RINGS OF QUOTIENTS OF GROUP RINGS

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1. Introduction. The *group ring* AG of a group G and a ring A is the ring of all formal sums $\sum_{g \in G} a_g g$ with $a_g \in A$ and with only finitely many non-zero a_g. Elements of A are assumed to commute with the elements of G. In (**2**), Connell characterized or completed the characterization of Artinian, completely reducible and (von Neumann) regular group rings ((**2**) also contains many other basic results). In (3, Appendix 3) Connell used a theorem of Passman (**6**) to characterize semi-prime group rings. Following in the spirit of these investigations, this paper deals with the complete ring of (right) quotients $Q(AG)$ of the group ring AG. It is hoped that the methods used and the results given may be useful in characterizing group rings with maximum condition on right annihilators and complements, at least in the semi-prime case.

We begin this paper with a few general remarks about modules over AG and AH, where H is a subgroup of G. Some injective AG-modules and some essential and rational extensions of AG_{AG} are found. Using these remarks it is seen that, for each normal subgroup H of G, the complete ring of quotients $Q(AG)$ of AG may be regarded as a ring of functions from G/H to an injective hull of AH_{AH}. The multiplication of these functions is examined. When H is central of finite index, $Q(AG)$ is given explicitly in terms of $Q(AH)$. A necessary condition is given for $Q(AG)$ to be Noetherian and it is shown that, when $Q(A)$ is regular, $Q(AG)$ is semi-prime if and only if $Q(A)G$ is. The main result is a characterization, for a large class of groups, of those group rings, whose complete rings of quotients are regular, that is, of those group rings whose singular ideals are zero.

Some basic definitions and results are quoted for the convenience of the reader. The terminology and notation is that of (**3**). Throughout, rings have identity elements and modules are unitary. The *complete ring of* (*right*) *quotients* $Q(R)$ of the ring R may be defined as the bi-commutator of the injective hull I of R_R, and there is a unique copy of $Q(R)_R$ containing R and contained in I given by

$$Q(R) = \{i \in I\colon \text{for all } h \in \mathrm{Hom}_R(I, I),\ h(R) = 0 \text{ implies } h(i) = 0\}.$$

Any ring containing R and contained in $Q(R)$ is called a *ring of* (*right*) *quotients of* R. All rings of quotients of R have the same complete ring of quotients. A module M_R is a *rational extension* of a module N_R if $N \subseteq M$ and for every

Received December 20, 1967. This article is an adaptation of part of the author's doctoral thesis which was written under the direction of Professor J. Lambek.

$m_1 \neq 0$ and m_2 from M there is an $r \in R$ so that $m_1 r \neq 0$ and $m_2 r \in N$. If S is a ring of quotients of R, S_R is a rational extension of R_R. $Q(R)$ is the largest rational extension of R_R, unique up to isomorphism. A right ideal D of R is called *dense* if R is a rational extension of D. M_R is an *essential extension* of N_R if $N \subseteq M$ and for every $0 \neq m \in M$ there is an $r \in R$ with $0 \neq mr \in N$. A right ideal L of R is *large* if R is an essential extension of L. The *injective hull* I of a module M_R is an injective module which is an essential extension of M. Such modules always exist and are unique up to isomorphism.

The (*right*) *singular ideal* $J(R)$ of a ring R is the set of elements of R whose right annihilators are large. This is an ideal. The following conditions on R are equivalent: $J(R) = 0$ and $Q(R)$ is regular.

A *transversal* of a subgroup H of G is a complete set of coset representatives for H in G. A *partially ordered* group G is a partially ordered set, where $g \leqq h$ implies $kgk' \leqq khk'$ for all $k, k' \in G$. The *positive cone* of a partially ordered group G is $\{g \in G: g > 1\}$.

2. Modules over group rings. Since $Q(R)$ can be represented as the bicommutator of the injective hull of R_R, it would be useful to have some information about the injective hull of AG_{AG}. We first consider some generalities about modules over group rings. Throughout, $\mathfrak{M}_R$ will denote the category of unitary right modules over a ring R.

For any subgroup H of G there are two obvious functors from $\mathfrak{M}_{AH}$ to $\mathfrak{M}_{AG}$, namely $\otimes_{AH} AG$ and $\mathrm{Hom}_{AH}(AG, \ )$. It will be seen that in the present case these functors have convenient equivalent forms.

Since ${}_{AH}AG$ is a free module with the elements of a transversal $\{g_i\}$, $i \in \Lambda$, of H as a basis; $M \otimes {}_{AH}AG$, M in $\mathfrak{M}_{AH}$, is AG-isomorphic to the set of functions with finite support from $\{g_i\}$, $i \in \Lambda$, to M. This module will be denoted by $M_H G$ and its elements will be written as sums $m = \sum_{i \in \Lambda} m_i g_i$, where $m_i = m\langle g_i \rangle$. The AG-operation is defined as follows.

2.1. $mg_i g = mhg_k$ and $mg_i a = mag_i$ for $m \in M$, $g \in G$, $a \in A$ and $g_i g = hg_k$ for some $h \in H$.

The isomorphism θ: $M \otimes AG \rightarrow M_H G$ is given by:

2.2. $\theta(\sum m_i \otimes r_i) = \sum m_i r_i$ for $m_i \in M$ and $r_i \in AG$.

In particular, if $M = AH$, then $AH_H G = AG$.

Since ${}_{AH}AG$ is free, standard properties of the tensor product (**3**, p. 131) yield:

2.3. *If M is AH-projective, then $M_H G$ is AG-projective, and if M is AH-flat, then $M_H G$ is AG-flat.*

Similarly, $\mathrm{Hom}_{AG}(AG, M)$ may be represented as the module of all functions from $\{g_i\}$, $i \in \Lambda$, to M. This module will be denoted by $M_H G_\infty$ and its elements as sums. The AG-operation is the same as for $M_H G$. The correspondence takes $f \in \mathrm{Hom}(AG, M)$ to $\sum_{i \in \Lambda} f(g_i^{-1}) g_i$.

2.4. *If M is AH-injective, then M_HG_∞ is AG-injective.*

2.5. *If H is normal, then M_{AH} is an essential extension of N_{AH} if and only if M_HG is an essential extension of N_HG. In particular, the right ideal of AG generated by a large right ideal of AH is large in AG.*

Proof. Sufficiency. Let $m = m_1g_1 + \ldots + m_kg_k \in M_HG$ with $m_1, \ldots, m_k \neq 0$; then there is an $r_1 \in AH$ so that $0 \neq m_1r_1 \in N$. Since H is normal, there is an $r_1' \in H$ so that $g_1r_1' = r_1g_1$. Now consider mr_1' and repeat the process for its first non-zero term. Eventually we have an $r \in AH$ so that $0 \neq mr \in N_HG$. Note that M_HG is, in fact, an essential extension of N_HG as an AH-module.

Necessity. Assume for convenience that $1 \in \{g_i\}$ so that we may identify elements $m \in M$ with $m = m1 \in M_HG$. If M is not an essential extension of N, there is $0 \neq m \in M$ so that for all $s \in AH$ either $ms = 0$ or $ms \notin N$. However, there is an $r \in AG$ so that $0 \neq mr \in N_HG$. This is a contradiction.

Clearly, 2.5 may be generalized, but the present form is all that is required.

2.6. *If H is central, M_HG is a rational extension of N_HG if and only if M is a rational extension of N. In particular, the right ideal of AG generated by a dense right ideal of AH is dense in AG.*

Proof. Sufficiency. If $0 \neq \sum m_ig_i \in M_HG$ and $\sum m_j'g_j \in M_HG$, assume that $m_1 \neq 0$. Then there is an $s \in AH$ so that $m_1s \neq 0$ and $m_j's \in N$ for all j. Thus, since H is central, $\sum m_ig_is \neq 0$ and $\sum m_j'g_js \in N_HG$. The statement is true for M_HG and N_HG considered as AH-modules.

The converse is similar to that of 2.5.

2.7. *If H is normal and I is the injective hull of AH_{AH}, we have $AG \subseteq I_HG \subseteq I_HG_\infty$ as AG-modules, where I_HG is an essential extension of AG and I_HG_∞ is AG-injective. There is a copy of the injective hull of AG_{AG} between I_HG and I_HG_∞.*

We will ask when I_HG is injective and when I_HG_∞ is an essential extension of AG_{AG}.

2.8. *If I is the injective hull of AH_{AH} and H is normal of finite index, then I_HG is the injective hull of AG_{AG}.*

Proof. Since a transversal of H is finite, $I_HG = I_HG_\infty$.

If $H = \{1\}$, we write $I_HG = IG$ and $I_HG_\infty = IG_\infty$. As a special case of 2.8, we have the following well-known observation.

2.9. *If G is finite and I is the injective hull of A_A, then IG is the injective hull of AG_{AG}. In particular, if A is self-injective then so is AG.*

Continuing with $H = \{1\}$, a slight modification of a theorem of Connell (**3**, p. 162, exercise 7) yields:

https://doi.org/10.4153/CJM-1969-094-0 Published online by Cambridge University Press

2.10. *If I is the injective hull of A_A and IG is AG-injective, then G is locally finite.*

Next, if G is infinite, we shall see that IG_∞ is never an essential extension of AG, and the following theorem indicates a submodule of IG_∞ which is in the complement of every copy of the injective hull of AG_{AG} which lies between IG and IG_∞.

2.11. THEOREM. *Let H be an infinite subgroup of G with a transversal $\{g_i\}$, $i \in \Lambda$, so that for all finite subsets Φ of Λ, $\bigcap_{i\in\Phi} g_i^{-1}Hg_i$ is infinite, then a non-zero element of IG_∞ constant on the right cosets of H is not in any essential extension of AG in IG_∞.*

Proof. Let H be such a subgroup and $0 \neq m = \sum_{g\in G} m\langle g\rangle g \in IG_\infty$ be such that for all $i \in \Lambda$, $h \in H$, $m\langle g_i h\rangle = m\langle g_i\rangle = m_i$. Then

$$m = \sum_{i\in\Lambda} \sum_{h\in H} m_i g_i h.$$

If m is in an essential extension of AG, there is an $r \in AG$ so that $0 \neq mr \in AG$. Thus, for some $g \in G$, $(mr)\langle g\rangle \neq 0$; i.e.,

$$\sum_{i\in\Lambda} \sum_{h\in H} m_i r\langle h^{-1}g_i^{-1}g\rangle = \sum_{j=1}^{n} \sum_{h\in H} m_j r\langle h^{-1}g_j^{-1}g\rangle \neq 0,$$

where the support of r meets Hg_j^{-1} for $j = 1, \ldots, n$. Now, if $Hg_j^{-1}g = Hg_j^{-1}g'$ for $j = 1, \ldots, n$, then $(mr)\langle g\rangle = (mr)\langle g'\rangle$. However, this means that $g_j^{-1}g' \in Hg_j^{-1}g$ for $j = 1, \ldots, n$; or $g'g^{-1} \in g_jHg_j^{-1}$ for $j = 1, \ldots, n$. Thus $(mr)\langle g\rangle = (mr)\langle g'\rangle$ if $g'g^{-1} \in \bigcap_{j=1}^{n} g_jHg_j^{-1}$. By assumption, this intersection is infinite. However, $mr \in AG$ has finite support, which is a contradiction.

As a corollary we have the following result.

2.12. *IG_∞ is the injective hull of AG_{AG} if and only if G is finite.*

Every infinite normal subgroup satisfies the conditions of H in 2.11; however, H need not be normal. Consider the restricted symmetric group on an infinite set. The subgroup of permutations fixing a particular object is a non-normal subgroup satisfying the requirement of 2.11.

3. The complete ring of quotients of AG.

Let H be a normal subgroup of G, and I the injective hull of AH_{AH}. We have seen that there are copies of the injective hull of AG_{AG} between I_HG and I_HG_∞. Assume that $I(AG)$ is one of these and that $I_HG_\infty = I(AG) \oplus N$ as an AG-module. The meaning of H, I, $I(AG)$, and N will be fixed throughout the following discussion. There is a canonical, unique copy of $Q(AG)$, the complete ring of quotients of AG, lying in $I(AG)$, given by (**3**, p. 94)

$$Q(AG) = \{i \in I(AG)\colon \text{for all } AG \text{ endomorphisms } h \text{ of } I(AG),\ h(AG) = 0 \text{ implies } h(i) = 0\}.$$

https://doi.org/10.4153/CJM-1969-094-0 Published online by Cambridge University Press

If $R = \mathrm{Hom}_{AG}(I(AG), I(AG))$ and $S = \mathrm{Hom}_{AG}(I_H G_\infty, I_H G_\infty)$, then $R = eSe$, where e is the projection of $I_H G$ onto $I(AG)$. Thus, we have the following result.

3.1. $Q(AG) = \{m \in I(AG)$: for all $s \in S$, $ese(AG) = 0$ implies $ese(m) = 0\} = \{m \in I(AG)$: for all $s \in S$, $s(AG) \in N$ implies $s(m) \in N\}$.

If $T = \mathrm{Hom}_{AH}(I, I)$, then there is a ring injection of T into S given by $\theta: T \to S$, where $\theta(t)(\sum m_i g_i) = \sum t(m_i) g_i$, where $\{g_i\}$, $i \in \Lambda$, is a transversal of H. We assume below that $Q(AH)$ is the copy given by I; $Q(AH) = \{m \in I$: for all $t \in T$, $t(AH) = 0$ implies $t(m) = 0\}$.

3.2. THEOREM. *If $\sum m_i g_i \in Q(AG)$, then either $m_i \in Q(AH)$ for all $i \in \Lambda$ or $m_i \notin Q(AH)$ for infinitely many i.*

Proof. Assume that $\sum m_i g_i \in Q(AG)$ and that all but finitely many m_i are in $Q(AH)$ but that $m_j \notin Q(AH)$. Thus, there is a $t \in T$ so that $t(AH) = 0$ but $t(m_j) \neq 0$. It follows that $\theta(t)(AG) = 0$ but $0 \neq \theta(t)(\sum m_i g_i)$, and this has finite support. By 3.1, $\theta(t)(\sum m_i g_i) \in N$ and non-zero elements of N have infinite support.

3.3. *If H is central, then $Q(AH)_H G \subseteq Q(AG)$.*

Proof. By 2.6, $Q(AH)_H G$ is a rational extension of AG_{AG} contained in $I_H G \subseteq I(AG)$. However, $Q(AG)$ is the unique largest rational extension of AG in $I(AG)$.

3.4. *If H has finite index, then $Q(AG) \subseteq Q(AH)_H G$ and if, in addition, H is central, then $Q(AG) \cong Q(AH)_H G$.*

Proof. The first statement follows from 3.2, and the second from the first and 3.3.

These inclusions are as AG-modules but this can be improved. Assuming only that H is normal, if $\sum q_i g_i$ and $\sum q_j' g_j$ are elements of $Q(AG)_H G_\infty$, then these may be formally multiplied

$$(\sum q_i g_i) * (\sum q_j' g_j) = \left(\sum_{i,j} q_i q_{ij} g_i g_j\right),$$

where $q_{ij} g_i = g_i q_j'$. The associativity of this formal multiplication is clear. Now the formal product is an element of $Q(AH)_H G$ if and only if for each $k \in \Lambda$, the sum $\sum_{g_i g_j \in H g_k} q_i q_{ij}$ has only finitely many non-zero terms.

3.5. THEOREM. *If $q, q' \in Q(AG) \cap Q(AH)_H G_\infty$ and $q * q' \in Q(AH)_H G_\infty$, then either $q * q' \notin Q(AG)$ or $q * q' = qq'$, where the juxtaposition represents multiplication in $Q(AG)$.*

Proof. We note first that the $*$ multiplication and the $Q(AG)$ multiplication coincide when the right factor is in AG since both are just the AG-action in the module $Q(AH)_H G$; see (**3**, p. 94). Since $q' \in Q(AG)$, there is a dense right ideal D of AG (**3**, p. 97) so that for all $d \in D$, $q'd \in AG$. Thus, $(q * q' - qq')d = (q * q')d - q(q'd) = q(q'd) - q(q'd) = 0$. Therefore, $q * q' - qq'$ is annihilated by a dense right ideal of AG; hence if it is in $Q(AG)$ it must be zero.

3.6. *If H is central of finite index, then $Q(AG) \cong Q(A)_H G$ as rings, where $Q(AH)_H G$ has $*$ multiplication. In particular, if G is finite, then $Q(AG) \cong Q(A)G$. For any central H, $Q(AH)_H G \subseteq Q(AG)$ as rings.*

Although 3.5 gives a connection between multiplication in $Q(AG)$ and the $*$ multiplication, we shall see that elements of $Q(AG)$ need not have $*$ products. Improvements may be made by an appropriate choice of $I(AG)$ in IG_∞ (we assume below that $H = \{1\}$). We need the following definition.

3.7. An element $q \in Q(A)G$ has a $*$ *inverse* if there is an $s \in Q(A)G_\infty$ so that $s * q = q * s = 1 \in AG$.

3.8. THEOREM. *Assume that $J(AG) = 0$. If $\{s_i\}$, $i \in \Phi$, are elements of AG with inverses in $Q(AG)$ and $*$ inverses $\{s_i^{-1}\}$, $i \in \Phi$, and if $\{s_i^{-1}\}$, $i \in \Phi$, forms a commutative semigroup under $*$, then there is a copy of $Q(AG)$ in IG_∞ containing $\{s_i^{-1}\}$, $i \in \Phi$, as a sub-semigroup of its multiplicative semigroup.*

Proof. We first note that $\sum_{i\in\Phi} s_i^{-1}AG + AG$ is an essential extension of AG. In any ring R, if $r \in R$ has a right inverse in $Q(R)$, then rR is dense. For, if $rq = 1$, $q \in Q(R)$ and $0 \neq s \in R$, and $t \in R$, there is $u \in R$ so that $qsu \neq 0$ and $qtu \in R$, then $su \neq 0$ and $tr = r(qtu) \in rR$. Now assume that

$$0 \neq t = \sum_{j=1}^{n} s_j^{-1} r_j + r_0 \in \sum_{i\in\Phi} s_i^{-1}AG + AG.$$

By (**3**, p. 96),

$$L = \bigcap_{j=1}^{n} r_j^{-1}(s_j AG)$$

is dense in AG (here $r_j^{-1}(s_j AG) = \{u \in AG\colon r_j u \in s_j AG\}$). If $tL = 0$, then $(s_1 \ldots s_n) * tL = 0$. However, the s_i commute with each other, hence

$$(s_1 \ldots s_n) * t = \sum_{j=1}^{n} s_1 \ldots \hat{s}_j \ldots s_n r_j + s_1 \ldots s_n r_0 \in AG,$$

where $s_1 \ldots \hat{s}_j \ldots s_n$ means the product $s_1 \ldots s_n$ with the factor s_j omitted. Since L is dense, $(s_1 \ldots s_n) * t = 0$ and, consequently, $s_1^{-1} * (s_1 \ldots s_n) * t = (s_2 \ldots s_n) * t = 0$ and, ultimately, $t = 0$, which is a contradiction. Hence, $tL \neq 0$ and by the choice of L, $tL \subseteq AG$.

From this it follows that there is a copy of the injective hull of AG containing $\{s_i^{-1}\}$, $i \in \Phi$, lying in IG_∞. Now each s_i^{-1} is in the copy of $Q(AG)$ in this copy of the injective hull. Let q_i be the ring inverse of s_i in $Q(AG)$; then $(s_i^{-1} - q_i)s_i AG = 0$ and $s_i AG$ is dense (and therefore large). Thus $s_i^{-1} = q_i$ since no non-zero element of an essential extension of a ring with zero singular ideal can annihilate a large right ideal. By 3.5, $s_i^{-1} * s_j^{-1} = s_i^{-1} s_j^{-1}$ for all $i, j \in \Phi$.

The following special case of a theorem of Neumann (**4**, Theorems 3.4 and 3.5) yields many examples.

3.9. *If G is partially ordered and S is a totally ordered sub-semigroup of G not containing* 1, *then each element of the form* $1 - s$, *where* $s \in AG$ *and the support of s is in S, has a* $*$ *inverse* $(1 - s)^{-1}$ *and these* $*$ *inverses form a semigroup under* $*$; *the inverses are, in fact, in* AG_∞.

3.10. *Assume that G is partially ordered and S is a central totally ordered sub-semigroup of the positive cone of G. Let C be the centre of a ring A. Then* $T = \{1 - s : s \in CG\}$, *with support in S, satisfies the condition of* 3.8 *if* $J(AG) = 0$.

Proof. By 3.9, elements of T have $*$ inverses and these form a commutative semigroup under $*$. We need only show that $1 - s \in T$ has an inverse in $Q(AG)$. It suffices to show that $1 - s$ is a central non-zero-divisor in AG. However, $1 - s$ is central and if $0 \neq x \in AG$, then let g be minimal in the support of x and

$$((1 - s)x)\langle g\rangle = x\langle g\rangle - \sum_{hk=g} s\langle h\rangle x\langle k\rangle = x\langle g\rangle$$

since the support of s is strictly positive and well-ordered. Similarly, $x(1 - s) \neq 0$.

As an example where $*$ is not the multiplication in $Q(AG)$, take A to be a field and $g \in G$ central of infinite order, then $1 - g$ and $1 - g^2$ have inverses in $Q(AG)$. However, $1 - g$ and $1 - g^2$ have $*$ inverses $1 + g + g^2 + \ldots$ and $-g^{-2} - g^{-4} - \ldots$, respectively. By the proof of 3.8, there is a copy of $Q(AG)$ which contains the two $*$ inverses but these clearly cannot be $*$ multiplied.

4. Properties of $Q(AG)$. In this section we attempt to find necessary and sufficient conditions on A and G so that $Q(AG)$ has certain properties. The representation of $Q(AG)$ as a submodule of IG_∞, where I is the injective hull of A_A will be used throughout; elements of IG_∞ will be thought of as functions or formal sums, whichever is more convenient.

4.1. *If H is a locally finite subgroup of G, then* $1 - g \in \sum_{h \in H} (1 - h)Q(AG)$ *if and only if* $g \in H$; *and* $1 - g \in \sum_{h \in H} Q(AG)(1 - h)$ *if and only if* $g \in H$.

Proof. If $1 - g = \sum_{i=1}^{n} q_i(1 - h_i)$, $\{h_1, \ldots, h_n\}$, a subgroup of H, then $(1 - g)(h_1 + \ldots + h_n) = 0$, and hence $g \in H$. The other statement follows similarly because of 3.5.

This is not true for all subgroups; take $g \in G$ central of infinite order, then $1 - g$ is invertible in $Q(AG)$.

Thus, the partially ordered set of locally finite subgroups of G is mapped order monomorphically into the lattice of right (or left) ideals of $Q(AG)$ by assigning to the locally finite subgroup H the right ideal $\sum_{h\in H} (1 - h)Q(AG)$ (the left ideal $\sum_{h\in H} Q(AG)(1 - h)$). Consequently, we have the following result.

4.2. *If $Q(AG)$ is right (left) Noetherian, then G has the ascending chain condition on finite subgroups.*

In general, $q \in Q(AG)$ may have values in I but not in $Q(A)$ so that it is especially difficult to study left multiplication by elements of AG. This problem is eased if $Q(A) = I$ and this occurs if the (right) singular ideal of A, $J(A)$, is zero.

4.5. *If $J(A) = 0$, then $Q(AG)$ left Noetherian (Artinian) implies that A is left Noetherian (Artinian). Assuming that A is commutative and $J(A) = 0$, if $Q(AG)$ has maximum (minimum) condition on ideals, so does A. In any of these cases, $Q(A)$ is completely reducible.*

Proof. This follows from the observation that, if an element $a \in Q(A)$ is representable in the form

$$a = \sum_{i=1}^{n} q_i b_i, \qquad q_i \in Q(AG), \quad b_i \in Q(A),$$

there are elements $a_i \in Q(A)$ such that $a = \sum a_i b_i$. Simply take a_i to be the value of q_i at $1 \in G$.

If $B_1 \subseteq B_2 \subseteq \ldots$ is a chain of left ideals of A, consider

$$Q(AG)B_1 \subseteq Q(AG)B_2 \subseteq \ldots .$$

The chain in $Q(AG)$ terminates at stage n, say, and the remark yields: $B_n = B_{n+1}$; similarly if $Q(AG)$ is Artinian. The second statement follows in the same fashion. For the third statement, $Q(A)$ has a chain condition, as above (using 3.6), and $Q(A)$ is regular since $J(A) = 0$.

4.4. *If $J(A) = 0$, $Q(AG)$ semi-prime implies that the order of every finite normal subgroup of G is a non-zero-divisor in $Q(A)$. That is, when $J(A) = 0$, $Q(AG)$ is semi-prime if and only if $Q(A)G$ is semi-prime.*

Proof. Let $H = \{h_1, \ldots, h_n\}$ be a normal subgroup of G and assume that $na = 0$ for some $0 \neq a \in Q(A)$. We wish to show that

$$a(h_1 + \ldots + h_n)Q(AG)a(h_1 + \ldots + h_n) = 0,$$

and by 3.5, it suffices to show that

$$a(h_1 + \ldots + h_n) * Q(AG)a(h_1 + \ldots + h_n) = 0.$$

For $q \in Q(AG)$,

$$a(h_1 + \ldots + h_n) * qa(h_1 + \ldots + h_n) = \sum_{g\in G} \sum_{i,j=1}^{n} aq\langle h_i^{-1} g h_j \rangle ag.$$

However, for each $j = 1, \ldots, n$,

$$\{h_i^{-1}gh_j;\ i = 1, \ldots, n\} = \{gh_k\colon k = 1, \ldots, n\}$$

since H is normal. Therefore,

$$\sum_{i,j=1}^{n} aq\langle h_i^{-1}gh_j\rangle a = n \sum_{k=1}^{n} aq\langle gh_k\rangle a = 0.$$

In any ring R, R semi-prime implies $Q(R)$ semi-prime. Thus, the second statement is an immediate consequence of the characterization of semi-prime group rings (Connell-Passman in (**3**, Appendices 2 and 3)) as those AG, where A is semi-prime and the order of every finite normal subgroup is a non-zero-divisor in A.

Next, we look for necessary and sufficient conditions so that $Q(AG)$ be regular, or, equivalently, that $J(AG) = 0$.

If H is a subgroup of G, there is a mapping ϕ of AG to AH defined by $\phi(r) = r/H$, r restricted to H. In terms of sums, $\phi(\sum_{g\in G} a_g g) = \sum_{g\in H} a_g g$.

4.5. *ϕ is an AH-AH-homomorphism.*

Proof. If $r, r' \in AH$ and $s \in AG$, then for $g \in H$,

$$\begin{aligned}\phi(rsr')\langle g\rangle &= \textstyle\sum r\langle h\rangle s\langle k\rangle r'\langle h'\rangle \qquad (h, h' \in H, k \in G, \text{ and } hkh' = g)\\ &= \textstyle\sum r\langle h\rangle \phi(s)\langle k\rangle r'\langle h'\rangle \qquad (h, k, h' \in H \text{ and } hkh' = g)\\ &= (r\phi(s)r')\langle g\rangle.\end{aligned}$$

4.6. *ϕ preserves large and dense left and right ideals.*

Proof. The statement will be proved for dense right ideals, the others being similar. If D is dense in AG, $0 \neq r \in AH$, $r' \in AH$, there is a $u \in AG$ so that $ru \neq 0$ and $r'u \in D$; u may be chosen so that $\phi(ru) \neq 0$. Then $r\phi(u) = \phi(ru) \neq 0$ and $r'\phi(u) = \phi(r'u) \in \phi D$.

4.7. *If H is a subgroup of G, then $J(AH) = 0$ implies $J(AG) \cap AH = 0$.*

Proof. If $r \in AH$ and L is large in AG with $rL = 0$, then $\phi(rL) = r\phi L = 0$, hence $r = 0$.

A necessary condition that $J(AG) = 0$ can be given immediately.

4.8. THEOREM. *If $J(AG) = 0$, then $J(A) = 0$ and the order of every finite normal subgroup of G is a non-zero-divisor in A.*

Proof. More generally, if H is normal in G, then $J(AG) = 0$ implies $J(AH) = 0$; for, if $r \in AG$ and $rL = 0$, where L is a large right ideal of AG, then $rLAG = 0$ and by 2.5, LAG is large in AG. Thus $r = 0$. Furthermore, if $J(AG) = 0$, then $Q(AG)$ is regular and hence semi-prime. Then, by 4.4, the order of every finite normal subgroup is a non-zero-divisor in $Q(A)$, and in A.

We next show that for a large class of groups, the condition of 4.8 is also sufficient. A *locally normal group* is one in which every finite subset is contained in a finite normal subgroup. If R is regular, then $J(R) = 0$.

4.9. *If $J(A) = 0$, G is locally normal, and the order of every finite normal subgroup of G is a non-zero-divisor in A, then $J(AG) = 0$.*

Proof. Auslander, McLaughlin, and Connell (**2**; **3**) have characterized regular group rings as those AG where A is regular and the order of every finite subgroup of G is a unit in A. Note that a central non-zero-divisor in any ring is a unit in its complete ring of quotients (**3**, p. 96). Thus, we see that $Q(A)G$ is regular, and hence $J(Q(A)G) = 0$. However, by 3.3, $Q(A)G$ is a ring of quotients of AG so that $Q(A)G$ and AG have the same complete ring of quotients and $J(AG) = 0$.

4.10. *If G has a normal subgroup H so that G/H is totally ordered, then $J(AG) = 0$ if $J(AH) = 0$.*

Proof. Following Connell (**3**, Appendix 3), we order a transversal $\{g_i\}$, $i \in \Lambda$, of H and note that there is an AH-homomorphism from AG to AH given by representing $r \in AG$ by $\sum_{i\in\Lambda} r_i g_i$, $r_i \in AH$, and defining $\psi(r) = r_j$, where g_j is the least g_i so that $r_i \neq 0$; if $r = 0$, then $\psi(r)$ is defined as zero. Now, ψ preserves large (also dense) left and right ideals. For, if L is large in AG, assume that $\psi L \cap B = 0$ for some right ideal B of AH. However, BAG consists of elements of the form $\sum r_i g_i$ with $r_i \in B$. Thus, if $r \in L \cap BAG$, $\psi(r) \in \psi L \cap B = 0$, hence $\psi(r) = 0$ and $r = 0$.

Assume that $J(AG) \neq 0$; then for some large L there is a $0 \neq r \in AG$ with $rL = 0$. Thus, for any $s \in L$, $rs = 0$. Let $r = \sum r_i g_i$, $s = \sum s_j g_j$ with $\psi(r) = r_p$ and $\psi(s) = s_q$. By the ordering of the transversal, $r_p g_p s_q g_q = 0$, and $r_p g_p \psi(s) = 0$ for any $s \in L$. It follows that $r_p g_p \psi L = 0$ and if $r_p g_p = g_p r_p'$ (r_p' exists since H is normal), then $r_p' L = 0$. However, $r_p' \neq 0$ if and only if $r_p \neq 0$; hence $r_p' \neq 0$, contradicting the assumption that $J(AH) = 0$.

Summarizing, we have the following result.

4.11. Theorem. *If G has a normal, locally normal subgroup H so that G/H is totally ordered, then $J(AG) = 0$ if and only if $J(A) = 0$ and the order of every finite normal subgroup of G is a non-zero-divisor in A.*

All totally ordered groups are in this class of groups and in (**5**) it is shown that all FC groups are in this class also (an FC group is one in which each element has only finitely many distinct conjugates). One would conjecture that the theorem holds for all groups. The development given above is analogous to that of the characterization of semi-prime group rings; all the steps are analogous and the only one missing in the present case is the analogue of the Passman theorem (**3**, p. 158).

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