaginary components are i.i.d. Gaussian. In this case we often speak about the *Rayleigh fading model* because the norm (absolute value) of such a complex RV is Rayleigh distributed; see, e.g. [22, pp. 50 and 501]. To complete the probabilistic description of the model, we assume that, given Φ, the random elements $\{\boldsymbol{e}_i\}_i$, $\{\boldsymbol{W}_i\}_i$, and $\{\boldsymbol{F}_i\}_i$ are independent. For more on this framework, which is classical, see, e.g. [3], [4], or [5].

Our stationary i.m. PP $\tilde{\Phi}$ is considered on some probability space with probability P. We will denote by P^0 the Palm probability with respect to Φ; see [8, Chapter 13]. Recall that it can be interpreted as the conditional probability, given Φ has a point at the origin 0 of the plane. We will denote this point (considered under P^0) by X_0 and call it the *typical node*. Under P^0, $\tilde{\Phi}$ is also an i.m. PP with marks distributed as in the original law. Moreover, in the case of

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Poisson PPs, the distribution of Φ under P^0 is equal to the distribution of $\Phi \cup \{X_0 = 0\}$ under the stationary probability P (cf. the Slivnyak–Mecke theorem [8, p. 281]).

2.2. SINR marks

Given the i.m. PP $\tilde{\Phi}$ described above, we construct another family of random variables $\{\mathrm{SINR}_{ij}(n) \colon i, j, n\}$, which will be interpreted as the SINR observed in the channel from $X_i \in \Phi$ to $X_j \in \Phi$ at time n. These variables, which have an information theoretic background, will be used to assess the success of transmissions. For defining these variables, we give ourselves some nondecreasing function $l \colon \mathbb{R}^+ = \{t \colon t \geq 0\} \to \mathbb{R}^+$ that we call the *path-loss function*. A special example considered in this paper (and commonly accepted in the wireless communication context) is

$$l(r) = (Ar)^\beta \quad \text{with some } A > 0 \text{ and } \beta > 2. \tag{2.1}$$

Denote by $\Phi^1(n) = \{X_i \colon e_i(n) = 1\}$ the PP of *transmitters* at time slot n and by $\Phi^0(n) = \{X_i \colon e_i(n) = 0\}$ the PP of (potential) *receivers*. For a given receiver $X_j \in \Phi^0$ and transmitter $X_i \in \Phi^1(n)$ pair, we will assume that X_j receives a signal from X_i with power $F_{i,j}(n)/l(|X_j - X_i|)$ at time n. Node X_j also receives signals from *other* transmitters $X_k \in \Phi^1(n)$, $X_k \neq X_i$, at time n. The total received power is equal to

$$I_{i,j}(n) = \sum_{X_k \in \Phi^1(n) \setminus \{X_i\}} \frac{F_{k,j}(n)}{l(|X_k - X_j|)}.$$

Also, let

$$I_j(n) = \sum_{X_k \in \Phi^1(n) \setminus \{X_j\}} \frac{F_{k,j}(n)}{l(|X_k - X_j|)}.$$

Both $I_{i,j}(n)$ and $I_j(n)$ are *shot noise* RVs generated by $\Phi^1(n)$, the fading marks and the path-loss function. The are infinite sums of nonnegative RVs. In order to check whether these RVs are almost surely (a.s.) finite, we use the Campbell–Little–Mecke formula (Campbell for short; cf. [8, Proposition 13.3.II]), which implies that

$$\mathrm{E}^0\biggl[\sum_{X_k \in \Phi^1(n),\, |X_k| > \varepsilon} \frac{F_{k,0}(n)}{l(|X_k|)}\biggr] = p\,\mathrm{E}[F] \int_{\mathbb{R}^2 \setminus [0,\varepsilon]^2} \frac{1}{l(|x|)} \breve{M}_{[2]}(\mathrm{d}x), \tag{2.2}$$

where $\breve{M}_2(\cdot)$ is the *reduced second-order moment measure* of Φ (cf. [8, p. 238]). In what follows, we will always tacitly assume that $l(\cdot)$ and Φ are such that the integral on the right-hand side of (2.2) is finite for some $\varepsilon \geq 0$, which implies that $I_0(n)$ is a.s. finite under P^0 for all n as well as all $I_j(n)$ and $I_{i,j}(n)$ under P. If Φ is the homogeneous Poisson PP, we have $\breve{M}_{[2]}(\mathrm{d}x) = \lambda\,\mathrm{d}x$ and it is easy to see that the we have finiteness for $l(\cdot)$ given by (2.1) for all $\varepsilon > 0$. It is also relatively easy to see that it holds for the Poisson+grid PP $\Phi = \Phi_M + \Phi_G$.

The SINR at the receiver $X_j \in \Phi^0(n)$ with respect to transmitter $X_i \in \Phi^1(n)$ at time n is defined as

$$\mathrm{SINR}_{i,j}(n) = \frac{F_{i,j}(n)/l(|X_i - X_j|)}{W_j(n) + I_{i,j}(n)}. \tag{2.3}$$

2.3. Space–time SINR graph

Let

$$\delta_{i,j}(n) = \begin{cases} \mathbf{1}(\mathrm{SINR}_{i,j} \geq T) & \text{if } e_i(n) = 1, e_j(n) = 0,\ i \neq j, \\ 1 & \text{if } i = j, \\ 0 & \text{otherwise,} \end{cases} \tag{2.4}$$

where $T > 0$ is a threshold assumed to be some given constant throughout the paper. We define the space–time SINR graph $\mathbb{G}$ as the *directed graph* with the set of vertices $\Phi \times \mathbb{Z}$ and a directed edge from (X_i, n) to $(X_j, n+1)$ if $\delta_{i,j}(n) = 1$.

Let us stress an important convention in our terminology. By a network node, or point, we understand a point of Φ. A (graph) vertex is an element of $\Phi \times \mathbb{Z}$, i.e. it represents some network node at some time. The existence of a graph edge is to be interpreted as the possibility of a successful communication between two network nodes (those involved in the edge) at time n. This can be rephrased as follows. Suppose that at time n the network node X_i has a packet (containing some information). Then the set of graph neighbors of the vertex (X_i, n) describes all the nodes that can decode this packet at time $n+1$. Thus, any path on the graph $\mathbb{G}$ represents some possible route of the packet in space and time.

3. Results

In this section we present our results on $\mathbb{G}$.

3.1. Existence of paths

All the results of this section are obtained under the general PP and fading assumptions of Section 2, under the assumption that the finiteness of the expression in (2.2) is granted.

Note first that $\mathbb{G}$ has no isolated nodes in the usual sense. Indeed, we have always (X_i, n) connected to $(X_i, n+1)$. We will consider directed paths on $\mathbb{G}$ and call them paths for short. Note that these paths are self-avoiding due to the fact that there are no loops in the time dimension.

Denote by $\mathcal{H}_i^{\mathrm{out},k}(n)$ the number of paths of length k (i.e. with k edges) *originating* from (X_i, n). Similarly, denote by $\mathcal{H}_i^{\mathrm{in},k}(n)$ the number of such paths *terminating* at (X_i, n). In particular, $\mathcal{H}_i^{\mathrm{out}}(n) = \mathcal{H}_i^{\mathrm{out},1}(n)$ and $\mathcal{H}_i^{\mathrm{in}}(n) = \mathcal{H}_i^{\mathrm{in},1}(n)$ are respectively the out- and in-degrees of the node (X_i, n).

Lemma 3.1. *For a general PP Φ and a general fading model, the in-degree $\mathcal{H}_i^{\mathrm{in}}$ of any node of $\mathbb{G}$ is bounded from above by the constant $\xi = 1/T + 2$.*

Proof. Assume that there is an edge to node (X_j, n) from nodes $(X_{i_1}, n-1), \dots, (X_{i_k}, n-1)$ for some $k > 1$ and $i_p \neq j$ $(p = 1, \dots, k)$. Then, for all such p,

$$\frac{F_{i_p,j}}{l(|X_{i_p} - X_j|)} \geq \frac{T}{1+T} \sum_{q=1}^{k} \frac{F_{i_q,j}}{l(|X_{i_q} - X_j|)}.$$

When summing up all these inequalities, we obtain $Tk \leq 1 + T$, that is, $k \leq 1/T + 1$. Considering the edge from $(X_i, n-1)$ to (X_i, n), the in-degree of any node is bounded from above by $\xi = 1/T + 2$.

Let

$$h^{\mathrm{out},k} = \mathrm{E}^0[\mathcal{H}_0^{\mathrm{out},k}(n)] = \mathrm{E}^0[\mathcal{H}_0^{\mathrm{out},k}(0)]$$

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and

$$h^{\mathrm{in},k} = \mathrm{E}^0[\mathcal{H}_0^{\mathrm{in},k}(n)] = \mathrm{E}^0[\mathcal{H}_0^{\mathrm{in},k}(0)]$$

be the expected numbers of paths of length k originating or terminating at the typical node, respectively. In particular, $h^{\mathrm{out}} = h^{\mathrm{out},1}$ and $h^{\mathrm{in}} = h^{\mathrm{in},1}$ are the mean out- and in-degrees of the typical node, respectively.

Lemma 3.2. *For a general PP Φ and a general fading model,*

$$h^{\mathrm{in},k} = h^{\mathrm{out},k}.$$

Proof. We use the mass transport principle to obtain $\mathrm{E}^0[\mathcal{H}_0^{\mathrm{out},k}(0)] = \mathrm{E}^0[\mathcal{H}_0^{\mathrm{in},k}(0)]$, which implies the desired result. Indeed, Campbell's formula and stationarity give

$$\begin{aligned}
\lambda h^{\mathrm{out},k} &= \lambda \int_{[0,1)^2} \mathrm{E}^0[\mathcal{H}_0^{\mathrm{out},k}(0)]\,\mathrm{d}x \\
&= \mathrm{E}\bigg[\sum_{X_i \in \Phi\cap[0,1)^2} \mathcal{H}_i^{\mathrm{out},k}(0)\bigg] \\
&= \sum_{v\in\mathbb{Z}} \mathrm{E}\bigg[\sum_{X_i\in[0,1)^2}\ \sum_{X_j\in[0,1)^2+v} (\#\text{ of paths from } (X_i,0) \text{ to } (X_j,k))\bigg] \\
&= \sum_{v\in\mathbb{Z}} \mathrm{E}\bigg[\sum_{X_i\in[0,1)^2-v}\ \sum_{X_j\in[0,1)^2} (\#\text{ of paths from } (X_i,0) \text{ to } (X_j,k))\bigg] \\
&= \lambda \int_{[0,1)^2} \mathrm{E}^0[\mathcal{H}_0^{\mathrm{in},k}(k)]\,\mathrm{d}x \\
&= \lambda h^{\mathrm{in},k},
\end{aligned}$$

where # denotes the cardinality. This completes the proof.

Immediate consequences of the two above lemmas are as follows.

Corollary 3.1. *Under the assumptions of Lemma 3.1,*

- $\mathbb{G}$ *is locally finite (both on in- and out-degrees of all nodes are* P*-a.s. finite),*
- $\mathcal{H}_i^{\mathrm{in},k}(n) \le \xi^k$*,* P*-a.s. for all i, n, k,*
- $h^{\mathrm{in},k} = h^{\mathrm{out},k} \le \xi^k$ *for all k.*

For all $X_i, X_j \in \Phi$ and $n \in \mathbb{Z}$, we define the *local delay* from X_i to X_j at time n by the quantity

$$L_{i,j}(n) = \inf\{k \ge n\colon \delta_{i,j}(k) = 1\},$$

with the usual convention that $\inf\varnothing = \infty$. Note that $L_{i,j}(n)$ is the length (number of edges) of the shortest path (with the smallest number of edges) from (X_i, n) to $\{X_j\}\times\mathbb{Z}$ among the paths contained in the subgraph $\mathbb{G}\cap\{X_i, X_j\}\times\mathbb{Z}$ of $\mathbb{G}$, which is of the form

$$\begin{aligned}
((X_i, n), (X_i, n+1)), \dots, ((X_i, n+L_{i,j}(n)-1), (X_i, n+L_{i,j}(n)))\,, \\
((X_i, n+L_{i,j}(n)), (X_j, n+L_{i,j}(n)+1)).
\end{aligned}$$

Our next result gives a condition for the local delays to be a.s. finite.

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Lemma 3.3. *Consider a general PP Φ and a general fading model with F having unbounded support ($\mathrm{P}\{F > s\} > 0$ for all $0 < s < \infty$). Then, given Φ, all local delays $L_{i,j}(n)$ are P-a.s. finite geometric RVs.*

Proof. Owing to our assumption on the independence of marks in successive time slots, given Φ, the variables $\{\delta_{i,j}(n) \colon n \in \mathbb{Z}\}$ are (i.i.d.) Bernoulli RVs and, thus, $L_{i,j}(n)$ is a geometric RV. It remains to show that $\mathrm{P}\{\delta_{i,j}(0) = 1 \mid \Phi\} := \pi_{i,j}(\Phi) > 0$ for P-almost all Φ. For this, note that

$$\pi_{i,j}(\Phi) = p(1-p)\,\mathrm{P}\{F_{i,j}(0) \geq l(|X_j - X_i|)(W_j(0) + I_{i,j}(0))\}.$$

Under our general assumptions (including finiteness of the expression in (2.2)), $I_{i,j}(0)$ is a finite RV P-a.s. The result follows from the assumption that $0 < p < 1$ and the fact that $F_{i,j}(0)$ is independent of $I_{i,j}(0)$ and $W_{i,j}(0)$ and has infinite support.

The next result directly follows from Lemma 3.3.

Corollary 3.2. *Under the assumptions of Lemma 3.3, $\mathbb{G}$ is* P-a.s. connected *in the following* weak sense*: for all $X_i, X_j \in \Phi$ and all $n \in \mathbb{Z}$, there exists a path from (X_i, n) to the set $\{(X_j, n + l) \colon l \in \mathbb{N}\}$, where $\mathbb{N} = \{1, 2, \dots\}$.*

We denote by $L_i(n) = \inf_{j \neq i} L_{i,j}$ the length of a shortest directed path from (X_i, n) to $(\{\Phi \setminus X_i\}) \times \mathbb{Z}$. We will call $L_i(n)$ the *exit delay* from X_i at time n. Finally, we denote by $P_{i,j}(n)$ the length of a shortest path of $\mathbb{G}$ from (X_i, n) to $\{X_j\} \times \mathbb{Z}$. We call $P_{i,j}(n)$ the *delay* from X_i to X_j at time n. Obviously, for $i \neq j$, we have

$$L_i(n) \leq P_{i,j}(n) \leq L_{i,j}(n), \tag{3.1}$$

and, thus, it follows immediately from Lemma 3.3 that all the three collections of delays are finite RVs P-a.s.

3.2. Optimal paths: Poisson PP case

We have seen in the previous section that, under very general assumptions, all the delays are P-a.s. finite RVs. In this section we show that, under some natural assumptions (such as the Poisson PP and exponential fading), the averaging over Φ may lead to *infinite* mean values. This averaging is expressed in terms of the expectation for the typical node under the Palm probability. The proofs of the results stated in what follows are given in Section 4.1.

Define $\ell = \mathrm{E}^0[L_0(n)] = \mathrm{E}^0[L_0(0)]$.

Proposition 3.1. *Assume that Φ is a Poisson PP, F is exponential, and that the noise W is bounded away from 0: $\mathrm{P}\{W > w\} = 1$ for some $w > 0$. Let the path-loss function be given by (2.1). Then $\mathrm{P}^0\{L_0(0) \geq q\} \geq 1/q$ for large enough q.*

Corollary 3.3. *Under the assumptions of Proposition 3.1, the following statements hold.*

- *The mean exit delay from the typical node is infinite, i.e. $\ell = \infty$.*
- *In any given subset of the plane with positive Lebesgue measure, at a given time, the expected number of points of Φ which have exit delays larger than q decreases not faster than $1/q$ asymptotically for large q.*

The fact that the mean exit delay from the typical point is infinite ($\ell = \infty$) seems to be a consequence of the potential existence of arbitrarily large 'voids' (disks without points of Φ) around this point. Indeed, when conditioning on the existence of another point in the

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configuration Φ, we obtain finite-mean local delays. This will be shown in Proposition 3.2 below.

Before stating it we need to formalise the notion of the existence of two given points $X, Y \in \mathbb{R}^2$ of Φ. For this, we consider Φ under the two-fold Palm probability $\mathrm{P}^{X,Y}$. Since our results on the matter bear only on the Poisson PP case, we can assume (by Slivnyak's theorem) that the following version of the Palm probability of the Poisson PP Φ holds:

$$\mathrm{P}^{X,Y}\{\Phi \in \cdot\} = \mathrm{P}\{\Phi \cup \{X, Y\} \in \cdot\}. \tag{3.2}$$

Moreover, under $\mathrm{P}^{X,Y}$, the marked Poisson PP $\tilde{\Phi}$ is obtained by an independent marking of the points of $\Phi \cup \{X, Y\}$ according to the original distribution of marks. Slightly abusing the notation, we denote by $L_{X,Y}(n)$ the local delay from X to Y at time n when considered under $\mathrm{P}^{X,Y}$. A similar convention will be adopted in the notation of other types of delay under the Palm probability P^X or $\mathrm{P}^{X,Y}$.

Proposition 3.2. *Assume that Φ is a Poisson PP, F is exponential, and that the noise W has a general distribution. Then, for all $X, Y \in \mathbb{R}^2$, the mean local delay from X to Y is finite given the existence of these two points in Φ. More precisely,*

$$\mathrm{E}^{X,Y}[L_{X,Y}(0)] < \infty.$$

The next result follows immediately from (3.1).

Corollary 3.4. *Under the assumptions of Proposition 3.2,*

$$\mathrm{E}^{X,Y}[L_X(0)] \le \mathrm{E}^{X,Y}[P_{X,Y}(0)] < \mathrm{E}^{X,Y}[L_{X,Y}(0)] < \infty.$$

The following result is our main 'negative' result concerning $\mathbb{G}$ in the Poisson PP case.

Proposition 3.3. *Under the assumptions of Proposition 3.1, we have*

$$\lim_{|X-Y|\to\infty} \frac{\mathrm{E}^{X,Y}[P_{X,Y}(0)]}{|X - Y|} = \infty.$$

In other words, the expected shortest delay necessary to send a packet between two given points of the Poisson PP grows faster than the Euclidean distance between these two points. However, we do not know the exact asymptotic of this delay. This latter problem seems to be much more complicated.

3.3. Filling in Poisson voids

In this section we show that adding an independent periodic pattern of points to the Poisson PP allows us to get a linear scaling of the shortest path delay with Euclidean distance. In order to prove the *existence and finiteness* of the associated time constant, we adopt a slightly different approach to the notion of paths on $\mathbb{G}$, which will allow us to exploit a subadditive ergodic theorem. The proofs of the results stated in what follows are given in Section 4.2.

For $x \in \mathbb{R}^2$, let $X(x)$ be the point of Φ which is closest to x. The point $X(x) \in \Phi$ is a.s. well defined for all given $x \in \mathbb{R}^2$ since Φ is assumed to be a simple and stationary PP. For all $x, y \in \mathbb{R}^2$, define $P(x, y, n) = P_{X(x),X(y)}(n)$ to be the length of a shortest path of $\mathbb{G}$ from vertex $(X(s), n)$ to the set $\{(X(y), n + l),\ l \in \mathbb{N}\}$. We will call $P(x, y, n)$ the *delay* from x to y at time n. For all triples of points $x, y, z \in \mathbb{R}^2$, we have

$$P(x, z, n) \le P(x, y, n) + P(y, z, n + P(x, y, n)). \tag{3.3}$$

Let

$$p(x, y, \Phi) = \mathrm{E}[P(x, y, 0) \mid \Phi].$$

Using the strong Markov property, we find that, conditionally on Φ, the law of $P(y, z, n + P(x, y, n))$ is the same as that of $P(y, z, n)$. Then, the last relation and (3.3) give

$$p(x, z, \Phi) \le p(x, y, \Phi) + p(y, z, \Phi). \tag{3.4}$$

We are now in a position to use the subadditive ergodic theorem to show the existence of the time constant

$$\kappa_{\boldsymbol{d}} = \lim_{t \to \infty} \frac{p(0, t\boldsymbol{d}, \Phi)}{t},$$

where $\kappa_{\boldsymbol{d}}$ may depend on the *unit vector* $\boldsymbol{d} \in \mathbb{R}^2$ representing the direction in which the delay is measured. Here is the main result of this section.

Proposition 3.4. *Consider the Poisson+grid PP defined in Section 2.1 with exponential fading F and with path-loss function given by (2.1). Then, for all unit vectors $\boldsymbol{d} \in \mathbb{R}^2$, the nonnegative limit $\kappa_{\boldsymbol{d}}$ exists and is* P*-a.s.* finite. *The convergence also holds in L_1.*

Note that $\kappa_{\boldsymbol{d}}$ is not a constant. Indeed, the superposition of the PPs $\Phi = \Phi_M$ and Φ_G is ergodic but not mixing due to the fact that Φ is a (stationary) grid. For $\boldsymbol{d}$ parallel to say the horizontal axis of the grid Φ_G, the limit $\kappa_{\boldsymbol{d}}$ will depend on the distance from the line $\{t\boldsymbol{d} : t \in \mathbb{R}\}$ to the nearest parallel (horizontal) line of the grid Φ_G, i.e. on the shift U_G of the grid. A more precise formulation of the result is a follows.

Proposition 3.5. *Under the assumptions of Proposition 3.4, the limit $\kappa_{\boldsymbol{d}} = \kappa_{\boldsymbol{d}}(U_G)$ is measurable with respect to the shift U_G of the grid PP Φ_G and* does not *depend on the Poisson component Φ_M of the PP Φ. Moreover, the set of vectors $\boldsymbol{d}$ in the unit sphere for which $\kappa_{\boldsymbol{d}}(U_G)$ is not* P*-a.s. a constant is at most countable.*

The final result on this case is as follows.

Proposition 3.6. *Under the assumptions of Proposition 3.4, suppose that W is constant and strictly positive. Then* $\mathrm{E}[\kappa_{\boldsymbol{d}}] > 0$.

Finally, let us remark that the method used in this section cannot be used in the case of the Poisson PP (without the addition of the grid PP). The main problem is the lack of integrability of $p(x, y, \Phi)$ as stated in the following result. Note, however, that this does *not* immediately imply that $\kappa_{\boldsymbol{d}} = \infty$.

Corollary 3.5. *Under the assumptions of Proposition 3.1,* $\mathrm{E}[p(x, y, \Phi)] = \infty$ *for all x and y in* $\mathbb{R}^2$.

4. Proofs

Consider the shortest path from (X_i, n) to $(\Phi \setminus \{X_i\}) \times \mathbb{Z}$. Let $\mathcal{T}_i(n)$ be the number of edges $(X_i, k), (X_i, k + 1)$ in this path such that $e_i(k) = 1$. These variables are the *number of trials* before the first exit form X_i at time n. Obviously,

$$\mathcal{T}_i(n) \le L_i(n). \tag{4.1}$$

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We will also consider an auxiliary graph $\widehat{\mathbb{G}}$, called the *(space–time) signal to noise ratio (SNR) graph*, defined exactly in the same manner as the SINR graph $\mathbb{G}$ except that the variables $\mathrm{SINR}_{i,j}(n)$ defined in (2.3) are replaced by the variables

$$\mathrm{SNR}_{i,j}(n) = \frac{F_{i,j}(n)/l(|X_i - X_j|)}{W_j(n)}. \tag{4.2}$$

Note that this modification consists in suppressing the interference term $I_{i,j}(n)$ in the SINR condition in (2.4). The edges of $\mathbb{G}$ form a subset of the edges of $\widehat{\mathbb{G}}$ (both graphs share the same vertices), which will be denoted by

$$\mathbb{G} \subset \widehat{\mathbb{G}}. \tag{4.3}$$

In what follows, we will denote the delays, local delays, exit delays, and numbers of trials related to $\widehat{\mathbb{G}}$ by $\widehat{P}_{i,j}(n)$, $\widehat{L}_{i,j}(n)$, $\widehat{L}_i(n)$, and $\widehat{\mathcal{T}}_i(n)$, respectively. The inclusion $\mathbb{G} \subset \widehat{\mathbb{G}}$ immediately implies that $\widehat{P}_{i,j}(n) \le P_{i,j}(n)$, and the same inequalities hold for the three other families of variables mentioned above.

4.1. Proofs of the results of Section 3.2

Proof of Proposition 3.1. The inclusion (4.3) and the inequality (4.1) yield

$$\widehat{\mathcal{T}}_i(n) \le \mathcal{T}_i(n) \le L_i(n),$$

which holds for all i, n. The results follow from the above inequalities and the next lemma.

Lemma 4.1. *Under the assumptions of Proposition 3.1,* $\mathrm{P}^0\{\widehat{\mathcal{T}}_0(0) \ge q\} \ge 1/q$ *for large enough* q.

Proof. Under P^0, denote by τ_k the kth time slot in $\{0, 1, \ldots\}$, such that $e_0(k) = 1$. For all $q \ge 0$, we have

$$\begin{aligned}
&\mathrm{P}^0\{\widehat{\mathcal{T}}_0(0) > q \mid \Phi\} \\
&\quad = \mathrm{P}^0\{\text{for all } 0 \le k \le q \text{ and } 0 \ne X_i \in \Phi, \delta_{0,i}(\tau_k) = 0 \mid \Phi\} \\
&\quad = \mathrm{P}^0\{\text{for all } 0 \le k \le q \text{ and } 0 \ne X_i \in \Phi, e_i(\tau_k) = 1 \text{ or } \mathrm{SNR}_{0,i}(\tau_k) < T \mid \Phi\},
\end{aligned}$$

and, by the conditional independence of marks given Φ,

$$\begin{aligned}
\mathrm{P}^0\{\widehat{\mathcal{T}}_0(0) > q \mid \Phi\} &= \prod_{0 \ne X_i \in \Phi} (p + (1-p)\,\mathrm{P}\{F < Tl(|X_i|)W\})^q \\
&= \exp\biggl\{q \sum_{0 \ne X_i \in \Phi} \log(p + (1-p)(1 - \mathrm{e}^{-\mu T l(|X_i|)W}))\biggr\},
\end{aligned}$$

where F and W are independent generic RVs representing fading and thermal noise, independent of Φ, and F is exponential with mean $1/\mu$, Using the Laplace functional formula for Φ and the assumption that $W > w$ a.s., we have

$$\begin{aligned}
\mathrm{P}^0\{\widehat{\mathcal{T}}_0(0) \ge q\} &\ge \exp\biggl\{-2\pi\lambda \int_{v>0} (1 - (1 - (1-p)\,\mathrm{e}^{-w\mu l(v)T})^q) v \,\mathrm{d}v\biggr\} \\
&= \exp\biggl\{-\pi\lambda \int_{v>0} (1 - (1 - f(v))^q)\,\mathrm{d}v\biggr\},
\end{aligned} \tag{4.4}$$

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where

$$f(v) := (1-p)\exp\{-K v^{\beta/2}\} \quad \text{and} \quad K = w\mu T A^{\beta}.$$

In what follows, we will show that the expression in (4.4) is not smaller than $1/q$ for large enough q. To this end, denote by v_q the unique solution of $f(v) = 1/q$. We have

$$v_q = \frac{1}{A^2(\mu T w)^{2/\beta}}(\log(q(1-p)))^{2/\beta}.$$

It is clear that $f(v)$ tends to 0 when v tends to ∞ and that v_q tends to ∞ as q tends to ∞. Therefore, there exists a constant $Q = Q(\mu, w, A, T) < \infty$ such that, for all $q \geq Q$ and $v \geq v_q$,

$$(1 - f(v)) \geq \exp\{-f(v)\}.$$

Hence, for all $q \geq Q$,

$$\begin{aligned}
\int_{v>0} (1-(1-f(v))^q)\,\mathrm{d}v &\leq v_q + \int_{v_q}^{\infty} (1-(1-f(v))^q)\,\mathrm{d}v \\
&\leq v_q + \int_{v=v_q}^{\infty} (1-\exp\{-qf(v)\})\,\mathrm{d}v \\
&\leq v_q + \int_{v_q}^{\infty} qf(v)\,\mathrm{d}v \\
&= v_q + \int_{u=0}^{\infty} qf(u+v_q)\,\mathrm{d}u.
\end{aligned}$$

The third inequality follows from the fact that $1 - \exp\{-x\} \leq x$. Now using the fact that $(u + v_q)^{\beta/2} \geq u + v_q^{\beta/2}$ (for large enough q, say again $q \geq Q$), we obtain

$$\begin{aligned}
\int_{u=0}^{\infty} qf(u+v_q)\,\mathrm{d}u &= \int_{u=0}^{\infty} q(1-p)\exp\{-K(u+v_q)^{\beta/2}\}\,\mathrm{d}u \\
&\leq \int_{u=0}^{\infty} q(1-p)\exp\{-Ku - Kv_q^{\beta/2}\}\,\mathrm{d}u \\
&= \frac{1}{K},
\end{aligned}$$

since $(1-p)\exp\{-Kv_q^{\beta/2}\} = 1/q$. Hence, for $q \geq Q$,

$$\int_{v>0} (1-(1-f(v))^q)\,\mathrm{d}v \leq v_q + \frac{\alpha}{K}.$$

Also, it is not difficult to see that $\beta > 2$ implies that

$$v_q \leq \frac{\log q}{\pi\lambda} - \frac{1}{K}$$

for large enough q. This implies that, for large enough q, say again $q \geq Q$,

$$\exp\left\{-\pi\lambda \int_{v>0} (1-(1-f(v))^q)\,\mathrm{d}v\right\} \geq \exp\left\{-\pi\lambda\left(v_q + \frac{1}{K}\right)\right\} \geq \frac{1}{q}, \tag{4.5}$$

which completes the proof.

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Proof of Proposition 3.2. Assume without loss of generality that $Y = 0$ and $|X| = r$. Under P, consider the PP $\Phi \cup \{X, 0\}$ and its independent marking. Given Φ, the RV $L_{X,0}(0)$ associated with the i.m. PP $\Phi \cup \{X, 0\}$ has a geometric distribution with parameter

$$\pi_{X,0}(\Phi) = p(1-p)\,\mathrm{P}\{F \geq l(r)(W + I)\},$$

where F, W, and I are independent RVs, F and W are generic fading and noise variables, and $I = \sum_{X_i \in \Phi} e_i(0) F_{i,0}(0)/l(|X_i|)$. Using the exponential distribution of F and the independence, we obtain

$$\pi_{X,0}(\Phi) = \mathrm{E}[\mathrm{e}^{-\mu l(r) T W}]\,\mathrm{E}[\mathrm{e}^{-\mu l(r) T I} \mid \Phi].$$

The mean of the geometric RV is known to be $\mathrm{E}^{X,0}[L_{X,0}(0) \mid \Phi] = 1/\pi_{X,0}(\Phi)$. By unconditioning with respect to Φ, we obtain

$$\mathrm{E}^{X,0}[L_{X,0}(0)] = \frac{1}{\mathcal{L}_W(\mu l(r) T)}\,\mathrm{E}\bigg[\frac{1}{\mathrm{E}[\mathrm{e}^{-\mu l(r) T I} \mid \Phi]}\bigg].$$

The first factor in the above expression is obviously finite. In what follows, we will evaluate the second factor.

By the conditional independence of marks and denoting by $\mathcal{L}_{eF}(\cdot)$ the Laplace transform of eF, where e and F are independent generic variables for $e_i(0)$ and $F_{i,0}(0)$, we have

$$\begin{aligned}(\mathrm{E}[\mathrm{e}^{-\mu l(r) T I} \mid \Phi])^{-1} &= \bigg(\mathrm{E}\bigg[\exp\bigg\{-\mu l(r) T \sum_{X_i \in \Phi} \frac{e_i(0) F_{i,0}(n)}{l(|X_i|)}\bigg\} \biggm| \Phi\bigg]\bigg)^{-1} \\ &= \exp\bigg\{\sum_{X_i \in \Phi} \log \mathcal{L}_{eF}\bigg(\frac{\mu T l(r)}{l(|X_i|)}\bigg)\bigg\}.\end{aligned}$$

Note that $\mathcal{L}_{eF}(\xi) = 1 - p + p\mathcal{L}_F(\xi) = 1 - p + p\mu/(\mu + \xi)$. Using this and the Laplace functional formula for Φ (cf. [8, Equation 9.4.17]), we obtain

$$\mathrm{E}\bigg[\frac{1}{\mathrm{E}[\mathrm{e}^{-\mu l(r) T I} \mid \Phi]}\bigg] = \exp\bigg\{2\pi p \lambda \int_0^\infty \frac{v T l(r)}{l(v) + (1-p) T l(r)}\,\mathrm{d}v\bigg\};$$

cf. (2.2). Now, using the fact that, for the Poisson PP, $\breve{M}_{[2]}(\mathrm{d}x) = \lambda\,\mathrm{d}x$, it is easy to see that, for any path-loss function satisfying $\int_\varepsilon^\infty v/l(v)\,\mathrm{d}v < \infty$, the integral in the exponent of the last expression is finite. This completes the proof.

Proof of Proposition 3.3. Using inclusion (4.3), inequality (4.1), and the left-hand side of (3.1), we have

$$\widehat{\mathcal{T}}_i(n) \leq \mathcal{T}_i(n) \leq L_i(n) \leq P_{i,j}(n).$$

Thus, it is enough to show that

$$\lim_{|X-Y| \to \infty} \frac{\mathrm{E}^{X,Y}[\widehat{\mathcal{T}}_X(0)]}{|X - Y|} = \infty.$$

Without loss of generality, assume that $X = 0$ and $|Y| = r$. Using the same arguments as in the proof of Lemma 4.1 and representation (3.2) of the Palm probability with respect to Poisson PP, we obtain

$$\begin{aligned}
&\mathrm{P}^{0,Y}\{\widehat{\mathcal{T}}_0(0) > q \mid \Phi\} \\
&\quad \geq \prod_{0,Y\neq X_i\in\Phi} (p + (1-p)\,\mathrm{P}\{F < Tl(|X_i|)W\})^q (p + (1-p)\,\mathrm{P}\{F < Tl(|Y|)W\})^q \\
&\quad \geq \exp\biggl\{-\pi\lambda \int_{v>0} (1-(1-f(v))^q)\,\mathrm{d}v\biggr\}\alpha(r)^q,
\end{aligned}$$

where $\alpha(r) = 1 - (1-p)\,\mathrm{e}^{-w\mu A^{\alpha} T r^{\beta}}$. Using (4.5), which holds for large q, or, more precisely, for $q > Q = Q(\mu, w, A, T)$, we obtain

$$\frac{\mathrm{E}^{0,Y}[\widehat{\mathcal{T}}_0(0)]}{r} \geq \frac{1}{r}\sum_{q>Q}\frac{\alpha(r)^q}{q}.$$

It is now easy to see that

$$\lim_{r\to\infty}\frac{1}{r}\sum_{q>Q}\frac{\alpha(r)^q}{q} = \infty.$$

4.2. Proofs of the results of Section 3.3

Denote by $B_x(R)$ the ball centered at $x \in \mathbb{R}^2$ of radius R. Similarly as for the delays, we extend the definition of the local delays to arbitrary pairs of points $x, y \in \mathbb{R}^2$ by taking $L(x, y, n) = L_{X(x),X(y)}(n)$. We first establish the following technical result.

Lemma 4.2. *Under the assumptions of Proposition 3.4, let $X_i, X_j \in \Phi \cap B_0(R)$ for some $R > 0$, where $\Phi = \Phi_M + \Phi_{G_s}$. Then the conditional expectation of the local delay $L_{i,j}(0)$ given Φ satisfies*

$$\begin{aligned}
&\mathrm{E}[L_{i,j}(0) \mid \Phi] \\
&\quad = \frac{1}{p(1-p)\mathcal{L}_W(T\mu A^{\beta}|X_i - X_j|^{\beta})}\exp\biggl\{-\sum_{\Phi\ni X_k,\,k\neq i,k}\log\mathcal{L}_{eF'}\biggl(\frac{T|X_i - X_j|^{\beta}}{|X_j - X_k|^{\beta}}\biggr)\biggr\} \qquad (4.6)\\
&\quad \leq \frac{1}{p(1-p)\mathcal{L}_W(T\mu(A2R)^{\beta})} \\
&\qquad \times \exp\{-49\log(1-p) + (2R)^{\beta}pTC(s,\beta)\} \qquad (4.7)\\
&\qquad \times \exp\{-\Phi_M(B_0(2R))\log(1-p)\} \qquad (4.8)\\
&\qquad \times \exp\biggl\{-\sum_{X_k\in\Phi_M,\,|X_k|>2R}\log\biggl(1-p+\frac{p(|X_k|-R)^{\beta}}{(|X_k|-R)^{\beta}+T(2R)^{\beta}}\biggr)\biggr\}, \qquad (4.9)
\end{aligned}$$

where $C(s,\beta) < \infty$ is some constant (which depends on s and β but not on Φ), F' is an exponential RV of mean 1, and $\mathcal{L}_{eF'}(\cdot)$ is the Laplace transform of eF'.

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Proof. We first prove the equality in (4.6). When using the independence assumptions, we have

$$\begin{aligned}
&\mathrm{P}\{L_{i,j}(0) > m \mid \Phi\} \\
&\quad = \mathrm{P}\{\text{for all } 1 \le n \le m, e_j(n) = 1 \text{ or} \\
&\qquad e_j(n) = 0 \text{ and } e_i(n)F_{i,j}(n) \le Tl(|X_i - X_j|)(W_j(n) + I_{i,j}(n))\} \\
&\quad = \prod_{n=1}^{m}\biggl(p + (1-p)\biggl(1 - p + p\biggl(1 - \mathcal{L}_W(T\mu A^\beta |x-y|^\beta) \\
&\qquad\qquad \times \prod_{\Phi \ni X_k,\, k \ne i,j} \mathcal{L}_{eF'}\biggl(\frac{T|X_i - X_j|^\beta}{|X_j - X_k|^\beta}\biggr)\biggr)\biggr)\biggr).
\end{aligned}$$

The result then follows from the evaluation of

$$\mathrm{E}[L_{i,j}(0) \mid \Phi] = \sum_{m=0}^{\infty} \mathrm{P}\{L_{i,j}(0) > m \mid \Phi\}.$$

The bound $|X_i - X_j| \le 2R$ used in the Laplace transform of W leads to the first factor of the upper bound. We now factorize the exponential function in (4.6) as the product of three exponential functions:

$$\begin{aligned}
\alpha &:= \exp\biggl\{-\sum_{\Phi_{G_s} \ni X_k,\, k \ne i,j}\biggr\}, \\
\beta &:= \exp\biggl\{-\sum_{\Phi_M \ni X_k,\, k \ne i,\, j |X_k| \le 2R}\biggr\}, \\
\gamma &:= \exp\biggl\{-\sum_{\Phi_M \ni X_k,\, |X_k| > 2R}\biggr\}.
\end{aligned}$$

Next we prove that the last three exponentials are upper bounded by (4.7), (4.8), and (4.9), respectively.

We use $|X_i - X_j| \le 2R$ and Jensen's inequality to obtain

$$\begin{aligned}
\log \mathcal{L}_{eF'}\biggl(\frac{T|X_i - X_j|^\beta}{|X_j - X_k|^\beta}\biggr) &\ge \log \mathcal{L}_{eF'}\biggl(\frac{T(2R)^\beta}{|X_j - X_k|^\beta}\biggr) \\
&\ge \frac{-T(2R)^\beta\, \mathrm{E}[eF']}{|X_j - X_k|^\beta} \\
&= -pT(2R)^\beta |X_j - X_k|^{-\beta}.
\end{aligned}$$

We now prove that

$$\sum_{\{\Phi_{G_s} \ni X_k :\, |X_j - X_k| > 3\sqrt{2}s\}} |X_j - X_k|^{-\beta} \le C(s, \beta)$$

for some constant $C(s, \beta)$. This follows from an upper bounding of the value of $|X_j - X_k|^{-\beta}$ by the value of the integral $1/s^2 \int (|X_j - x| - \sqrt{2}s)^{-\beta}\, \mathrm{d}x$ over the square with corner points

X_k, $X_k + (s, 0)$, $X_k + (0, s)$, and $X_k + (s, s)$. In this way we obtain

$$\sum_{\{\Phi_{G_s} \ni X_k :\, |X_j - X_k| > 3\sqrt{2}s\}} |X_j - X_k|^{-\beta} \le \frac{1}{s^2} \int_{|x - X_j| > 2\sqrt{2}s}^{\infty} (|X_j - x| - \sqrt{2}s)^{-\beta}\, \mathrm{d}x$$
$$= \frac{2\pi}{s^2} \int_{\sqrt{2}s}^{\infty} \frac{t + \sqrt{2}s}{t^\beta}\, \mathrm{d}t$$
$$=: C(s, \beta)$$
$$< \infty.$$

Combining this and what precedes, we obtain

$$\exp\biggl\{- \sum_{X_k \in \Phi_{G_s},\, |X_j - X_k| > 2\sqrt{s}} \log \mathcal{L}_{eF'}\biggl(\frac{T|X_j - X_i|^\beta}{|X_j - X_k|^\beta}\biggr)\biggr\} \le \exp(T(2R)^\beta C(s, \beta)).$$

We also have

$$\log \mathcal{L}_{eF'}\biggl(\frac{T(2R)^\beta}{|y - X_i|^\beta}\biggr) \ge \log \mathcal{L}_{eF'}(\infty) = \log(1 - p)$$

for all $X_k \in \Phi_{G_s}$ and, in particular, for $|X_j - X_k| \le 3\sqrt{2}s$. Hence, we obtain

$$\exp\biggl\{- \sum_{X_k \in \Phi_{G_s}} (\cdots)\biggr\} \le \exp\{-49 \log(1 - p) + T(2R)^\beta C(s, \beta)\},$$

where 49 upper bounds the number of points $X_k \in \Phi_{G_s}$ such that $|X_j - X_k| \le 3\sqrt{2}s$.

Using the bound $|X_j - X_i| \le 2R$ and the inequality

$$\log \mathcal{L}_{eF'}(\xi) \ge \log \mathcal{L}_{eF'}(\infty) = \log(1 - p),$$

we obtain

$$\exp\biggl\{- \sum_{\Phi_M \ni X_k,\, k \ne i, j,\, |X_i| \le 2R} (\cdots)\biggr\} \le \exp\{-\Phi_M(B_0(2R)) \log(1 - p)\}.$$

Using the bounds $|X_j - X_i| \le 2R$ and $|X_j - X_k| \ge |X_k| - R$ (the latter follows from the triangle inequality), and the expression $\mathcal{L}_{eF'}(\xi) = 1 - p + p/(1 + \xi)$, we obtain

$$\exp\biggl\{- \sum_{\Phi_M \ni X_k, |X_k| > 2R} (\cdots)\biggr\}$$
$$\le \exp\biggl\{- \sum_{X_k \in \Phi_M,\, |X_k| > 2R} \log\biggl(1 - p + \frac{p(|X_k| - R)^\beta}{(|X_k| - R)^\beta + T(2R)^\beta}\biggr)\biggr\}.$$

This completes the proof.

We can now prove the following auxiliary result.

Lemma 4.3. *Under the assumptions of Proposition 3.4 for all points x, y of $\mathbb{R}^2$,*

$$\mathrm{E}\Bigl[\sup_{x_1, y_1 \in [x, y]} p(x_1, y_1, \Phi)\Bigr] < \infty,$$

where the supremum is taken over x_1 and y_1 belonging to the interval $[x, y] \subset \mathbb{R}^2$.

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Proof. Without loss of generality, we assume that $(x+y)/2 = 0$ is the origin of the plane. Let $B = B_0(R)$ be the ball centered at 0 of radius R such that no modification of the points in the complement of B modifies $X(z)$ for any $z \in [x, y]$ (recall that $X(z)$ is the point of Φ which is the closest from z). Since $\Phi = \Phi_M + \Phi_{G_s}$, with Φ_{G_S} the square lattice PP with intensity $1/s^2$, it suffices to take $R = |u - v|/2 + \sqrt{2}s$. Let $B' = B_0(2R)$. By the above choice of B and inequality (3.1), we have, for all $x_1, y_1 \in [x, y]$,

$$P(x_1, y_1, 0) \le \sum_{X_i, X_j \in \Phi \cap B} L_{i,j}(0),$$

and, consequently,

$$\sup_{x_1, y_1 \in [x,y]} p(x_1, y_1, \Phi) \le \sum_{X_i, X_j \in \Phi \cap B} \mathrm{E}[L_{i,j}(0) \mid \Phi].$$

Using (4.6), we obtain

$$\begin{aligned}
&\sup_{x_1, y_1 \in [x,y]} |p(x_1, y_1, \Phi)| \\
&\quad \le \frac{\exp\{-49 \log(1-p) + (2R)^\beta pTC(s, \beta)\}}{p(1-p)\mathcal{L}_W(T\mu A(2R)^\beta)} \\
&\quad\quad \times \exp\left\{-\sum_{X_k \in \Phi_M,\, |X_k| > 2R} \log\left(1 - p + \frac{p(|X_k| - R)^\beta}{(|X_k| - R)^\beta + T(2R)^\beta}\right)\right\} \\
&\quad\quad \times \left(\Phi_M(B) + \frac{\pi(R + \sqrt{2}s)^2}{s^2}\right) \exp\{-\Phi_M(B') \log(1-p)\},
\end{aligned}$$

where $\pi(R + \sqrt{2}s)^2/s^2$ is an upper bound of the number of points of Φ_{G_s} in B. The first factor in the above upper bound is deterministic. The two other factors are random and independent due to the independence property of the Poisson PP. The finiteness of the expectation of the last expression follows from the finiteness of the exponential moments (of any order) of the Poisson RV $\Phi_M(B')$. For the expectation of the second (exponential) factor, we use the known form of the Laplace transform of the Poisson shot noise to obtain the expression

$$\mathrm{E}\left[\exp\left\{-\sum(\cdots)\right\}\right] = \exp\left\{2\pi p \lambda_M \int_R^\infty \frac{T(2R)^\beta}{v^\beta + (1-p)T(2R)^\beta}(v + R)\,\mathrm{d}v\right\} < \infty.$$

Proof of Proposition 3.4. The existence and finiteness of the limit $\kappa_{\boldsymbol{d}}$ follows from the subadditivity (3.4) and Lemma 4.3 by the continuous-parameter subadditive ergodic theorem (see [18, Theorem 4]).

Proof of Proposition 3.5. First, we prove the second statement, i.e. that $\kappa_{\boldsymbol{d}}$ is constant for all $\boldsymbol{d}$ in the unit sphere of some countable subset. Note that the PP Φ is ergodic since it is an independent superposition of the mixing Poisson PP Φ_M and the ergodic grid process Φ_G. This can be easily proved using, e.g. the respective characterizations of the above properties by means of Laplace transforms of PPs (see [8, Proposition 12.3.VI]). From the ergodicity of Φ we *cannot* conclude the desired property for any vector $\boldsymbol{d}$ since the limit $\kappa_{\boldsymbol{d}} = \kappa_{\boldsymbol{d}(\Phi)}$ is not necessarily invariant with respect to translations of Φ by *any* vector $x \in \mathbb{R}^2$, but only $x = \alpha \boldsymbol{d}$ for any scalar $\alpha \in \mathbb{R}$. The announced result follows from [21, Theorem 1].

For the first statement, consider a product space on which two independent PPs (Φ_M, Φ_G) are defined. Fix some vector $\boldsymbol{d}$, and define the operator $T = T_1 \times T_2$ on this product space as the product of two operators which correspond to the shift in the direction $\boldsymbol{d}$ of Φ_M and Φ_G, respectively. The σ-field invariant with respect to T is the product of the respective σ-fields invariant with respect to T_1 and T_2. The latter is trivial since Φ_M is mixing (as a Poisson PP). Consequently, every function of (Φ_M, Φ_G) that is invariant with respect to the shift in the direction $\boldsymbol{d}$ of its first argument (Φ_M) is a.s. constant. This concludes the proof that $\kappa_{\boldsymbol{d}}$ is constant in Φ_M and, thus, depends only on U_G.

Proof of Proposition 3.6. For a given path $\sigma = \{(X_0, n_0), (X_1, n_0 + 1), \dots, (X_k, n_0 + k)\}$ on $\mathbb{G}$ denote by $|\sigma| = \sum_{i=1}^{k} |X_i - X_{i-1}|$ the Euclidean length of the projection of σ on $\mathbb{R}^2$; let us call it the Euclidean length of σ for short and recall that the (graph) length of σ is equal to k. For fixed $\varepsilon > 0$ and all $n \geq 1$, denote by $\Pi(n) = \Pi_\varepsilon(n)$ the event that there exists a path on $\mathbb{G}$ starting at $(X(0), 0)$ that has (graph) length n and Euclidean length larger than n/ε.

Assume that $\mathrm{E}[\kappa_{\boldsymbol{d}}] = 0$. We show first that this implies that, for any $\varepsilon > 0$, P^0-a.s., the event $\Pi_\varepsilon(n)$ holds for infinitely many n:

$$\mathrm{P}^0\bigg\{\bigcap_{n\geq 1}\bigcup_{k\geq n} \Pi_\varepsilon(k)\bigg\} = 1. \tag{4.10}$$

Indeed, $\mathrm{E}[\kappa_{\boldsymbol{d}}] = 0$ implies that $\kappa_{\boldsymbol{d}} = 0$, P-a.s. and, by Palm–Matthes' definition of the Palm probability, P^0-a.s. as well. This means that $\mathrm{E}^0[P(0, t\boldsymbol{d}, 0) \mid \Phi]/t \to 0$ when $t \to \infty$, which implies that

$$\lim_k \frac{P(0, t_k\boldsymbol{d}, 0)}{t_k} \to 0 \tag{4.11}$$

P^0-a.s. for some subsequence $\{t_k : k \geq 1\}$, with $\lim_k t_k = \infty$. Recall that $P(0, t_k\boldsymbol{d}, 0)$ is the length of a shortest path from $(X(0), 0)$ (with $X(0) = 0$ under P^0) to $\{(X(t_k\boldsymbol{d}), n) : n \geq 0\}$. Denote one of these shortest paths by σ_k. By the triangle inequality, its Euclidean length satisfies

$$|\sigma_k| \geq |0 - X(t_k\boldsymbol{d})| \geq t_k - \sqrt{2}s. \tag{4.12}$$

From (4.11) and (4.12), we conclude that, for any $\varepsilon > 0$ and large enough k, the length of the path σ_k is smaller than ε times its Euclidean length $|\sigma_k|$. Now, (4.10) follows from the fact that the length of the path σ_k tends to ∞ with k, which is a consequence of $t_k \to \infty$ and the local finiteness of the graph $\mathbb{G}$ (cf. Corollary 3.1).

We conclude the proof by showing that, for small enough ε,

$$\sum_n \mathrm{P}^0\{\Pi_\varepsilon(n)\} < \infty, \tag{4.13}$$

which, by the Borel–Cantelli lemma, implies that $\Pi(n)$ holds P^0-a.s. only for a finite number of integers n and, thus, contradicts (4.10). To this end, assume that the constant $W = w > 0$ and let $\mathcal{P}_w^n$ denote the set of paths σ in $\mathbb{G}$ of length n, originating from $(X(0) = 0, 0)$. Also, denote by $\mathcal{P}_0^n$ the analogous set of paths on the graph constructed under the assumption that $W = 0$. Note that, by monotonicity,

$$\mathcal{P}_w^n \subset \mathcal{P}_0^n.$$

By definition,

$$\mathrm{P}^0\{\Pi_\varepsilon(n) \mid \Phi\} = \mathrm{P}^0\bigg\{\bigcup_\sigma \bigg\{\sigma \in \mathcal{P}_w^n \text{ and } |\sigma| \geq \frac{n}{\varepsilon}\bigg\} \;\bigg|\; \Phi\bigg\},$$

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where the sum bears on all possible n-tuples $\sigma = ((X_{j_1}, 1), \dots, (X_{j_n}, n))$, with $X_{j_i} \in \Phi$. From this we have

$$\begin{aligned}
&\mathrm{P}^0\{\Pi_\varepsilon(n) \mid \Phi\} \\
&\quad \le \sum_\sigma \mathrm{P}^0\left\{\sigma \in \mathcal{P}_W^n,\ |\sigma| \ge \frac{n}{\varepsilon} \;\middle|\; \Phi\right\} \\
&\quad = \sum_\sigma \mathrm{P}^0\left\{\sigma \in \mathcal{P}_W^n,\ |\sigma| \ge \frac{n}{\varepsilon} \;\middle|\; \Phi, \sigma \in \mathcal{P}_0^n\right\} \mathrm{P}^0\{\sigma \in \mathcal{P}_0^n \mid \Phi\} \\
&\quad \le \mathrm{E}^0[\mathcal{H}_0^{\mathrm{out},n;\,W=0}(0) \mid \Phi] \sup_\sigma \mathrm{P}^0\left\{\sigma \in \mathcal{P}_W^n,\ |\sigma| \ge \frac{n}{\varepsilon} \;\middle|\; \Phi, \sigma \in \mathcal{P}_0^n\right\}, \qquad (4.14)
\end{aligned}$$

where $\mathcal{H}_0^{\mathrm{out},n;\,W=0}(0)$ denotes the number of paths of length n originating from $(X_0 = 0, 0)$ under the assumption that $W = 0$. But

$$\begin{aligned}
&\sup_\sigma \mathrm{P}^0\left\{\sigma \in \mathcal{P}_w^n,\ |\sigma| \ge \frac{n}{\varepsilon} \;\middle|\; \Phi, \sigma \in \mathcal{P}_0^n\right\} \\
&\qquad \le \sup_{\substack{\sigma = ((X_{j_1},1),\dots,(X_{j_n},n)) \\ \sum_{i=1}^n |X_{j_i} - X_{j_{i-1}}| \ge n/\varepsilon}} \mathrm{E}^0\left[\prod_{i=1}^n \delta_{j_{i-1},j_i}(i-1, w) \;\middle|\; \Phi, \sigma \in \mathcal{P}_0^n\right],
\end{aligned}$$

where $X_{j_0} = 0$ and $\delta_{j_{i-1},j_i}(i-1, w) = \delta_{j_{i-1},j_i}(i-1)$ is the indicator of the existence of the edge from $(X_{j_{i-1}}, i-1)$ to (X_{j_i}, i) defined by (2.4), and where we add in the notation the dependence on the noise $W = w$. Using the conditional independence of marks, (2.4), (2.3), and the lack of memory of the exponential distribution of F of parameter μ, we have, for the path-loss function (2.1),

$$\begin{aligned}
&\mathrm{E}^0\left[\prod_{i=1}^n \delta_{j_{i-1},j_i}(i-1, w) \;\middle|\; \Phi, \sigma \in \mathcal{P}_0^n\right] \\
&\qquad = \prod_{i=1}^n \mathrm{E}^0[\delta_{j_{i-1},j_i}(i-1, w) \mid \Phi, \delta_{j_{i-1},j_i}(i-1, 0) = 1] \\
&\qquad = \prod_{i=1}^n \exp\{-\mu (A|X_{j_{i-1}} - X_{j_i}|)^\beta T w\}.
\end{aligned}$$

Hence,

$$\sup_\sigma \mathrm{P}^0\left\{\sigma \in \mathcal{P}_w^n,\ |\sigma| \ge \frac{n}{\varepsilon} \;\middle|\; \Phi, \sigma \in \mathcal{P}_0^n\right\} \le \exp\{-\mu A^\beta T w n \varepsilon^{-\beta}\},$$

where the last inequality follows from a convexity argument. Using this and (4.14), we obtain

$$\begin{aligned}
\mathrm{E}^0[\Pi_\varepsilon(n)] &\le \mathrm{E}^0[\mathcal{H}_0^{\mathrm{out},n;\,W=0}(0)] \exp\{-\mu A^\beta T w n \varepsilon^{-\beta}\} \\
&\le \xi^n \exp\{-\mu A^\beta T w n \varepsilon^{-\beta}\} \\
&\le \exp\left\{n\left(\log(\xi) - \frac{K}{\varepsilon^\beta}\right)\right\},
\end{aligned}$$

where K is a positive constant and in the second inequality we used the following result of Corollary 3.1:

$$\mathrm{E}^0[\mathcal{H}_0^{\mathrm{out},n;\,W=0}(0)] = h^{\mathrm{out},n;\,W=0} = h^{\mathrm{in},k,\,W=0} \le \xi^k.$$

This shows (4.13) for small enough ε, and, thus, concludes the proof.

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Proof of Corollary 3.5. Without loss of generality, assume that $x = 0$. We use the left inequality in (3.1), (4.1), and the inclusion (4.3) to obtain

$$P_{X(0),X(y)}(0) \geq L_{X(0)}(0) \geq \mathcal{T}_{X(0)}(0) \geq \widehat{\mathcal{T}}_{X(0)}(0),$$

and as a consequence,

$$p(0, y, \Phi) \geq \mathrm{E}[\widehat{\mathcal{T}}_{X(0)}(0) \mid \Phi].$$

Using the isotropy and the strong Markov property of the Poisson PP,

$$\mathrm{E}[\widehat{\mathcal{T}}_{X(0)}(0) \mid \Phi] = \mathrm{E}^0[\widehat{\mathcal{T}}_0(0) \mid \Phi|_{\overline{B}}],$$

where $\Phi|_{\overline{B}}$ is the restriction of Φ to the complement of the open ball $B = B_{(0,R)}(R)$, centered at $(0, R)$ of radius $R \geq 0$, where R is an RV independent of Φ and having density

$$\frac{\mathrm{d}\theta}{2\pi} 2\pi\lambda r \exp\{-\lambda\pi r^2\}.$$

However, since we consider here the SNR graph $\widehat{\mathbb{G}}$,

$$\mathrm{E}^0[\widehat{\mathcal{T}}_0(0) \mid \Phi|_{\overline{B}}] \geq \mathrm{E}^0[\widehat{\mathcal{T}}_0(0) \mid \Phi].$$

The result now follows from Lemma 4.1.

5. SINR space–time graph and routeing

Let us now translate our results regarding the SINR graph into properties of routeing in ad-hoc networks.

Firstly, it makes sense to assume that any routeing algorithm builds paths on $\mathbb{G}$. This takes two key phenomena into account: contention for the channel (nodes have to wait for some particular time slots to transmit a packet) and collisions (lack of capture due to insufficient SINR).

Our time constant gives bounds on the delays that can be attained in the ad-hoc network by any routeing algorithms. Of course, realistic routeing policies cannot use information about future channel conditions. In the case of the Poisson PP there is hence no routeing algorithm with a finite time constant. The existence of such an algorithm in the case of the Poisson+grid PP remains an open question. In the Poisson PP case, we can ask about the exact asymptotics of the optimal delay (we know it is not linear) and about the delay realizable by some nonanticipating algorithm.

Let us now discuss the relation of our results to those obtained in [11] and [14]. In these papers the so-called delay-tolerant networks are considered and modeled by a spatial SINR or SNR graph with no time dimension. In these models, the time constant (defined there as the asymptotic ratio of the graph distance to the Euclidean distance) is announced to be finite, even in the pure Poisson case. The reason for the different performance of these models lies in the fact that they do not take the time required for a successful transmission from a given node into the evaluation of the end-to-end delay. The heavy tailedness of this time (which follows from that of the exit time (cf. Proposition 3.1)) makes the time constant infinite in the space–time Poisson scenario. The reason for the heavy tailedness of the successful transmission time is linked to the so called 'RESTART' algorithm (see, e.g. [1], [15]–[17]). In our case the spatial irregularities in the ad-hoc network play a role similar to that of the file size variability in the RESTART scenario.

References

[1] Asmussen, S. *et al.* (2008). Asymptotic behavior of total times for jobs that must start over if a failure occurs. *Math. Operat. Res.* **33,** 932–944.
[2] Baccelli, F. and Bordenave, C. (2007). The radial spanning tree of a Poisson point process. *Ann. Appl. Prob.* **17,** 305–359.
[3] Baccelli, F., Błaszczyszyn, B. and Mühlethaler, P. (2006). An aloha protocol for multihop mobile wireless networks. *IEEE Trans. Inf. Theory* **52,** 421–436.
[4] Baccelli, F., Błaszczyszyn, B. and Mühlethaler, P. (2009). Stochastic analysis of spatial and opportunistic aloha. *IEEE J. Sel. Areas Commun.* **27,** 1109–1119.
[5] Baccelli, F., Błaszczyszyn, B. and Mühlethaler, P. (2010). Time–space opportunistic routing in wireless ad hoc networks: algorithms and performance optimization by stochastic geometry. *Comput. J.* **53,** 592–609.
[6] Bordenave, C. (2008). Navigation on a Poisson point process. *Ann. Appl. Prob.* **18,** 708–746.
[7] Broadbent, S. R. and Hammersley, J. M. (1957). Percolation processes. I. Crystals and mazes. *Proc. Camb. Phil. Soc.* **53,** 629–641.
[8] Daley, D. J. and Vere-Jones, D. (2008). *An Introduction to the Theory of Point Processes*, Vol. II. Springer, New York.
[9] Dousse, O., Baccelli, F. and Thiran, P. (2005). Impact of interferences on connectivity in ad hoc networks. *IEEE/ACM Trans. Networking* **13,** 425–436.
[10] Dousse, O. *et al.* (2006). Percolation in the signal to interference ratio graph. *J. Appl. Prob.* **43,** 552–562.
[11] Ganti, R. K. and Haenggi, M. (2009). Bounds on information propagation delay in interference-limited ALOHA networks. In *Proc. Workshop Modeling and Optimization in Mobile, Ad Hoc and Wireless Networks*, IEEE Press, pp. 513–519.
[12] Gilbert, E. N. (1961). Random plane networks. *J. SIAM* **9,** 533–543.
[13] Howard, C. D. and Newman, C. M. (1997). Euclidean models of first-passage percolation. *Prob. Theory Relat. Fields* **108,** 153–170.
[14] Jacquet, P., Mans, B., Mühlethaler, P. and Rodolakis, G. (2009). Opportunistic routing in wireless ad hoc networks: upper bounds for the packet propagation speed. *IEEE J. Sel. Areas Commun.* **27,** 1192–1202.
[15] Jelenkovic, P. R. and Tan, J. (2009). Stability of finite population ALOHA with variable packets. Preprint. Available at http://arxiv.org/abs/0902.4481v2.
[16] Jelenković, P. R. and Tan, J. (2007). Can retransmissions of superexponential documents cause subexponential delays? In *Proc. INFOCOM* 2007 (Anchorage, AL, May 2007), pp. 892–900.
[17] Jelenković, P. R. and Tan, J. (2007). Is ALOHA causing power law delays? In *Managing Traffic Performance in Converged Networks* (Lecture Notes Comput. Sci. **4516**), eds L. Mason, T. Drwiega and J. Yan, Springer, Berlin, pp. 1149–1160.
[18] Kingman, J. F. C. (1973). Subadditive ergodic theory. *Ann. Prob.* **1,** 883–909.
[19] Kleinberg, J. (2000). The small-world phenomenon: an algorithmic perspective. In *Proc. 32nd Ann. ACM Symp. Theory of Computing*, ACM, New York, pp. 163–170.
[20] Pimentel, L. P. R. (2006). The time constant and critical probabilities in percolation models. *Electron Commun. Prob.* **11,** 160–167.
[21] Pugh, C. and Shub, M. (1971). Ergodic elements of ergodic actions. *Compositio Math.* **23,** 115–122.
[22] Tse, D. and Viswanath, P. (2005). *Foundamentals of Wireless Communication*. Cambridge University Press.
[23] Vahidi-Asl, M. Q. and Wierman, J. C. (1990). First-passage percolation on the Voronoi tessellation and Delaunay trangulation. In *Random Graphs '87* (Poznań, 1987), eds M. Karośki, J. Jaworski and A. Ruciński, John Wiley, Chichester, pp. 341–359.

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