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On Russell-Type Modular Equations

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Abstract. In this paper, we revisit Russell-type modular equations, a collection of modular equations first studied systematically by R. Russell in 1887. We give a proof of Russell's main theorem and indicate the relations between such equations and the constructions of Hilbert class fields of imaginary quadratic fields. Motivated by Russell's theorem, we state and prove its cubic analogue which allows us to construct Russell-type modular equations in the theory of signature 3.

1 Introduction

Let $(a)_k = a \cdot (a+1) \cdots (a+k-1)$ and define the *ordinary hypergeometric series*

$$ {}_2F_1(a, b; c; z) := \sum_{k=0}^{\infty} \frac{(a)_k (b)_k}{(c)_k} \frac{z^k}{k!}, \quad |z| < 1. $$

We say that the *modulus* β has degree n, $n \in \mathbb{N}$, over the *modulus* α when

$$ \frac{{}_2F_1\left(\frac12, \frac12; 1; 1-\beta\right)}{{}_2F_1\left(\frac12, \frac12; 1; \beta\right)} = n \frac{{}_2F_1\left(\frac12, \frac12; 1; 1-\alpha\right)}{{}_2F_1\left(\frac12, \frac12; 1; \alpha\right)}. \tag{1.1} $$

A *modular equation of degree n* is a relation between α and β which is induced by (1.1).

In general, for a given $n > 1$, there exist more than one modular equation of degree n and they appear in various forms. For example, the relations

$$ (\alpha\beta)^{1/4} + \{(1-\alpha)(1-\beta)\}^{1/4} = 1, \tag{1.2} $$

$$ \{\alpha(1-\beta)\}^{1/2} + \{\beta(1-\alpha)\}^{1/2} = 2\{\alpha\beta(1-\alpha)(1-\beta)\}^{1/8}, $$

and

$$ (\alpha\beta^5)^{1/8} + \{(1-\alpha)(1-\beta)^5\}^{1/8} = (\alpha^5\beta)^{1/8} + \{(1-\alpha)^5(1-\beta)\}^{1/8} $$

are all modular equations of degree 3 (see [1, pp. 230–231]). As a result, in order to understand the constructions of modular equations, it is necessary first to classify these modular equations (see [2, Chapter 36]) and investigate these classes separately. The purpose of this paper is to study modular equations of prime degree p similar to (1.2). In other words, we will be interested in modular equations which involve only

$$ (\alpha\beta)^{1/8} \quad \text{and} \quad \{(1-\alpha)(1-\beta)\}^{1/8}. $$

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These modular equations were first studied systematically by R. Russell [17], and we shall refer to them as *Russell-type modular equations*.

In the aforementioned paper, Russell devised an effective way of computing Russell-type modular equations of prime degrees and constructed many new modular equations. However, his proof of this general method lacks of detail and is incomplete. In Section 2, we will state and prove Russell's result. We will also indicate the relations between Russell-type modular equations and the Hilbert class fields of the imaginary quadratic fields.

We say that the *modulus* β^* has degree n over the *modulus* α^* in the theory of signature 3 when

$$\frac{{}_2F_1\left(\frac{1}{3},\frac{2}{3};1;1-\beta^*\right)}{{}_2F_1\left(\frac{1}{3},\frac{2}{3};1;\beta^*\right)} = n\frac{{}_2F_1\left(\frac{1}{3},\frac{2}{3};1;1-\alpha^*\right)}{{}_2F_1\left(\frac{1}{3},\frac{2}{3};1;\alpha^*\right)}. \tag{1.3}$$

A *modular equation of degree n* in the theory of signature 3 is a relation between α^* and β^* which is induced by (1.3). The first few modular equations in the theory of signature 3 are recorded by S. Ramanujan and they have been proved recently by B. C. Berndt, S. Bhargava, and F. G. Garvan [3]. Among these modular equations, there are three which involve only

$$(\alpha^*\beta^*)^{1/6} \quad \text{and} \quad \{(1-\alpha^*)(1-\beta^*)\}^{1/6}.$$

These are

$$(\alpha^*\beta^*)^{1/3} + \{(1-\alpha^*)(1-\beta^*)\}^{1/3} = 1, \tag{1.4}$$

$$(\alpha^*\beta^*)^{1/3} + \{(1-\alpha^*)(1-\beta^*)\}^{1/3} + 3\{\alpha^*\beta^*(1-\alpha^*)(1-\beta^*)\}^{1/6} = 1, \tag{1.5}$$

and

$$\begin{aligned}(\alpha^*\beta^*)^{1/3} &+ \{(1-\alpha^*)(1-\beta^*)\}^{1/3} + 6\{\alpha^*\beta^*(1-\alpha^*)(1-\beta^*)\}^{1/6} \\ &+ 3\sqrt{3}\{\alpha^*\beta^*(1-\alpha^*)(1-\beta^*)\}^{1/12}\{(\alpha^*\beta^*)^{1/6} + \{(1-\alpha^*)(1-\beta^*)\}^{1/6}\} = 1,\end{aligned} \tag{1.6}$$

which are of degrees 2, 5, and 11, respectively. These modular equations are clearly the cubic analogues of the Russell-type modular equations and we shall refer to them as *cubic Russell-type modular equations*. In Section 3, we will state and prove a cubic analogue of Russell's theorem. We will then apply our result to construct new cubic Russell-type modular equations, which are used to evaluate some cubic singular moduli in Section 4.

2 Russell-Type Modular Equations and Their Applications

Theorem 2.1 **(Russell)** *Let p be an odd prime and $(p+1)/8 = n/s$ in lowest terms. Suppose β has degree p over α, then the relation between*

$$x = (\alpha\beta)^{s/8} \quad \textit{and} \quad y = \{(1-\alpha)(1-\beta)\}^{s/8}$$

can be given in the form

$$A_0(y)x^n + A_1(y)x^{n-1} + \cdots + A_n(y) = 0,$$

where $A_0(y), \ldots, A_n(y)$ are polynomials of degrees at most n in y.

To prove Theorem 2.1, we first recall that [1, p. 101, Entry 6, and pp. 122–123, Entries 10 and 11] when

$$q = \exp\left(-\pi \frac{{}_2F_1\left(\frac{1}{2}, \frac{1}{2}; 1; 1-\alpha\right)}{{}_2F_1\left(\frac{1}{2}, \frac{1}{2}; 1; \alpha\right)}\right),$$

x and y can be expressed in terms of classical theta functions (see Theorem 2.2). It turns out that if

$$\varphi(q) := \sum_{n=-\infty}^{\infty} q^{n^2} \quad \text{and} \quad \psi(q) := \sum_{n=0}^{\infty} q^{n(n+1)/2},$$

then Theorem 2.1 follows immediately from

Theorem 2.2 *Let p be an odd prime and $(p+1)/8 = n/s$ in lowest terms and $q = e^{\pi i \tau}$, $\operatorname{Im}\tau > 0$. Then the relation between*

$$x(\tau) = \left(4q^{(p+1)/4} \frac{\psi(q^2)\psi(q^{2p})}{\varphi(q)\varphi(q^p)}\right)^{s/2}$$

and

$$y(\tau) = \left(\frac{\varphi(-q)\varphi(-q^p)}{\varphi(q)\varphi(q^p)}\right)^{s/2}$$

can be given in the form

$$A_0(y)x^n + A_1(y)x^{n-1} + \cdots + A_n(y) = 0,$$

where $A_0(y), \ldots, A_n(y)$ are polynomials of degrees at most n in y.

To prove Theorem 2.2, we need two lemmas.

Lemma 2.3 *Let*

$$W_p := \begin{pmatrix} e\sqrt{p} & 2f/\sqrt{p} \\ 2g\sqrt{p} & h\sqrt{p} \end{pmatrix},$$

where $peh - 4fg = 1$. Set

$$\Gamma := \langle \Gamma(2) \cap \Gamma_0(p), W_p \rangle.$$

Then the functions $x(\tau)$ and $y(\tau)$ are modular functions invariant under Γ, i.e., $x(\tau),\ y(\tau) \in M(\Gamma, 0, 1)$.

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Proof Set $g_0(\tau) = \varphi(-q)$, $g_1(\tau) = \varphi(q)$, and $g_2(\tau) = q^{1/4}\psi(q^2)$, with $q = e^{\pi i\tau}$. Suppose $A = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma(2)$. Then the multiplier systems associated with g_0, g_1, and g_2 are given by [1, p. 331]

$$v_{g_0}(A) = \left(\frac{c}{d}\right)_* e^{\pi i(d-1)/4} e^{-\pi icd/4},$$
$$v_{g_1}(A) = \left(\frac{c}{d}\right)_* e^{\pi i(d-1)/4},$$

and

$$v_{g_2}(A) = \left(\frac{c}{d}\right)_* e^{\pi i(d-1)/4} e^{\pi ibd/4},$$

where for $(c, d) = 1$, d odd, and $c \neq 0$,

$$\left(\frac{c}{d}\right)_* := \begin{cases} -\left(\frac{c}{|d|}\right), & \text{if } c < 0, d < 0, \\ \left(\frac{c}{|d|}\right), & \text{otherwise,} \end{cases}$$

the right hand side being the Legendre-Jacobi symbols. Also,

$$\left(\frac{0}{-1}\right)_* := -1.$$

Let $B = \left(\begin{smallmatrix} a & 2b \\ 2pc & d \end{smallmatrix}\right) \in \Gamma(2) \cap \Gamma_0(p)$, where a and d are odd and $ad - 4pbc = 1$.

Now,

$$x(B\tau) = \left(\frac{\left(\frac{2pc}{d}\right)_* e^{\pi i(d-1)/4} e^{2\pi ibd/4} \left(\frac{2c}{d}\right)_* e^{\pi i(d-1)/4} e^{2\pi ipbd/4}}{\left(\frac{2pc}{d}\right)_* e^{\pi i(d-1)/4} \left(\frac{2c}{d}\right)_* e^{\pi i(d-1)/4}} \right)^{s/2} x(\tau).$$

Since $(2bd + 2pbd)/4 \cdot (s/2) = 2nbd$, we find that the multiplier system v_x associated with $x(\tau)$ is 1. Clearly, $x(\tau)$ has weight 0. Therefore, $x(\tau)$ is invariant under $\Gamma(2) \cap \Gamma_0(p)$.

Next, set

$$\mu(\tau) = \frac{g_2(\tau)}{g_1(\tau)}.$$

Note that the action on τ by W_p is the same as $\left(\begin{smallmatrix} e & 2f \\ 2g & ph \end{smallmatrix}\right)\left(\begin{smallmatrix} p & 0 \\ 0 & 1 \end{smallmatrix}\right)$. Therefore,

$$\begin{aligned} x(W_p\tau) &= \big(4\mu(W_p\tau)\mu(pW_p\tau)\big)^{s/2} \\ &= \left(4\mu\left(\begin{pmatrix} e & 2f \\ 2g & ph \end{pmatrix} p\tau\right)\mu\left(p\frac{pe\tau+2f}{2gp\tau+ph}\right)\right)^{s/2} \\ &= \left(4\mu\left(\begin{pmatrix} e & 2f \\ 2g & ph \end{pmatrix} p\tau\right)\mu\left(\begin{pmatrix} pe & 2f \\ 2g & h \end{pmatrix}\tau\right)\right)^{s/2} \\ &= \left(\frac{\left(\frac{2g}{ph}\right)_* e^{\pi i(ph-1)/4}e^{2\pi i pfh/4}\left(\frac{2g}{h}\right)_* e^{\pi i(h-1)/4}e^{2\pi i fh/4}}{\left(\frac{2g}{ph}\right)_* e^{\pi i(ph-1)/4}\left(\frac{2g}{h}\right)_* e^{\pi i(h-1)/4}}\right)^{s/2} x(\tau) \\ &= x(\tau), \end{aligned}$$

since both

$$\begin{pmatrix} e & 2f \\ 2g & ph \end{pmatrix} \quad\text{and}\quad \begin{pmatrix} pe & 2f \\ 2g & h \end{pmatrix}$$

belong to $\Gamma(2)$ and $2sfh(p+1)/8 = 2fhn$. Hence, $x(\tau) \in M(\Gamma, 0, 1)$. A similar argument gives $y(\tau) \in M(\Gamma, 0, 1)$. ■

Lemma 2.4 *Let G be a real discrete group. Let f_1 and f_2 be non-constant functions in $M(G,0,1)$ with q_1 and q_2 poles, respectively, in a fundamental region of G. Then there exists an irreducible polynomial $\Phi(x_1,x_2)$ of degrees at most q_2 and q_1 in x_1 and x_2, respectively, and with complex coefficients not all of which are zero, such that $\Phi(f_1,f_2)=0$ identically.*

The proof of Lemma 2.4 can be found in [13, Section 2H, p. 90].

Proof of Theorem 2.2 In view of Lemma 2.4, it suffices to determine the total number of poles of $x(\tau)$ and $y(\tau)$. Now, $x(\tau)$ and $y(\tau)$ are analytic on the upper half plane $\operatorname{Im}\tau > 0$ with possible poles or zeros at the cusps associated with $\Gamma = \langle \Gamma(2)\cap\Gamma_0(p), W_p\rangle$. Now, a complete set of representatives of inequivalent cusps associated with $\Gamma(2)\cap\Gamma_0(p)$ is given by [1, p. 370]

$$\mathcal{C} := \left\{\frac{1}{0}, \frac{1}{2}, \frac{1}{1}, \frac{1}{p}, \frac{2}{p}, \frac{0}{1}\right\}.$$

Under W_p, the elements in $\mathcal{C}$ are grouped in pairs, namely,

$$\begin{aligned} W_p\left(\frac{0}{1}\right) &\sim \frac{2}{p} \\ W_p\left(\frac{1}{1}\right) &\sim \frac{1}{p} \qquad (2.1) \\ W_p\left(\frac{1}{2}\right) &\sim \frac{1}{0}. \end{aligned}$$

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Hence, there are exactly three cusps associated with Γ, say

$$\mathcal{C}' := \left\{ \frac{0}{1}, \frac{1}{1}, \frac{1}{2} \right\}.$$

In the following table, we indicate $24 \cdot \operatorname{ord}(F;\zeta)$ for each of the functions g_i, where $i = 0, 1, 2$ and $\zeta \in \mathcal{C}'$. The width $N(\Gamma;\zeta)$ is defined as the smallest positive integer k such that

$$\sigma \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}^k \sigma^{-1} \in \Gamma,$$

where $\sigma \in \mathrm{SL}_2(\mathbb{R})$ such that $\sigma(\zeta) = \frac{1}{0}$.

Function $F \setminus$ Cusp ζ	$\frac{0}{1}$	$\frac{1}{1}$	$\frac{1}{2}$
$g_0(\tau)$	3	0	0
$g_1(\tau)$	0	3	0
$g_2(\tau)$	0	0	3
$g_0(p\tau)$	$3/p$	0	0
$g_1(p\tau)$	0	$3/p$	0
$g_2(p\tau)$	0	0	$3/p$
width $N(\Gamma;\zeta)$	$2p$	$2p$	$2p$

The above table shows that $x(\tau)$ has a pole at $\frac{0}{1}$ in the fundamental region associated with Γ and the number of poles are given by

$$\frac{2p}{24}\left(\frac{3(p+1)}{p}\right)\frac{s}{2} = n.$$

Similarly, $y(\tau)$ has a total of n poles. Hence, we conclude that the relation between $x(\tau)$ and $y(\tau)$ is of the form

$$A_0(x)y^n + A_1(x)y^{n-1} + \cdots + A_n(x) = 0,$$

where $A_0(x), \ldots, A_n(x)$ are polynomials of degrees at most n in x. This completes the proof of Theorem 2.2. ∎

Remarks One can show that $A_0(x)$ is a constant and that the degrees of $A_i(x)$ are at most i, for $0 \le i \le n$. For more details, refer to [17, pp. 93–96]. Examples of Russell-type modular equations can also be found in [17].

Recall [9, p. 224] that the *modular j-invariant* $j(\tau)$, for $\tau \in \mathbb{H} := \{\tau : \operatorname{Im}\tau > 0\}$, is defined by

$$j(\tau) = 1728\frac{g_2^3(\tau)}{\Delta(\tau)},$$

where

$$\Delta(\tau) = g_2^3(\tau) - 27g_3^2(\tau),$$

$$g_2(\tau) = 60 \sum_{\substack{m,n=-\infty \\ (m,n)\neq(0,0)}}^{\infty} (m\tau + n)^{-4},$$

and

$$g_3(\tau) = 140 \sum_{\substack{m,n=-\infty \\ (m,n)\neq(0,0)}}^{\infty} (m\tau + n)^{-6}.$$

Furthermore, the function $\gamma_2(\tau)$ is defined by [9, p. 249]

$$\gamma_2(\tau) = \sqrt[3]{j(\tau)},$$

where that branch is real when τ is purely imaginary is chosen.

In [10], Greenhill obtained, using Russell's modular equations, minimal polynomials for $\gamma_2(\tau)$ at

$$\tau = \tau_p := \begin{cases} \frac{3+\sqrt{-p}}{2}, & \text{if } p \equiv 3 \pmod 4, \\ \sqrt{-p}, & \text{if } p \equiv 1 \pmod 4, \end{cases}$$

for many values of p. Some of these polynomials have recently been used to verify certain infinite products in Ramanujan's Notebooks [4] and this is what motivates the authors to re-examine Russell's theorem.

It is known [9, p. 249, Theorem 12.2] that $\gamma_2(\tau_p)$ generates the Hilbert class field of $\mathbb{Q}(\sqrt{-p})$, for a prime p. Hence, the splitting field of Greenhill's polynomial associated with τ_p coincides with the Hilbert class field of $\mathbb{Q}(\sqrt{-p})$. However, to obtain Greenhill's polynomials from Russell's modular equations can be very tedious. Furthermore, these polynomials usually involve large coefficients.

For $|q| < 1$ let

$$(a;q)_\infty = \prod_{n=0}^{\infty}(1 - aq^n)$$

and

$$\chi(q) = (-q;q^2)_\infty.$$

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Ramanujan first introduced the class invariant

$$G_n = 2^{-1/4} e^{\pi\sqrt{n}/24} \chi(e^{-\pi\sqrt{n}})$$

in his famous paper "Modular equations and approximations to π" [16], [15, pp. 23–39]. Ramanujan's class invariant differs from Weber's class invariant by a factor of $2^{-1/4}$. In [19], Weber asserted without proof that a certain small power (depending on n) of G_n generates the Hilbert class field $K_n^{(1)}$ of $K_n = \mathbb{Q}(\sqrt{-n})$. These assertions were later used by Heegner to give a proof of Gauss' class number one problem. Heegner's proof was initially regarded as unacceptable since proofs of Weber's assertions were not available. Motivated by this, B. J. Birch [6] proved Weber's assertions using Söhngen's theorem [18] and showed that if $p > 3$ is a prime number, then:

1. If $p \equiv 1 \pmod 8$, then G_p^2 is a unit which generates $K_p^{(1)}$ over K_p.
2. If $p \equiv 5 \pmod 8$, then G_p^4 is a unit which generates $K_p^{(1)}$ over K_p.
3. If $p \equiv 7 \pmod 8$, then $2^{-3/4} G_p^3$ is a unit which generates $K_p^{(1)}$ over K_p.

It is known that [5, Eq. (1.1)]

$$G_n = \{4\alpha_n(1-\alpha_n)\}^{-1/24}, \tag{2.2}$$

where the singular modulus α_n is the unique number satisfying the relation

$$\frac{{}_2F_1\left(\frac{1}{2}, \frac{1}{2}; 1; 1-\alpha_n\right)}{{}_2F_1\left(\frac{1}{2}, \frac{1}{2}; 1; \alpha_n\right)} = \sqrt{n}. \tag{2.3}$$

From (2.3), we see that α_n is the unique number α satisfying (1.1) with $\beta = 1 - \alpha$. This means that if we set $\alpha = \alpha_n$ and $\beta = 1 - \alpha_n$ in a Russell-type modular equation of degree n, we obtain immediately a polynomial satisfied by $\{\alpha_n(1-\alpha_n)\}^{1/4} = 2^{-1/2} G_n^{-6}$. This gives us a way to obtain the minimal polynomial for some power of G_n, which in turn gives us a minimal polynomial for the generator of the Hilbert class field of $\mathbb{Q}(\sqrt{-n})$. As a result, Russell's theorem provides us (in some sense) a general way to determine the Hilbert class field of $\mathbb{Q}(\sqrt{-p})$ for $p \not\equiv 3 \pmod 8$. We end this section with some examples.

Examples 1. Suppose $p = 5$. In this case, $K = \mathbb{Q}(\sqrt{-5})$, the class number $h_K = 2$, and the Hilbert class field $K^{(1)} = \mathbb{Q}(\sqrt{-5}, i)$. The Russell-type modular equation of degree 5 is given by (see [1, p. 280, Entry 13(i)])

$$(\alpha\beta)^{1/2} + \{(1-\alpha)(1-\beta)\}^{1/2} + 2\{16\alpha\beta(1-\alpha)(1-\beta)\}^{1/6} = 1.$$

Setting $\beta = 1 - \alpha$, we find that

$$\big(4\alpha(1-\alpha)\big)^{1/2} + 2\big(4\alpha(1-\alpha)\big)^{1/3} = 1.$$

If now we put $t = G_5^4$, then t satisfies

$$t^3 - 2t - 1 = (t+1)(t^2 - t - 1) = 0.$$

Since $G_n > 1$, we deduce that

$$G_5^4 = \frac{1+\sqrt{5}}{2} \quad \text{and} \quad K^{(1)} = K(G_5^4).$$

2. Suppose $p = 7$. Then $K = \mathbb{Q}(\sqrt{-7})$, $h_K = 1$, and $K^{(1)} = K$. We have the Russell-type modular equation of degree 7 (see [1, p. 314, Entry 19(i)])

$$(\alpha\beta)^{1/8} + \{(1-\alpha)(1-\beta)\}^{1/8} = 1.$$

Letting $\beta = 1-\alpha$, we see immediately that $2^{3/4}G_7^{-3} = 1$, thereby verifying Weber's assertion.

3 Cubic Russell-Type Modular Equations

The cubic analogue of Theorem 2.1 is as follows:

Theorem 3.1 *Let $p > 3$ be a prime and $(p+1)/3 = n/s$ in lowest terms. Suppose β^* has degree p over α^* in the theory of signature 3. Then the relation between*

$$u = (\alpha^*\beta^*)^{s/6} \quad \textit{and} \quad v = \{(1-\alpha^*)(1-\beta^*)\}^{s/6}$$

can be given in the form

$$B_0(v)u^n + B_1(v)u^{n-1} + \cdots + B_n(v) = 0,$$

where $B_0(v), \ldots, B_n(v)$ are polynomials of degrees at most n in v.

Let

$$a(q) := \sum_{m,n=-\infty}^{\infty} q^{m^2+mn+n^2},$$

$$b(q) := \sum_{m,n=-\infty}^{\infty} \omega^{n-m} q^{n^2+mn+m^2}, \quad (\omega = e^{2\pi i/3})$$

and

$$c(q) := \sum_{m,n=-\infty}^{\infty} q^{(n+1/3)^2+(n+1/3)(m+1/3)+(m+1/3)^2}.$$

To prove Theorem 3.1, we first recall that if $0 < \alpha^* < 1$ and

$$q = \exp\left(-\frac{2\pi}{\sqrt{3}} \frac{{}_2F_1\left(\frac{1}{3}, \frac{2}{3}; 1; 1-\alpha^*\right)}{{}_2F_1\left(\frac{1}{3}, \frac{2}{3}; 1; \alpha^*\right)}\right),$$

then [3, p. 4172]

$$\alpha^* = \frac{c^3(q)}{a^3(q)}, \tag{3.1}$$

and

$$1 - \alpha^* = \frac{b^3(q)}{a^3(q)}. \tag{3.2}$$

Identity (3.2) follows from (3.1) and the Borweins' elegant identity [7]

$$a^3(q) = b^3(q) + c^3(q).$$

With the identifications (3.1) and (3.2), we see that Theorem 3.1 is an immediate consequence of

Theorem 3.2 *Let $p > 3$ be a prime and $(p+1)/3 = n/s$ in lowest terms and $q = e^{2\pi i\tau}$. Then the relation between*

$$u(\tau) = \left(\frac{c(q)c(q^p)}{a(q)a(q^p)}\right)^{s/2} \quad \text{and} \quad v(\tau) = \left(\frac{b(q)b(q^p)}{a(q)a(q^p)}\right)^{s/2}$$

can be given in the form

$$B_0(v)u^n + B_1(v)u^{n-1} + \cdots + B_n(v) = 0,$$

where $B_0(v), \ldots, B_n(v)$ are polynomials of degrees at most n in v.

Before proving Theorem 3.2, we need the following lemmas.

Lemma 3.3 *Let $q = e^{2\pi i\tau}$ with $\operatorname{Im}\tau > 0$ and let the Dedekind eta-function be defined by*

$$\eta(\tau) := q^{1/24}(q;q)_\infty := q^{1/24}\prod_{k=1}^{\infty}(1-q^k).$$

In the notation of Ramanujan, set

$$f(-q) := (q;q)_\infty = q^{-1/24}\eta(\tau).$$

Then

$$a(q) = 1 + 6\sum_{n=1}^{\infty}\left(\frac{n}{3}\right)\frac{q^n}{1-q^n} = \frac{f^3(-q^{1/3}) + 3q^{1/3}f^3(-q^3)}{f(-q)}, \tag{3.3}$$

$$b(q) = \frac{f^3(-q)}{f(-q^3)}, \tag{3.4}$$

and

$$c(q) = 3q^{1/3}\frac{f^3(-q^3)}{f(-q)}. \tag{3.5}$$

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Proof Identities (3.4) and (3.5) were first discovered by the Borweins and elementary proofs can be found in [8]. Identity (3.3) is due to Ramanujan [1, p. 346]. ■

Note that if we divide (3.3) by (3.5) and invoke (3.1), we obtain some useful representations for $(\alpha^*)^{1/3}$,

$$\frac{1}{(\alpha^*)^{1/3}} = 1 + \frac{f^3(-q^{1/3})}{3q^{1/3}f^3(-q^3)} = \left(1 + \frac{f^{12}(-q)}{27qf^{12}(-q^3)}\right)^{1/3}, \tag{3.6}$$

where the second equality is justified by [1, p. 345, Entry 1(iv)].

Lemma 3.4 *The functions $u(\tau)$ and $v(\tau)$ are modular functions invariant under $\Gamma_0(3p)$, i.e., $u(\tau), v(\tau) \in M(\Gamma_0(3p), 0, 1)$.*

Proof Let $A = \left(\begin{smallmatrix} a & b \\ 3c & d \end{smallmatrix}\right) \in \Gamma_0(3)$. Noting that

$$\begin{pmatrix} 3 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} a & b \\ 3c & d \end{pmatrix} = \begin{pmatrix} a & 3b \\ c & d \end{pmatrix}\begin{pmatrix} 3 & 0 \\ 0 & 1 \end{pmatrix}$$

and the fact that $\eta(\tau)$ is a modular form of weight $\frac{1}{2}$ on the full modular group $\Gamma(1)$ with multiplier system ν_η, we obtain

$$\begin{aligned} \frac{\eta^3(3A\tau)}{\eta(A\tau)} &= \frac{{\nu_\eta}^3 \left(\begin{smallmatrix} a & 3b \\ c & d \end{smallmatrix}\right) \big(c(3\tau) + d\big)^{3/2} \eta^3(3\tau)}{\nu_\eta \left(\begin{smallmatrix} a & b \\ 3c & d \end{smallmatrix}\right) (3c\tau + d)^{1/2} \eta(\tau)} \\ &= \frac{{\nu_\eta}^3 \left(\begin{smallmatrix} a & 3b \\ c & d \end{smallmatrix}\right)}{\nu_\eta \left(\begin{smallmatrix} a & b \\ 3c & d \end{smallmatrix}\right)} (3c\tau + d) \frac{\eta^3(3\tau)}{\eta(\tau)}. \end{aligned} \tag{3.7}$$

For $(c, d) = 1$ with c odd let $\left(\frac{d}{c}\right)$ be the Legendre-Jacobi symbol and define

$$\left(\frac{d}{c}\right)^* := \left(\frac{d}{|c|}\right).$$

By [1, p. 330], we consider separately two cases: if c is odd,

$$\frac{{\nu_\eta}^3 \left(\begin{smallmatrix} a & 3b \\ c & d \end{smallmatrix}\right)}{\nu_\eta \left(\begin{smallmatrix} a & b \\ 3c & d \end{smallmatrix}\right)} = \frac{\left(\left(\frac{d}{c}\right)^*\right)^3 e^{2\pi i\{-3c-3bd(c^2-1)+c(a+d)\}\cdot 3/24}}{\left(\frac{d}{3c}\right)^* e^{2\pi i\{-9c-bd(9c^2-1)+3c(a+d)\}/24}} = \left(\frac{d}{3}\right) e^{2\pi i bd/3};$$

if c is even,

$$\begin{aligned} \frac{{\nu_\eta}^3 \left(\begin{smallmatrix} a & 3b \\ c & d \end{smallmatrix}\right)}{\nu_\eta \left(\begin{smallmatrix} a & b \\ 3c & d \end{smallmatrix}\right)} &= \frac{\left(\frac{c}{d}\right)_*^3 e^{2\pi i\{3(d-1)-ac(d^2-1)+d(3b-c)\}\cdot 3/24}}{\left(\frac{3c}{d}\right)_* e^{2\pi i\{3(d-1)-3ac(d^2-1)+d(b-3c)\}/24}} \\ &= \left(\frac{3}{d}\right)_* e^{2\pi i(d-1)/4} e^{2\pi i bd/3} = \left(\frac{d}{3}\right) e^{2\pi i bd/3}. \end{aligned} \tag{3.8}$$

Thus, $c(q)$ is a modular form of weight 1 on $\Gamma_0(3)$ with multiplier system $\left(\frac{d}{3}\right)e^{2\pi ibd/3}$. In view of the second expression in (3.3) and [12, Eq. (1.9)], where $a(q)$ is denoted by $6V_{1,3}(\tau)$, it is known that $a(q)$ is a modular form of weight 1 on $\Gamma_0(3)$ with multiplier system $\left(\frac{d}{3}\right)$. Therefore,

$$\lambda(\tau) := (\alpha^*)^{1/3} \tag{3.9}$$

is a modular form of weight 0 on $\Gamma_0(3)$ with multiplier system v_λ given by

$$v_\lambda \begin{pmatrix} a & b \\ 3c & d \end{pmatrix} = e^{2\pi ibd/3}.$$

Note that $u(\tau) = (\alpha^*\beta^*)^{s/6}$ in Theorem 3.1 takes the form

$$u(\tau) = \bigl(\lambda(\tau)\lambda(p\tau)\bigr)^{s/2}. \tag{3.10}$$

The invariance of $u(\tau)$ under $\Gamma_0(3p)$ can now be established. Let $B = \left(\begin{smallmatrix} a & b \\ 3pc & d \end{smallmatrix}\right)$, then

$$\begin{aligned} u(B\tau) &= \bigl(\lambda(B\tau)\lambda(pB\tau)\bigr)^{s/2} \\ &= \left(\lambda\left(\begin{pmatrix} a & b \\ 3pc & d \end{pmatrix}\tau\right)\lambda\left(p\frac{a\tau+b}{3pc\tau+d}\right)\right)^{s/2} \\ &= \left(\lambda\left(\begin{pmatrix} a & b \\ 3pc & d \end{pmatrix}\tau\right)\lambda\left(\begin{pmatrix} a & pb \\ 3c & d \end{pmatrix}p\tau\right)\right)^{s/2} \\ &= \bigl(e^{2\pi ibd/3}\lambda(\tau)e^{2\pi ipbd/3}\lambda(p\tau)\bigr)^{s/2} \\ &= e^{\pi ibd(p+1)s/3}u(\tau) \\ &= u(\tau), \end{aligned} \tag{3.11}$$

since $bd(p+1)s/3 = bdn$ is even. This shows that $u(\tau) \in M\bigl(\Gamma_0(3p), 0, 1\bigr)$. Similarly, $v(\tau) \in M\bigl(\Gamma_0(3p), 0, 1\bigr)$. ■

Proof of Theorem 3.2 By Lemma 2.4, it remains to compute the total number of poles of $u(\tau)$ and $v(\tau)$. To this end, we obtain from (3.6), (3.9), and (3.10) that

$$u(\tau) = \left(\left(1 + \frac{\eta^{12}(\tau)}{27\eta^{12}(3\tau)}\right)\left(1 + \frac{\eta^{12}(p\tau)}{27\eta^{12}(3p\tau)}\right)\right)^{-s/6},$$

and observe that a complete set of inequivalent cusps for $\Gamma_0(3p)$ is given by [11, p. 18]

$$\mathcal{C}'' := \left\{\frac{0}{1}, \frac{1}{3}, \frac{1}{p}, \frac{1}{3p}\right\}.$$

By [1, p. 333],

$$\operatorname{ord}\left(\eta(n\tau); \frac{r}{s}\right) = \frac{(nr, s)^2}{24n},$$

the computation of $24 \cdot \operatorname{ord}(F; \zeta)$ for several functions F and $\zeta \in \mathcal{C}''$ are summarized in the following table:

Function $F \setminus$ Cusp ζ	$\frac{0}{1}$	$\frac{1}{3}$	$\frac{1}{p}$	$\frac{1}{3p}$
$\eta(\tau)$	1	1	1	1
$\eta(p\tau)$	$1/p$	$1/p$	p	p
$\eta(3\tau)$	$1/3$	3	$1/3$	3
$\eta(3p\tau)$	$1/3p$	$3/p$	$p/3$	$3p$
width $N(\Gamma_0(3p); \zeta)$	$3p$	p	3	1

According to a general formula in [11, p. 49, Lemma 3.5], the power series expansions of functions involving products or quotients of Dedekind eta-functions at the cusps are determined. With routine calculations, it can be shown that the functions $1+\eta^{12}(\tau)/\big(27\eta^{12}(3\tau)\big)$ and $1 + \eta^{12}(p\tau)/\big(27\eta^{12}(3p\tau)\big)$ are non-vanishing at the cusps. Hence, utilizing the table above, we find that $u(\tau)$ has zeros at $\frac{1}{3}$ and $\frac{1}{3p}$ and the number of poles of $u(\tau)$ is

$$\frac{1}{24}\left(p\frac{2(p+1)}{p} + 2(p+1)\right) 12\frac{s}{6} = n.$$

Similarly, $v(\tau)$ has the same number of poles. The proof of Theorem 3.2 is therefore complete. ■

In view of Theorem 3.1, we deduce that cubic Russell-type modular equations are simpler when $p \equiv 2 \pmod 3$, the reason being that the degrees of the polynomials satisfied by u and v are almost 3 times smaller than that of other p's. This condition is satisfied by all the degrees of Ramanujan's modular equations (see Section 1). Using Theorem 3.1, we have discovered two new modular equations with degrees 17 and 23.

For $p \equiv 1 \pmod 3$, the polynomial satisfied by u and v is of degree $p + 1$. The first prime satisfying this condition is 7. Although Ramanujan did supply a modular equation of degree 7 in the theory of signature 3, his equation is not of Russell-type. Using Theorem 3.1, we succeeded in constructing this "missing" modular equation of degree 7.

We record our findings in the following corollaries.

Corollary 3.5 *Suppose β^* has degree 7 over α^* in the theory of signature 3. Set $w = (u + v + 1)(u + v - 1)$ and $s = uv$, where $u = \sqrt{\alpha^*\beta^*}$ and $v = \sqrt{(1 - \alpha^*)(1 - \beta^*)}$. Then*

$$\begin{aligned} w^4 - 2^4 \cdot 3 \cdot 7sw^3 + (2^8 \cdot 3 \cdot 7^2 s^2 - 2^3 \cdot 3^5 \cdot 5s)w^2 \\ - (2^{12} \cdot 7^3 s^3 + 2^7 \cdot 3^5 \cdot 11s^2 + 3^7 \cdot 13s)w - 3^5 s(2^6 s + 3^2)^2 = 0. \end{aligned} \tag{3.12}$$

Corollary 3.6 *Suppose β^* has degree 17 over α^* in the theory of signature 3. Set $w = u^2 + v^2 - 1$ and $s = 3uv$, where $u = (\alpha^*\beta^*)^{1/6}$ and $v = \{(1-\alpha^*)(1-\beta^*)\}^{1/6}$. Then*

$$w^3 + 32sw^2 + 9(11s - 3s^2)w + 3s(s-3)(2s-9) = 0. \tag{3.13}$$

Corollary 3.7 *Suppose β^* has degree 23 over α^* in the theory of signature 3. Set $P = u^2 + v^2 + 6uv - 1$, $Q = u + v$ and $R = 3\sqrt{uv}$, where $u = (\alpha^*\beta^*)^{1/6}$ and $v = \{(1-\alpha^*)(1-\beta^*)\}^{1/6}$. Then*

$$P^2 - R^2(11Q^2 + 4) - \sqrt{3}RQ(4P + 9) = 0. \tag{3.14}$$

The constructions of these new modular equations are straightforward but tedious. For a fixed p, we use the power series expansions of the corresponding u and v to determine each of the coefficients in $B_0(v), \ldots, B_n(v)$ by solving systems of linear equations. These computations and the simplifications of the modular relations are done with the aid of MAPLE V.

Corollary 3.7 shows that when β^* has degree 23 over α^*, the modular equation is rather simple. This final form is derived from the relation

$$\begin{aligned} &w^4 - 2\cdot 101sw^3 + (7\cdot 19s - 2\cdot 3^2\cdot 7^2)sw^2 \\ &\quad + (2^2\cdot 3\cdot 7s^2 + 2\cdot 3^2\cdot 17s - 3^4\cdot 17)sw - 3s(s-3)(2^2\cdot 5s^2 - 3^4) = 0, \end{aligned} \tag{3.15}$$

where $w = u^2 + v^2 - 1$ and $s = 3uv$. More precisely, the left hand side of (3.15) splits into two factors in the polynomial ring $\mathbb{Q}(\sqrt{3})(t, z)$ if we make use of the change of variables, $u \mapsto t^2$ and $v \mapsto z^2$, and one of these factors yields the desired modular equation (3.14).

Classical modular equations expressed in the form (3.14) are referred to as Weber-type modular equations. Corollary 3.7 gives one of the first few cubic Weber-type modular equations. Note that with the same parameterizations, Ramanujan's cubic Russell-type modular equation of degree 11 (see (1.6)) can be expressed simply as

$$P + \sqrt{3}RQ = 0.$$

While computing the first few modular equations, we observe three significant features shared by these relations, namely,

(i) The polynomial is symmetric in u and v,
(ii) If $u^k v^l$ appear in the polynomial, then $k + l$ is even, and
(iii) The degree of $B_j(v)$ is at most j.

Proof of (i) From (1.3), we find that $1 - \alpha^*$ has degree p over $1 - \beta^*$ if β^* has degree p over α^*. Hence, Theorem 3.1 is true with u replaced by v and v replaced by u and this implies that the polynomial is symmetric in u and v. ∎

Statements (ii) and (iii) appear to be true but we are unable to provide any proofs. The truth of (ii) and (iii) will significantly reduce the number of constants to be determined in the relations between u and v.

Remark Cubic Russell-type modular equations of degrees 13, 19, 29, 41, 47, 53, and 59 have been found, see [14, pp. 26–34].

4 Analogue of G_n and the Cubic Singular Modulus

Define the cubic analogue of G_n (see (2.2)) as

$$H_n = \{\alpha_n^*(1-\alpha_n^*)\}^{-1/3}, \tag{4.1}$$

where α_n^* is the *cubic singular modulus* satisfying (1.3) when $\beta^* = 1-\alpha^*$. As in the classical case, we may evaluate H_n from its corresponding cubic Russell-type modular equation by setting $\beta^* = 1-\alpha^*$. The first few values or minimal polynomials of H_n which follow from the known modular equations are

$$H_2 = 2,$$
$$H_5 = 5,$$
$$H_7 = 2(34+13\sqrt{7})^{1/3},$$
$$H_{11} = 2(4+3\sqrt{3}),$$
$$H_{17}^3 - 48H_{17}^2 - 36H_{17} - 68 = 0,$$

and

$$H_{23} = (2+\sqrt{3})^3\left(1+3\sqrt{-83+48\sqrt{3}}\right).$$

The table above indicates that the degree of the polynomial satisfied by H_n is not related to the class number of the field $\mathbb{Q}(\sqrt{-n})$. As a result, H_n, unlike G_n (see Section 2), is not a class invariant. On the other hand, as in the classical case, we can solve a quadratic equation (4.1) and deduce the corresponding cubic singular modulus α_n^*. Such values are recently used to derive new values for the modular j-invariant [4]. We conclude this section with a list of some simple moduli:

$$\alpha_2^* = \frac{1}{2}\left(1-\frac{\sqrt{2}}{2}\right),$$
$$\alpha_5^* = \frac{1}{2}\left(1-\frac{11\sqrt{5}}{25}\right),$$
$$\alpha_7^* = \frac{1}{2}\left(1-\frac{13-\sqrt{7}}{6\sqrt{3}}\right),$$
$$\alpha_{11}^* = \frac{1}{2}\left(1-\frac{45\sqrt{3}-5}{22\sqrt{11}}\right),$$

and

$$\alpha_{23}^* = \frac{1}{2}\left(1-\frac{1}{2}\sqrt{4-2\left(\frac{1}{23}+\frac{3}{46}\sqrt{3}\right)^3\left(3\sqrt{-83+48\sqrt{3}}-1\right)^3}\right).$$

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