Glasgow Math. J. **41** (1999) 115–124. © Glasgow Mathematical Journal Trust 1999. Printed in the United Kingdom

THE SYMMETRIC GENUS OF 2-GROUPS

JAY ZIMMERMAN†

Department of Mathematics, Towson University, 8000 York Road, Towson, Maryland 21252, USA
e-mail: J.Zimmerman@towson.edu

(Received 10 March, 1997)

1. Introduction. A finite group G can be represented as a group of automorphisms of a compact Riemann surface, that is, G acts on a Riemann surface. The *symmetric genus* σ(G) is the minimum genus of any Riemann surface on which G acts (possibly reversing orientation).

The origins of the symmetric genus parameter can be traced to the work of Hurwitz, Poincare, Burnside and others (see [**1**] and [**5**]). The modern terminology was introduced in the important article [**14**]. There is now a considerable body of work on the symmetric genus parameter [**3**, Chapter 6]. Much of this has concentrated on non-solvable groups; see the survey article [**2**].

Another body of work has concentrated on solvable groups. The symmetric genus of each finite abelian group has been determined [**7**] and [**11**]; also relevant here is the work of Maclachlan [**6**]. The symmetric genus of metacyclic groups was considered in [**8**] and May and Zimmerman [**9**] obtained a general lower bound for the symmetric genus of a finite group G and in the same paper calculated the symmetric genus of all groups of order less than 48 (with the exception of groups of order 32).

The purpose of this paper is to produce a lower bound for the genus of a 2-group and to find the genus of all groups of order 32. We use the standard representation of G as a quotient of a non-euclidean crystallographic group Γ by a surface group K; then G acts on the Riemann surface U/K, where U is the open upper half-plane. In our work on groups of order 32, we frequently employ the computer algebra system GAP [**12**]. In particular, the groups of order 32 are accessed in GAP through the command AllSolvableGroups(Size,32). Table 1 gives the symmetric genus of each group of order 32. For example, the group G20 stands for the group labeled grp_32_20 in the GAP Library. The non-abelian groups in this listing are in the same order as in the Hall-Senior Table [**4**]. In addition, I either provide a descriptive name for the group, if one exists in GAP or the designation for that group in the Hall-Senior Table, if one does not.

2. Preliminaries. Non-euclidean crystallographic groups (NEC groups) have been quite useful in investigating group actions on surfaces. We shall assume that all surfaces are compact. Let $\mathcal{L}$ denote the group of automorphisms of the open upper half-plane U, and let $\mathcal{L}^+$ denote the subgroup of index 2 consisting of the orientation-preserving automorphisms. An NEC group is a discrete subgroup Γ of $\mathcal{L}$ (with the quotient space U/Γ compact). If $\Gamma \subseteq \mathcal{L}^+$, then Γ is called a *Fuchsian group*. Otherwise Γ is called a *proper NEC group*; in this case Γ has a canonical Fuchsian subgroup $\Gamma^+ = \Gamma \cap \mathcal{L}^+$ of index 2.

Associated with the NEC group Γ is its signature, which has the form

†This research was partially supported by a grant from the Faculty Development and Research Committee of Towson State University.

Table 1. The genus of the groups of order 32

Group	σ	Group	σ	Group	σ
G1–C_2^5	5	G18–Γ_2h	5	G35–Q_8+Q_8	17
G2–$C_2^3\times C_4$	9	G19–Γ_2i	9	G36–Γ_4b_1	1
G3–$C_2^2\times C_8$	1	G20–Γ_2j_1	1	G37–Γ_4b_2	9
G4–$C_4^2\times C_2$	9	G21–C_8+Q_8	1	G38–Γ_4c_1	5
G5–$C_{16}\times C_2$	0	G22–Γ_2k	1	G39–Γ_4c_2	9
G6–$C_8\times C_4$	1	G23–$D_{16}\times C_2$	1	G40–Γ_4c_3	17
G7–C_{32}	0	G24–$QD_{16}\times C_2$	5	G41–Γ_4d	13
G8–$D_8\times C_2^2$	1	G25–$Q_{16}\times C_2$	9	G42–D_8YD_8	1
G9–$Q_8\times C_2^2$	13	G26–$D_{16}YC_4$	1	G43–D_8YQ_8	9
G10–$(D_8YC_4)\times C_2$	5	G27–Γ_3c_1	1	G44–Γ_6a_1	1
G11–Γ_2c_1	5	G28–Γ_3c_2	7	G45–Γ_6a_2	7
G12–2×(2×4).2	9	G29–Γ_3d_1	7	G46–Γ_7a_1	1
G13–Γ_2d	7	G30–Γ_3d_2	7	G47–Γ_7a_2	5
G14–$D_8\times C_4$	1	G31–Γ_3e	1	G48–Γ_7a_3	9
G15–$Q_8\times C_4$	17	G32–Γ_3f	11	G49–D_{32}	0
G16–Γ_2f	9	G33–Γ_4a_1	1	G50–QD_{32}	1
G17–D_8YC_8	1	G34–Γ_4a_2	1	G51–Q_{32}	1

$$(p; \pm; [\lambda_1, \ldots, \lambda_r]; \{(\nu_{11}, \ldots, \nu_{1s_1}), \ldots, (\nu_{k1}, \ldots, \nu_{ks_k})\}). \tag{2.1}$$

The quotient space X = U/Γ is a surface with topological genus p and k holes. The surface is orientable if the plus sign is used and non-orientable otherwise. Associated with the signature (2.1) is a presentation for the NEC group Γ (See [**7**]).

Let Γ be an NEC group with signature (2.1). The non-euclidean area $\mu(\Gamma)$ of a fundamental region Γ can be calculated directly from its signature [**13**, p.235]:

$$\mu(\Gamma)/2\pi = \alpha p + k - 2 + \sum_{i=1}^{r}\left(1 - \frac{1}{m_i}\right) + \sum_{i=1}^{k}\sum_{j=1}^{s_i}\frac{1}{2}\left(1 - \frac{1}{n_{ij}}\right), \tag{2.2}$$

where $\alpha = 2$ if the plus sign is used and $\alpha = 1$ otherwise.

An NEC group K is called a *surface group* if the quotient map from U to U/K is unramified. Let X be a Riemann surface of genus g≥2. Then X can be represented as U/K where K is a Fuchsian surface group with $\mu(K) = 4\pi(g-1)$. Let G be a group of dianalytic automorphisms of the Riemann surface X. Then there is an NEC group Γ and a homomorphism ϕ:Γ→G onto G such that kernel ϕ = K.If Λ is a subgroup of finite index in Γ, then $[\Gamma:\Lambda] = \mu(\Lambda)/\mu(\Gamma)$. It follows that the genus of the surface U/K on which G≅Γ/K acts is given by

$$g = 1 + |G|.\mu(\Gamma)/4\pi. \tag{2.3}$$

Minimizing g is therefore equivalent to minimizing $\mu(\Gamma)$.

3. Groups of genus one. There are 51 groups of order 32 of which the first 7 are abelian. The genus of the abelian groups is computed from the formulas in [**7**] and [**11**]. In addition, the genus of some of the other groups of order 32 is known. The group G9 ($Q_8\times C_2^2$) has genus 13 [**9**]. Also the dicyclic group G51 (Q_{32}) has genus 1 [**15**] and the dihedral group G49 (D_{32}) has genus 0.

In this section, we will look at the groups of order 32 that have genus 1. They are given in Table 2. In order to show that a group has genus 1, we must show that it is a quotient of a group with presentation given in Tucker's Theorem 6.3.3 [**3**]. We call these partial presentations "Tucker classes". It should be noted that some of these groups are in more than one Tucker class.

The first step in finding the symmetric genus of the other groups of order 32 is to show that they are not toroidal (genus 1). Let Δ be a group with one of the presentations in Tucker's Theorem 6.3.3 [**3**, p. 291]. Suppose that there is an epimorphism from Δ onto a group G. Then there is an epimorphism between the abelianizations Δ_{ab} and G_{ab}. The invariants of the abelianization Δ_{ab} are given in Table 3.

There are 17 partial presentations in Tucker's Theorem 6.3.3. Five of these partial presentations (c,e,m,n and q) involve generators of order 3. So any homomorphism into a 2-group will take these generators to the identity and the result will be an abelian image. Another five of these partial presentations (b,j,k,l, and o) are generated entirely by involutions. Therefore, any group in which all of the involutions lie in a proper subgroup cannot be an image of one of these partial presentations. There are 21 non-toroidal groups having this property.

Fact 3.1: The groups G12, G16, G18, G19, G21, G24, G25, G28, G29, G30, G32, G35, G37, G38, G39, G40, G41, G45, G47, G48, and G50 have the property that all of their involutions are contained in a proper subgroup. These groups are not an image of one of the partial presentations (b), (c), (e), (j), (k), (1), (m), (n), (o) or (q).

Finally, five of these partial presentations (a,f,g,h and i) have one or more generators of infinite order. The group described by any presentation in class (a) is abelian and any finite group in class (h) is $Z_m \times D_n$. If we add relations to the presentation in class (h) to make it have exponent 8, then is isomorphic to $Z_8 \times D_{16}$. This

Table 2. The groups of order 32 with genus 1

Group	Tucker class	Group	Tucker class
G8–$D_8 \times C_2^2$	class (j)	G31	class (i)
G14–$D_8 \times C_4$	class (h)	G33	class (o)
G17–$D_8 Y C_8$	class (h)	G34	class (b_1)
G20	class (i)	G36	class (l_1)
G21–$8+Q_8$	class (f_2)	G42–$D_8 Y D_8$	class (j)
G22	class (f_2)	G44	class (k)
G23–$D_{16} \times C_2$	class (h)	G46	class (d)
G26–$D_{16} Y C_4$	class (h)	G50–QD_{32}	class (i)
G27	class (g_1)	G51–Q_{32}	

Table 3. Invariants of the abelianizations of the Tucker classes

Class	Invariants	Class	Invariant	Class	Invariants
a	(∞,∞)	g	(2,4)	m	(2)
b	(2,2,2)	h	$(2,2,\infty)$	n	(6)
c	(3,3)	i	$(2,\infty)$	o	(2,2,2)
d	(2,4)	j	(2,2,2,2)	p	(2,4)
e	(6)	k	(2,2,2)	q	(2,2)
f	$(2,\infty)$	l	(2,2,2)		

group has order 128 and 5 normal subgroups of order 4. The quotient groups of $Z_8 \times D_{16}$ by these 5 normal subgroups are the four groups with class (h) in Table 2 and the abelian group $C_8 \times C_2^2$. Therefore, G3, G14, G17, G23 and G26 are the only groups of order 32 in class (h). Suppose that we add enough relators to make the groups in classes (f), (g) and (i) nilpotent of class 2 and of exponent 8. The group that results in classes (f) and (i) are isomorphic to G21 and G20 respectively. The group that results in class (g) has order 16. All other classes are generated by elements of order 4 or less. Therefore, the only groups which are non-abelian, nilpotent of class 2, have exponent 8 and are in one of the classes (f), (g), (h), or (i) are G14, G17, G20 and G21. We conclude that any other nilpotent class 2 group with exponent 8 that cannot be generated by elements of order 2 or 4 is not toroidal. Thus groups G13 and G47, where all elements of order 2 and 4 are contained in a proper subgroup, have genus greater than 1.

Now we will show that the groups of order 32 that are not in Table 2 have genus greater than 1. Suppose that G is a group where all elements of order 2 are in the Frattini subgroup. If Δ maps onto G, then G is generated by the images of generators of Δ of order greater than 2. In all cases, except Tucker class (f), this would force G to be abelian. The groups G18, G19, G28, G29, G30, G32, G35, G40, and G48 have the property that all elements of order 2 are in the Frattini subgroup. These groups are all nilpotent of class 2 and have exponent 8. Since none of these groups is in class (f), it follows that G18, G19, G28, G29, G30, G32, G35, G40, and G48 have genus greater than 1.

The groups G11, G12, G15, and G16 all have abelianization equal to $C_4 \times C_2^2$. The only group in Tucker's Theorem 6.3.3 with an abelianization large enough to map onto $C_4 \times C_2^2$ is in class (h). As we have seen previously, they cannot be in class (h). It follows that G11, G12, G15 and G16 have genus greater than 1.

Next we consider the groups G24, G25, G37, G38, G39, G41 and G45 which have abelianization equal to C_2^3. We combine Fact 3.1 and the information in Table 2 to see that class (h) is the only possible Tucker class that these groups could be in. Therefore, their genus is greater than 1.

Finally, we consider the groups of order 32 which have abelianization C_2^4. The only class that they could possibly be in is class (j). Adding relators to the presentation in class (j) to make the group have exponent 4 results in the group $D_8 \times D_8$, which has order 64. This group has only 3 normal subgroups of order 2 and it is easy to verify that the quotients that result are isomorphic to either G8 or G42. Therefore, the groups G9, G10 and G43 have genus greater than 1.

4. Minimal generating sets for these groups. The next step is to list the orders of the elements in a minimal generating set for each of the remaining groups. Let n_q be the number of generators of order q in a minimal generating set. A minimal generating set has the smallest number of generators as specified by the Burnside Basis Theorem and as many as possible have order 2 and then order 4, etc. We will list these numbers as an ordered triple, (n_2,n_4,n_8). Therefore, we have a lexicographic order on the ordered triples.

We will show that there is a generating set which embodies each of the ordered triples in Table 4. The fact that a group is the image of an NEC group with generators whose order is that of a minimal generating set does not mean that the genus is obtained from that NEC group. Furthermore, the genus may not be attained by an NEC group with generators whose orders are in a minimal generating set. The

Table 4. Number of generators in a minimal generating set

Group	Generators	Group	Generators	Group	Generators
G10	(4,0,0)	G24	(2,1,0)	G38	(2,1,0)
G11	(2,1,0)	G25	(1,2,0)	G39	(2,1,0)
G12	(1,2,0)	G28	(0,2,0)	G40	(0,3,0)
G13	(2,0,1)	G29	(0,2,0)	G41	(1,2,0)
G15	(0,3,0)	G30	(0,2,0)	G43	(4,0,0)
G16	(1,2,0)	G32	(0,0,2)	G45	(2,1,0)
G18	(0,2,0)	G35	(0,3,0)	G47	(1,0,1)
G19	(0,1,1)	G37	(1,2,0)	G48	(0,1,1)

group G45 is an example. The ordered triple of G45 is (2,1,0) and it is an image of NEC groups with generators having these orders, namely the NEC groups with signatures (0; +; [4],{(4,4)}) and (0; +; [2,2,4,8]; {()}). However, its genus is realized by an NEC group having two generators of order 2 and one generator of order 8.

First, we must show that each of these triples is minimal in the lexicographic order. Clearly, if the ordered triple derives from the invariants of the abelianization, then it is minimal in the lexicographic order. This occurs for the groups G10, G11, G18, and G43.

Next, we consider the groups where all elements of order 2 are in the Frattini subgroup. Therefore, the groups G15, G28, G29, G30, G35 and G40 are generated by elements of order 4. In addition, the group G32 also has all elements of order 4 in the Frattini subgroup and so it is generated by elements of order 8. Finally, the groups G19 and G48 also satisfy the property that all elements of order 2 and 4 are in a subgroup of order 16 and therefore they are generated by an element of order 4 and one of order 8. The rest of the groups have at least one element of order 2 that is not in the Frattini subgroup.

The groups G12, G16, G25, G41 and G47 have the property that all elements of order 2 are contained in a proper subgroup H and the intersection of H and the Frattini subgroup is a subgroup of index 2 in H. Therefore, these groups can have at most one generator of order 2 in any minimal generating set. All of these groups, except G47 are generated by one element of order 2 and two of order 4. In the group G47, all elements of order 4 are in the Frattini subgroup and so it is generated by one element of order 2 and one of order 8.

The remaining groups, G13, G24, G37, G38, G39 and G45, have the property that all elements of order 2 are contained in a proper subgroup. Thus the groups G24, G38, G39 and G45 have minimal generating set consisting of two elements of order 2 and one element of order 4. In the group G37, all of the elements of order 2 are contained in a normal subgroup N of order 8. If we adjoin a generator of order 4 to N, we get a subgroup of order 16. Therefore, a minimal generating set will have two elements of order 4 and one of order 2. Finally, group G13 has all elements of orders 2 or 4 contained in a subgroup of order 16 and so its minimal generating set consists of two elements of order 2 and one element of order 8. These arguments establish the ordered triples given in Table 4.

5. NEC groups with small non-euclidean area. In this section, we will find some lower bounds for the genus of a 2-group. In the process, we list the signatures of the NEC groups which have area less than 1/2 and are not generated by involutions. This is needed to calculate the genus of the groups of order 32.

Proposition 5.1. *Let Γ be an NEC group whose signature has a non-empty period cycle with one link period. Suppose that Γ maps onto a finite 2-group* G *such that the kernel of the map is a Fuchsian surface group. Let* c *and* d *be the involutions associated with the period cycle (*n*). If* G *has nilpotence class* 2 *or* c *maps to the center of* G*, then* n = 2*. Furthermore, in all cases,* n *divides exponent(*G*).*

Proof. The generators c,d and e are associated with the period cycle (n). They satisfy the relations $c^2=d^2=(cd)^n=1$ and $ece^{-1}=d$. It follows that $[c,e^{-1}]^n=1$. Let ϕ:$\Gamma \to G$ be the onto homomorphism described above. Now $1=[c^2,e^{-1}]=[c,e^{-1}]^c \cdot [c,e^{-1}]$. It follows that $[c,e^{-1}]^2 \epsilon \mathrm{Ker}(\phi)$. If n≥4, then $(cd)^{n/2} \epsilon \mathrm{Ker}(\phi)$ and Ker(ϕ) would contain an analytic element of finite order. So n = 2. The same argument shows that in general n divides exponent (G′).

Theorem 5.2. *Let* G *be a non-abelian 2-group with $\sigma(G) \geq 2$. Suppose that* G *has rank 3 or greater and cannot be generated by involutions or if it is generated by involutions, then it has rank 5 or greater. If $|G|=2^n$, then $\sigma(G) \leq 1+2^{n-3}$.*

Proof. Suppose that Γ is an NEC group which maps onto G and the kernel is a Fuchsian surface group. We must show that $\mu(\Gamma)/2\pi \geq 1/4$. Therefore, suppose that Γ is a non-abelian NEC group with Non-Euclidean area $\mu(\Gamma)/2\pi < 1/2$. We will list the possible signatures of such groups below.

Case 1: Suppose that k≥1. Suppose that U/Γ is orientable. It is clear that k≤2 and p = 0. Now suppose that k = 2. In this case, r = 0 and exactly one of the period cycles must be non-empty. This gives the signature (0; +; [ ]; {(), (n)}), which has area (n−1)/2n. Next suppose that k = 1. Clearly, this forces r≤2. Suppose that r = 2. If the period cycle is empty, then the signature is (0; +;[2,m]; {()}), with m > 2, which has area (m−2)/2m. If the period cycle is non-empty, then the area inequality forces both ordinary periods to be 2. So the signature is (0; +; [2,2], {(n)}), which has area (n−1)/2n. In each of these cases, $\mu(\Gamma)/2\pi \geq 1/4$.

Now let r = 1. The signature for this type of NEC group is (0; +;[m]; {C}). If C is empty, then the group G is abelian. Suppose that C = $(n_1,\ldots,n_t)$. Since G has rank 3 or greater, it follows that t≥2. If m≥4, then t = 2 and m = 4. If m = 2, then t≤3 and G has rank 4 or less and is generated by involutions. In each of these cases, $\mu(\Gamma)/2\pi \geq 1/4$. Finally, suppose that r = 0. The signature in this case is (0; +; [ ]; $\{(n_1,\cdots,n_t)\}$) with t≤5. Since G is generated by involutions, t = 5 and $\mu(\Gamma)/2\pi \geq 1/4$.

Suppose that U/Γ is non-orientable. In this case, p = 1 and r = 0. The period cycle must be non-empty for the area to be positive and the only possibility is (1;−; [ ];{(n)}), which has area (n−1)/2n.

Case 2: Suppose that k = 0. Suppose that U/Γ is orientable. It is clear p = 0 and the signature must be of the form (0; +; $[m_1,\cdots,m_r]$,{}) with 3≤r≤4. Since G has rank 3 or higher, r = 4 and $\mu(\Gamma)/2\pi \geq 1/4$. If U/$\Gamma$ is non-orientable, then p = 1 and r≤2. If r = 1, then we get a cyclic group. So the signature that we need to consider is (1;−;[2,m];{}), with area (m−2)/2m. Since m≥4, $\mu(\Gamma)/2\pi \geq 1/4$. This concludes the list of possible signatures and the proof of the theorem.

It is worthwhile to summarize the signatures that result in the following special case. Suppose that G is a 2-group with at least one generator of order larger than 2 in all generating sets. If G is the image of an NEC group with area less than 1/2 then it has one of the following signatures:

(1) $(0;+;[2,m];\{()\})$ and $m\geq 4$
(2) $(0;+;[\lambda];\{(m,n)\})$ and $\lambda\geq 4$
(3) $(0;+;[m_1,\cdots,m_r];\{\})$
(4) $(0;+;[\lambda];\{(n)\})$ and $\lambda\geq 4$
(5) $(1;-;[\ ];\{(n)\})$
(6) $(1;-;[2,m];\{\})$ and $m\geq 4$

Groups with the signatures (3) with $r=3$, (4), (5) and (6) are two generator groups. Finally, we note the following corollary.

COROLLARY 5.3. *A 3-generator group of order 32 with at least one generator of order 4 or higher that has genus greater than one, has genus at least five.*

This corollary comes from applying Theorem 5.2 to the groups of order 32 and it is best possible. There are several examples, but I will mention G47 which is the image of $(0;+;[2,8,8];\{\})$ and hence has genus 5 by Corollary 5.3.

THEOREM 5.4. *Let* G *be a non-abelian 2-group with* $\sigma(G)\geq 2$. *If* $|G|=2^n$, *then* $\sigma(G)\geq 1+2^{n-5}$.

Proof. Suppose that Γ is an NEC group which maps onto G, the kernel is a Fuchsian surface group and $\mu(\Gamma)/2\pi\leq 1/16$.

Case 1: Suppose that $k=0$. Suppose that U/Γ is orientable. It is clear that $p=0$ and $r=3$. In particular, the NEC group with the smallest positive area has signature $(0;+;[2,4,8];\{\})$ and the area is 1/8 (see [**16**]). If U/Γ is non-orientable then the NEC group has signature $(1;-;[m,n];\{\})$ and the smallest positive non-euclidean area is 1/4.

Case 2: Suppose that $k\geq 1$. Suppose that U/Γ is orientable. It is clear that $p=0$, $k=1$ and $r\leq 1$. Suppose that $r=1$. Therefore, the NEC group has signature $(0;+;[m];\{C\})$ where $C=(n_1,\cdots,n_t)$ is non-empty. It is easy to see that $t\leq 2$. If $t=2$, then $m=2$ and the NEC group with the smallest positive area has signature $(0;+;[2];\{(2,4)\})$ and the area is 1/8. If $t=1$, then $m\geq 4$ and an NEC group with the smallest positive area has signature $(0;+;[4];\{(4)\})$ or $(0;+;[8];\{(2)\})$. The area in both cases is 1/8.

Now suppose that $r=0$. Therefore, the NEC group has signature $(0;+;[\ ];\{C\})$ where $C=(n_1,\cdots,n_t)$. Since any group generated by two involutions is dihedral and has genus 0, we see that $3\leq t\leq 4$. If $t=4$, the NEC group with the smallest positive area has signature $(0;+;[\ ];\{(2,2,2,4)\})$ and the area is 1/8. Therefore, $t=3$ and Γ is an extended triangle group. Since the area of an extended triangle group is half the area of the corresponding triangle group, by case 1, the NEC group with the smallest positive area is the extended triangle group with signature $(0;+;[\ ];\{(2,4,8)\})$ and area 1/16. It is easily checked that the non-orientable case has area that is too large. Thus, we conclude that $\mu(\Gamma)/2\pi\geq 1/16$ and equality occurs only for the extended triangle group $\Gamma[2,4,8]$. This proves the theorem.

There are groups which are the image of the extended triangle group. Zomorrodian [**16**] has shown that there are 2-groups of every order greater than or equal to 16 which are images of the triangle group Γ with signature $(0;+;[2,4,8],\{\})$ by a Fuchsian surface group. Suppose Γ maps onto a group G generated by two elements R and S satisfying

$$R^2 = S^4 = (RS)^8 = 1.$$

Adjoin an element W of order 2 that transforms the elements of G according to the automorphism $\alpha(R)=R^{-1}$, $\alpha(S)=S^{-1}$. The new group G* will be an image of the extended triangle group. Thus there are images of the extended triangle group $\Gamma[2,4,8]$ by a Fuchsian surface group of every possible order greater than 16. Unfortunately, the examples of this construction which have order 32 or 64 are all groups of genus 1. It is unlikely that this will always be the case, but showing that groups of order 128 or higher are not toroidal is a difficult problem.

6. Genus of the remaining groups. There are 5 groups (G12, G16, G25, G37 and G41) in Table 4 with ordered triple (1,2,0). The groups G12, G16, G25 and G37 are all images of the NEC group Γ with signature $(0;+;[4,4];\{()\})$. In addition, Theorem 2 [**9**, p. 121] asserts that their genus is greater than or equal to 9. Thus the groups G 12, G 16, G25 and G37 all have symmetric genus equal to 9.

The group G41 is the image of the NEC group Γ with signature $(0;+;[2,4,4,4];\{\})$ and $\mu(\Gamma)/2\pi=3/4$ and thus has genus at most 13. The group G41 cannot be an image of the NEC group with signature $(0;+;[4,4];\{()\})$ because the product of the generators of order 4 (which must have order 4) must commute with the generator of order 2. However the centralizer of any generator of order 2 is the subgroup of order 8 consisting of the elements of order 2 and the identity. It is easy to show that there are no other possibilities with $k\geq 1$. If $k=0$, then there are no possibilities with smaller area and sufficient generators. Thus G41 has genus 13.

There are 3 groups (G15, G35 and G40) in Table 4 with ordered triple (0,3,0). The groups G15, G35 and G40 are all images of the NEC group Γ with signature $(0;+;[4,4,4,4];\{\})$. In addition, Theorem 2 [**9**, p. 121] asserts that their genus is greater than or equal to 17. Thus the groups G15, G35 and G40 all have symmetric genus equal to 17.

There are 4 groups (G18, G28, G29 and G30) in Table 4 with ordered triple (0,2,0). These groups have minimal genus of 5 by Theorem 2 [**9**, p. 121] and G18 attains this genus as an image of the triangle group $\Gamma(4,4,4)$. Each of the groups G28, G29 and G30 has a subgroup N of order 16 which is the union of the Frattini subgroup and the set of elements of order 8. So any generators of order 4 are in the coset of N. It follows that the product of two generators of order 4 must have order 8. Indeed, each of these groups is the image of the triangle group $\Gamma(4,4,8)$ and has genus 7.

Now we must show that G28, G29 and G30 cannot have genus less than 7. The non-euclidean area associated with the triangle group $\Gamma(4,4,8)$ is 3/8. Since $n_2=0$, we need only consider the triangle groups, signature (3). As we have seen the triangle group with minimal area which maps onto any of these groups is $\Gamma(4,4,8)$. This shows that G28, G29 and G30 cannot have genus less than 7.

There are five groups (G11, G24, G38, G39 and G45) in Table 4 with ordered triple (2,1,0). By the corollary, each of these groups has minimal genus equal to 5. The groups G11 and G38 are images of the NEC group with signature $(0;+;[2,4];\{()\})$ and

G24 is the image of the NEC group with signature (0; + ;[4];{(2,2)}). Thus G11, G24 and G38 have genus 5.

The group G39 is an image of the NEC group with signature (0; + ;[2,2,4,4];{}) and G45 is the image of the NEC group with signature (0; + ;[2,8];{()}). Thus the genus of G39 is 9 or less and the genus of G45 is 7 or less. In order to show that their genus is equal to 9 and 7, respectively, we must show that they are not images of the NEC groups with signatures (1) or (2). In G39, the center and Frattini subgroup coincide. Every generator of order 2 is a product of one of two elements (A or AC) and an element of the Frattini subgroup. Since the order of A*(AC) is 4, in any generating set the product of the elements of order 2 must have order 4. This shows that NEC groups with signature (2) and genus less than 9 cannot map onto G39. In addition, the centralizer of any generator of order 2 is the group of order 8 generated by that element and the Frattini subgroup, and thus all elements of the centralizer have order 2. Any generator of order 4 is in the coset of the normal subgroup of order 16 which contains all elements of order 2. This implies that the product of a generator of order 2 and one of order 4 has order 4 and cannot centralize a generator of order 2. This rules out G39 being the image of any NEC group with signature (1) and genus less than 9.

In G45, the product of two generators of order 2 has order 4. Thus there is no NEC group with signature of Type (2) that maps onto G45 and has small enough area. Finally, a computer search shows that the NEC group with signature (0; + ;[2,4],{()}) cannot map onto G45.

The remaining groups that cannot be generated by involutions are G13, G19, G32 and G48. If G13 has genus less than 9, then it must be the image of an NEC group with signature (0; + ;[2,8];{()}), (0; + ;[8];{(m,n)}) or (0; + ;[2,2,8,8];{}). It is the image of all of them (where m = n = 2) and hence has genus 7. The groups G19, G32 and G48 all have rank 2 and all involutions are non-generators (being in the Frattini subgroup). Therefore, we need only consider NEC groups with signature of Type (3). The groups G19 and G48 are images of the triangle group (0; + ;[4,8,8];{}) and this is obviously minimal. Thus G19 and G48 have genus 9. Similarly, G32 is the image of the triangle group (0; + ;[8,8,8];{}) and has genus 11.

Finally, there are two non-toroidal groups which are generated by involutions, G10 and G43. Both groups have genus less than or equal to 9 and have rank 4 or larger. If a group G has genus less than 9, has rank 4 and is generated by involutions, then it is the image of the NEC group with signature either (0; + ;[2];{(2,2,n)}) or (0; + ;[ ];{($a_1,\cdots,a_t$)}). The group G10 is the image of the NEC group with signature (0; + ;{(2,2,4,4)}) and hence has genus 5 or less. It is easy to see that G10 has genus 5, since it does not have genus 3 [**10**]. The group G43 is the image of the NEC group with signature (0; + ;[2,2,2,2,2];{}) and also (0; + ;[ ];{(4,4,4,4)}) and thus has genus 9 or less. The group G43 also has the property that two elements of order 2, say x and y, have product xy of order 2 if and only if y is x times the element in both the center and the Frattini subgroup. This rules out all signatures above which have genus less than 9 and so G43 has genus 9.

This completes the calculation of the symmetric genus of all the groups of order 32. The lower bound formulas given in Section 5 are better in some cases than the formula in Theorem 2 [**9**]. It would be desirable to improve the bound given in Theorem 2 [**9**] in the case when G is a 2-group. However, it appears that a general formula, involving a parameter, that is better than Theorem 2 [**9**] in the case of 2-groups is extremely difficult or impossible to obtain.

REFERENCES

1. W. Burnside, *Theory of groups of finite order*, (Cambridge University Press, 1911).

2. M. D. E. Conder, Hurwitz groups: a brief survey, *Bull. Amer. Math. Soc. (N.S.)* **23** (1990), 359–370.

3. J. L. Gross and T. W. Tucker, *Topological graph theory* (Wiley, 1987).

4. M. Hall and J. K. Senior, *The groups of order* 2^n $(n \leq 6)$, (MacMillan, 1964).

5. A. Hurwitz, Über algebraische Gebilde mit eindeutigen Transformationen in sich, *Math. Ann.* **41** (1893), 403–442.

6. C. Maclachlan, Abelian groups of automorphisms of compact Riemann surfaces, *Proc. London Math. Soc.* **15** (1965), 699–712.

7. C. L. May and J. Zimmerman, The symmetric genus of finite abelian groups, *Illinois J. Math.* **37** (3) (1993), 400–423.

8. C. L. May and J. Zimmerman, The symmetric genus of metacyclic groups, *Topology Appl.* **66** (1995), 101–115.

9. C. L. May and J. Zimmerman, Groups of small symmetric genus, *Glasgow Math. J.* **37** (1995), 115–129.

10. C. L. May and J. Zimmerman, The groups of symmetric genus three, *to appear*.

11. C. L. May and J. Zimmerman, Correction to the symmetric genus of finite abelian groups, *to appear*.

12. M. Schönert et al., *GAP—Groups, algorithms, and programming*. (Lehrstuhl D für Mathematik, Rheinisch Westälische Technische Hochschule, Aachen, Germany, fourth edition, 1994).

13. D. Singerman, On the structure of non-Euclidean crystallographic groups, *Proc. Cambridge Philos. Soc.* **76** (1974), 233–240.

14. T. W. Tucker, Finite groups acting on surfaces and the genus of a group, *J. Combin. Theory Ser.* **B 34** (1983), 82–98.

15. A. T. White, *Graphs, groups and surfaces*, Revised edition, (North-Holland, 1984).

16. R. Zomorrodian, Nilpotent automorphism groups of Riemann surfaces, *Trans. Amer. Math. Soc.* **288** (1) (1985), 241–255.