Can. J. Math., Vol. XXXVI, No. 4, 1984, pp. 685-717

ORTHOGONAL POLYNOMIALS WITH SYMMETRY OF ORDER THREE

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The measure $(x_1x_2x_3)^{2a}dm(x)$ on the unit sphere in $\mathbf{R}^3$ is invariant under sign-changes and permutations of the coordinates; here dm denotes the rotation-invariant surface measure. The more general measure

$$x_1^{2a}x_2^{2b}x_3^{2c}dm(x)$$

corresponds to the measure

$$v_1^{\alpha}v_2^{\beta}(1 - v_1 - v_2)^{\gamma}dv_1dv_2$$

on the triangle

$$E: = \{(v_1, v_2):v_1, v_2 \geqq 0:v_1 + v_2 \leqq 1\}$$

(where $\alpha = a - \frac{1}{2}$, $\beta = b - \frac{1}{2}$, $\gamma = c - \frac{1}{2}$, $v_i: = x_i^2$, $1 \leqq i \leqq 3$). Appell ([**1**] Chap. VI) constructed a basis of polynomials of degree n in v_1, v_2 orthogonal to all polynomials of lower degree, and a biorthogonal set for the case $\gamma = 0$. Later Fackerell and Littler [**6**] found a biorthogonal set for Appell's polynomials for $\gamma \neq 0$. Meanwhile Proriol [**10**] had constructed an orthogonal basis in terms of Jacobi polynomials. Indeed there are three different families of this type which transform to each other under permutations of coordinates (and parameters). For example, the involution $v_1 \leftrightarrow v_2$ is diagonalized by one such basis, but all other possible nontrivial permutations are represented by matrices with Racah-Wilson polynomial (balanced ${}_4F_3$-series, see [**14**]) entries, in this basis.

The aim in this paper is to construct an orthogonal basis of polynomials on which the Abelian group of order three generated by cyclic permutations of the coordinates acts diagonally (where $\alpha = \beta = \gamma = a - \frac{1}{2}$). This basis will be realized as the eigenvector decomposition of a self-adjoint third-order differential operator.

The first stage of orthogonal decomposition is easy: fix $\alpha > -1$, let $\beta = \gamma = \alpha$ and define H_n^{α} (for $n \geqq 0$) to be the space of (complex) polynomials in v_1, v_2 of degree $\leqq n$ which are orthogonal to all polynomials of lower degree. (There is a second-order differential operator analogous to the spherical Laplacian which has each H_n^{α} as an eigenmanifold.)

Received June 8, 1983. During the preparation of this paper the author was partially supported by NSF Grant MCS 81-02581.

To more neatly represent cyclic permutations we introduce the complex coordinate

$$z: = x_1^2 + \omega x_2^2 + \overline{\omega} x_3^2 \quad \text{where } \omega = e^{2\pi i/3}.$$

The measure transforms to a multiple of

$$(z^3 + \overline{z}^3 - 3z\overline{z} + 1)^\alpha dm_2(z)$$

on the triangle with vertices 1, ω, $\overline{\omega}$ in $\mathbf{C}$ (where dm_2 is Lebesgue measure on $\mathbf{R}^2$). An approach that was used by Koornwinder [7] on the region bounded by a three-cusped deltoid (Steiner's hypocycloid) to find an orthogonal basis, namely, polynomials of the form $z^{n-m}\overline{z}^m + p_{n-1}(z, \overline{z})$ which are orthogonal to all polynomials of lower degree, does not work in our situation.

The construction of the third-order operator is based on infinitesimal rotations. An appropriately invariant operator which is self-adjoint for $dm(x)$ on S^2 is constructed, and then modified (in its first- and second-order terms) to become self-adjoint for $(x_1x_2x_3)^{2a}dm(x)$. Restricted to H_n^α for given n (and $\alpha = a - \frac{1}{2}$) the self-adjoint operator, called D_α, is represented by a Hermitian tridiagonal matrix with respect to the normalized Jacobi-type basis. Its characteristic polynomial is the end-product of a chain of three-term recurrences, whose intermediate results form a family of polynomials orthogonal with respect to a discrete measure supported by the eigenvalues, which are thus pairwise distinct. We will discuss the connections between these orthogonal polynomials and the eigenvectors of D_α. From the limiting behavior as $\alpha \rightarrow \infty$, which will be explicitly described, the effect of a cyclic permutation on any given eigenvector (they are labelled in order of magnitude of the eigenvalues) can be found.

Indeed for given α, n let the eigenvalues of $D_\alpha|H_n$ be $\lambda_0 < \lambda_1 < \lambda_2 \ldots < \lambda_n$ with the eigenvectors q_{nj}^α associated to λ_j, $0 \leqq j \leqq n$, then

$$Uq_{nj}^\alpha = \omega^{n-2j} q_{nj}^\alpha,$$

where u is the permutation

$$Uf(x_1, x_2, x_3): = f(x_3, x_1, x_2).$$

Here is an outline of the sections of this paper:

Section 1. Background: general theory of polynomials on the sphere orthogonal with respect to a measure invariant under a reflection group, and an associated differential operator; families of two-variable Jacobi polynomials orthogonal for

$$v_1^\alpha v_2^\beta (1 - v_1 - v_2)^\gamma dv_1 dv_2$$

on the triangle E; transformations of these families in terms of ${}_4F_3$-series.

Section 2. The symmetric case: specialize to the measure

$$(v_1v_2(1 - v_1 - v_2))^\alpha dv_1dv_2$$

on E, the complex coordinate system, a basis for H_n^α of polynomials in $(z, \bar{z})$ constructed by means of a differential operator; the limiting behavior as $\alpha \to \infty$.

Section 3. The self-adjoint third-order differential operator: the construction, tridiagonal matrix representation with respect to the Jacobi-type basis, the family of discrete orthogonal polynomials related to the characteristic polynomial (on each H_n^α), the eigenvector decomposition; behavior as $\alpha \to \infty$.

Section 4. Consequences and further problems: limiting behavior as $\alpha \to -1$, degeneracies of the eigenvectors and eigenvalues; a four-term contiguity relation for a certain balanced ${}_4F_3$-series implied by the permutation invariance of D_α.

1. Background. Here are the general results from [**5**] which give a foundation for this work. Suppose that h is a product of homogeneous linear functions on $\mathbf{R}^N$ and G is a finite reflection group (fixing the origin, a subgroup of $O(N)$), then say that h satisfies condition $(*)$ for G if the reflections in the zero-sets of the factors of h generate G, and

$$h(\sigma x) = \pm h(x), \quad (\sigma \in G, x \in \mathbf{R}^N).$$

Define the linear differential operator L_n by

$$L_h f: = \Delta(fh) - f\Delta h, \quad (f \in C^\infty(\mathbf{R}^N)),$$

where Δ is the Laplacian $\sum_{i=1}^N \left(\frac{\partial}{\partial x_i}\right)^2$.

1.1. THEOREM. *If f is a polynomial, h satisfies $(*)$ for G, and $L_h f = 0$ then f is invariant under G.*

This says that solutions of $L_h f = 0$ are to be found in the algebra of G-invariant polynomials. Thus define P_n^G to be the space of G-invariant polynomials, homogeneous of degree n, and let

$$H_n^h: = P_n^G \cap \ker L_h.$$

Further let $S: = \{x \in \mathbf{R}^N: |x| = 1\}$, the unit sphere, be furnished with the normalized rotation-invariant surface measure $d\omega$. The analysis of H_n^h takes place in $L^2(S; h^2 d\omega)$.

1.2. THEOREM. *If $f \in H_n^h$, $g \in H_m^h$, $n \neq m$, then*

$$\int_S fgh^2 d\omega = 0.$$

1.3. THEOREM. $P_n^G = \sum_{j=0}^{[n/2]} \oplus |x|^{2j} H_{n-2j}^h$ *(direct sum in $L^2(S; h^2 d\omega)$).*

This shows that each G-invariant polynomial has a unique expansion in terms of the form $|x|^{2m}p_k(x)$ with $p_k \in H_k^h$. Further, H_n^h consists exactly of those elements of P_n^G which are orthogonal to all G-invariant polynomials of degree $<n$. Thus if a self-adjoint operator (densely defined) on $L^2(h^2 d\omega)$ leaves each P_n^G invariant, then it leaves each H_n^h invariant.

We will need the infinitesimal rotations. For $j \neq k$ define

$$R_{jk}: = x_j\frac{\partial}{\partial x_k} - x_k\frac{\partial}{\partial x_j}.$$

The surface Laplacian (the Laplace-Beltrami or Casimir operator for S) is

$$\Delta_S: = \sum_{1 \leqq j < k \leqq n} R_{jk}^2.$$

It is closely related to Δ since

$$\Delta_S f(x) = |x|^2 f(x) - (N - 2)\sum_{i=1}^{N} x_i\frac{\partial}{\partial x_i} f(x) - \left(\sum_{i=1}^{N} x_i\frac{\partial}{\partial x_i}\right)^2 f(x).$$

There is an obvious extension of Δ_S to the L_h-theory; indeed let

$$\Delta_{S,h}: = \Delta_S(fh) - f\Delta_S h.$$

If f is homogeneous of degree m then

$$\Delta_{S,h}f = |x|^2 L_h f - m(m + N + 2 \deg h - 2)hf.$$

1.4. PROPOSITION. *If $f \in P_m^G$, then $f \in H_m^h$ if and only if f is an eigenfunction of the operator $(1/h)\Delta_{S,h}$ with eigenvalue $-m(m + N + 2 \deg h - 2)$.*

For the rest of the paper we will deal only with the situation $N = 3$, $h(x) = x_1^a x_2^b x_3^c$. The theory discussed above applies fully to the values a, b, $c = 1, 2, 3\ldots$ but other real values will occasionally be used in the development.

The corresponding reflection group G is $(Z_2)^3$ and the invariants are exactly the polynomials in x_1^2, x_2^2, x_3^2. Thus

$$\dim P_{2m}^G = \binom{m+2}{m} \quad \text{and}$$

$$\dim H_{2m}^h = \dim P_{2m}^G - \dim P_{2m-2}^G = m + 1.$$

We introduce the variables $y_i: = x_i^2$ and the derivations

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$$\partial_i : = \frac{\partial}{\partial v_i}, \quad 1 \leqq i \leqq 3.$$

For $h = x_1^a x_2^b x_3^c$ we have

$$(1/h)L_h f(v) = 4\Big(\sum_{i=1}^{3} v_i \partial_i^2 + (a + \tfrac{1}{2})\partial_1 + (b + \tfrac{1}{2})\partial_2 + (c + \tfrac{1}{2})\partial_3 \Big) f(v).$$

A G-invariant function is determined by its values on certain triangular sectors of S, such as the first octant ($x_i \geqq 0$); that is, the region

$$E: = \{ (v_1, v_2) : v_1, v_2 \geqq 0; v_1 + v_2 \leqq 1 \} \subset \mathbf{R}^2.$$

We convert $h^2 d\omega$ to a measure on E.

1.5. LEMMA.

$$\int\int_E v_1^\alpha v_2^\beta (1 - v_1 - v_2)^\gamma dv_1 dv_2 = \frac{\Gamma(\alpha + 1)\Gamma(\beta + 1)\Gamma(\gamma + 1)}{\Gamma(\alpha + \beta + \gamma + 3)},$$

for $\alpha, \beta, \gamma > -1$.

1.6. LEMMA. *For f continuous,*

$$\int_S f(x_1^2, x_2^2, x_3^2) d\omega(x) = \frac{1}{2\pi} \int\int_E f(v_1, v_2, 1 - v_1 - v_2) \times (v_1 v_2 (1 - v_1 - v_2))^{-\frac{1}{2}} dv_1 dv_2.$$

We see that the measure $h^2 d\omega$ corresponds to a scalar multiple of

$$v_1^{a-\frac{1}{2}} v_2^{b-\frac{1}{2}} (1 - v_1 - v_2)^{c-\frac{1}{2}} dv_1 dv_2$$

on E. A family of orthogonal polynomials for this weight is known (see [**8**], [**10**]) in terms of Jacobi polynomials. We use a shifted, normalized Jacobi polynomial:

$$R_n^{(\alpha,\beta)}(s): = {}_2F_1\left(\begin{matrix} -n, n + \alpha + \beta + 1 \\ \alpha + 1 \end{matrix} ; s \right),$$

then

$$\int_0^1 R_n^{(\alpha,\beta)}(s) R_m^{(\alpha,\beta)}(s) s^\alpha (1 - s)^\beta ds = 0 \quad \text{for } m \neq n.$$

Also we continue to use v_3 with the understanding that $v_1 + v_2 + v_3 = 1$ on E. Let

$$d\mu(v) = k_{\alpha\beta\gamma} v_1^\alpha v_2^\beta v_3^\gamma dv_1 dv_2.$$

where

$$k_{\alpha\beta\gamma} := (\Gamma(\alpha + 1)\Gamma(\beta + 1)\Gamma(\gamma + 1)/\Gamma(\alpha + \beta + \gamma + 3))^{-1},$$

so that $\alpha = a - \frac{1}{2}, \beta = b - \frac{1}{2}, \gamma = c - \frac{1}{2}$.

We define one of the possible families of polynomials, for $0 \leqq m \leqq n$,

$$\phi_{nm}(v) := (\alpha + 1)_{n-m}(\beta + 1)_m R_{n-m}^{(\alpha,\beta+\gamma+2m+1)}\left(v_1 \Big/ \sum_i v_i\right) \times \left(\sum_i v_i\right)^{n-m}(v_2 + v_3)^m R_m^{(\beta,\gamma)}(v_2/(v_2 + v_3)).$$

By Pfaff's transformation,

$$\phi_{nm}(v) = (\alpha + 1)_{n-m}(\beta + 1)_m(v_2 + v_3)^{n-m} \times {}_2F_1\left(\begin{matrix} m - n, -n - m - \beta - \gamma - 1 \\ \alpha + 1 \end{matrix}; \frac{-v_1}{v_2 + v_3}\right) \times v_3^m \, {}_2F_1\left(\begin{matrix} -m, -m - \gamma \\ \beta + 1 \end{matrix}; -\frac{v_2}{v_3}\right),$$

a useful form. Thus ϕ_{nm} is homogeneous of degree n in v_1, v_2, v_3 and of degree $\leqq n - m$ in v_1. By the use of known integrals of Jacobi polynomials (see [**12**], p. 68) we obtain

$$k_{\alpha\beta\gamma} \iint_E \phi_{nm}(v)\phi_{kl}(v)v_1^\alpha v_2^\beta v_3^\gamma \, dv_1 dv_2 = \delta_{nk}\delta_{ml}N_{nm}(\alpha, \beta, \gamma)$$

where

$$N_{nm}(\alpha, \beta, \gamma) := \frac{(\beta + \gamma + m + 1)(\beta + \gamma + m + 2)_n(\beta + 1)_m}{(\beta + \gamma + 2m + 1)(\alpha + \beta + \gamma + 3)_{n+m}} \times \frac{(\gamma + 1)_m(\alpha + 1)_{n-m}(\alpha + \beta + \gamma + n + m + 2)m!(n - m)!}{(\alpha + \beta + \gamma + 2n + 2)},$$

for $0 \leqq m \leqq n$. By cyclically permuting $(v_1, \alpha), (v_2, \beta), (v_3, \gamma)$ we obtain two other orthogonal bases:

$$\psi_{nm}(v) := (\beta + 1)_{n-m}(\gamma + 1)_m R_{n-m}^{(\beta,\gamma+\alpha+2m+1)}\left(v_2 \Big/ \sum_i v_i\right) \times \left(\sum_i v_i\right)^{n-m}(v_3 + v_1)^m R_m^{(\gamma,\alpha)}(v_3/(v_3 + v_1));$$

$$\theta_{nm}(v) := (\gamma + 1)_{n-m}(\alpha + 1)_m R_{n-m}^{(\gamma,\alpha+\beta+2m+1)}\left(v_3 \Big/ \sum_i v_i\right) \times \left(\sum_i v_i\right)^{n-m}(v_1 + v_2)^m R_m^{(\alpha,\beta)}(v_1/(v_1 + v_2)),$$

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with orthogonalities

$$k_{\alpha\beta\gamma}\int\int_{E}\psi_{nm}(v)\psi_{kl}(v)v_1^{\alpha}v_2^{\beta}v_3^{\gamma}dv_1dv_2 = \delta_{nk}\delta_{ml}N_{nm}(\beta, \gamma, \alpha)$$

and

$$k_{\alpha\beta\gamma}\int\int_{E}\theta_{nm}(v)\theta_{kl}(v)v_1^{\alpha}v_2^{\beta}v_3^{\gamma}dv_1dv_2 = \delta_{nk}\delta_{ml}N_{nm}(\gamma, \alpha, \beta).$$

The other three possible permutations lead to no new bases; from the relation

$$R_m^{(\beta,\alpha)}(1 - s) = (-1)^m((\alpha + 1)_m/(\beta + 1)_m)R_m^{(\alpha,\beta)}(s)$$

we see that ϕ_{nm} transforms to $(-1)^m\phi_{nm}$ under the transposition of (v_2, β) and (v_3, γ), as one example.

It is striking that the orthogonal matrices expressing the transformations between the normalized versions of these bases are given in terms of the Racah-Wilson polynomials (balanced ${}_4F_3$-series, orthogonal with respect to a finite discrete measure, see [**13**]).

1.7. THEOREM. *For* $0 \leqq m, k \leqq n$:

i) $$\psi_{nk}N_{nk}(\beta, \gamma, \alpha)^{-\frac{1}{2}} = \sum_{m=0}^{n}(-1)^k M_{nmk}(\alpha, \beta, \gamma)\phi_{nm}N_{nm}(\alpha, \beta, \gamma)^{-\frac{1}{2}};$$

ii) $$\theta_{nk}N_{nk}(\gamma, \alpha, \beta)^{-\frac{1}{2}} = \sum_{m=0}^{n}(-1)^k M_{nmk}(\beta, \gamma, \alpha)\psi_{nm}N_{nm}(\beta, \gamma, \alpha)^{-\frac{1}{2}};$$

iii) $$\phi_{nk}N_{nk}(\alpha, \beta, \gamma)^{-\frac{1}{2}} = \sum_{m=0}^{n}(-1)^k M_{nmk}(\gamma, \alpha, \beta)\theta_{nm}N_{nm}(\gamma, \alpha, \beta)^{-\frac{1}{2}};$$

where

$$M_{nmk}(\alpha, \beta, \gamma):$$

$$= (-1)^{n+m+k}n!\,{}_4F_3\left(\begin{matrix}-k, k + \alpha + \gamma + 1, -m, m + \beta + \gamma + 1\\ -n, \gamma + 1, n + \alpha + \beta + \gamma + 2\end{matrix}; 1\right)$$

$$\times\left(\frac{(\alpha + 1)_{n-m}(\beta + 1)_{n-k}(\gamma + 1)_m(\gamma + 1)_k(n + \alpha + \beta + \gamma + 2)_m}{m!k!(n - m)!(n - k)!(m + \beta + \gamma + 2)_n(k + \alpha + \gamma + 2)_n}\right.$$

$$\left.\times\frac{(n + \alpha + \beta + \gamma + 2)_k(\beta + \gamma + 2m + 1)(\alpha + \gamma + 2k + 1)}{(\alpha + 1)_k(\beta + 1)_m(\beta + \gamma + m + 1)(\alpha + \gamma + k + 1)}\right)^{\frac{1}{2}}.$$

For fixed n, α, β, γ *the matrix* $(M_{nmk}(\alpha, \beta, \gamma))_{m,k=0}^{n}$ *is orthogonal.*

Proof. This is a direct consequence of a similar fact about Hahn polynomials in two variables, which was proved in [**3**]. The idea is to set up the weight

$$((\alpha + 1)_{y_1}(\beta + 1)_{y_2}(\gamma + 1)_{K-y_1-y_2})/(y_1!y_2!(K - y_1 - y_2)!)$$

on the (integer) lattice points in the triangle $y_1, y_2 \geqq 0, y_1 + y_2 \leqq K$; for which there are three families of Hahn polynomials similar to ϕ, ψ, θ; replace y_1, y_2 by Kv_1, Kv_2 respectively, and let $K \to \infty$. The ${}_4F_3$ function stated here is obtained from the one in Proposition 5.4 of [**3**] by a standard transformation ([**2**], p. 56).

1.8. COROLLARY.

$$\sum_{k=0}^{n} (-1)^k M_{njk}(\beta, \gamma, \alpha)M_{nkl}(\gamma, \alpha, \beta) = (-1)^{j+l}M_{lj}(\alpha, \beta, \gamma),$$

for $0 \leqq j, l \leqq n$.

Proof. Transform from the ψ-basis to the ϕ-basis by using the composition of (ii) and (iii), and directly, by using the inverse (adjoint) of (i).

This leads to an orthogonal matrix of period 3 when $\alpha = \beta = \gamma$. It is this situation that will be studied in detail in the sequel. (We caution the reader that a permutation of $(v_1, \alpha)(v_2, \beta)(v_3, \gamma)$ is a transformation of identities, not a well-defined linear transformation. For example the cycle $((v_1, \alpha) \to (v_2, \beta) \to (v_3, \gamma) \to)$ maps ϕ_{nm} to ψ_{nm}, but the corresponding matrix

$$((-1)^k M_{nmk}(\alpha, \beta, \gamma))_{m,k=0}^{n}$$

is not in general of period three, indeed the $m = k = 0$ entry for $n = 1$ is

$$-((\alpha + 1)(\beta + 1)/(\alpha + \gamma + 2)(\beta + \gamma + 2))^{\frac{1}{2}} \neq -1/2 \text{ usually.})$$

2. The third order symmetry. Henceforth $\alpha = \beta = \gamma$ and we will refer to

$$M_{nmk}(\alpha) := M_{nmk}(\alpha, \alpha, \alpha);$$

$$N_{nm}(\alpha) := N_{nm}(\alpha, \alpha, \alpha); \quad \text{and}$$

$$k_\alpha := \Gamma(3\alpha + 3)/\Gamma(\alpha + 1)^3.$$

The measure on E is

$$d_{\mu_\alpha} := k_\alpha(v_1v_2v_3)^\alpha dv_1dv_2,$$

and S_3, the symmetric group on 3 letters, acts as a group of isometries in $L^2(E, \mu_\alpha)$ by permuting (v_1, v_2, v_3). This group is generated by the operators U and J where

$$Uf(v_1, v_2, v_3) := f(v_3, v_1, v_2)$$

and

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$$Jf(v_1, v_2, v_3) := f(v_1, v_3, v_2).$$

Thus U generates the group Z_3, and it is this action that we wish to diagonalize. Note that $U\phi_{nm} = \theta_{nm}$ so that the matrix of U in the $\phi_{nm}N_{nm}^{-\frac{1}{2}}$ basis is

$$U_{km} = (-1)^k M_{nkm}(\alpha).$$

We introduce the coordinate system $z := v_1 + \omega v_2 + \bar{\omega} v_3$, $t := v_1 + v_2 + v_3$, where $\omega := e^{2\pi i/3}$ (note $\omega^2 = \bar{\omega}$ and $\omega + \bar{\omega} + 1 = 0$). Let Ω be the closed convex hull of $1, \omega, \bar{\omega}$ in $\mathbf{C}$ then

$$E = \{ (v_1, v_2): v_1, v_2 \geqq 0, v_1 + v_2 \leqq 1, (v_3 = 1 - v_1 - v_2) \}$$

corresponds to $\{ (z, t): z \in \Omega, t = 1\}$. The inverse transformation is

$$v_1 = (z + \bar{z} + t)/3, \quad v_2 = (\bar{\omega} z + \omega \bar{z} + t)/3,$$

$$v_3 = (\omega z + \overline{\omega z} + t)/3.$$

2.1. PROPOSITION. *The space E and the measure μ_α correspond to $\Omega \subset \mathbf{C}$ with the measure*

$$c_\alpha(z^3 + \bar{z}^3 - 3z\bar{z} + 1)^\alpha dm_2(z),$$

where $c_\alpha := 2k_\alpha 3^{-3\left(\alpha + \frac{1}{2}\right)}$ *and m_2 is the $\mathbf{R}^2$-Lebesgue measure on $\mathbf{C}$.*

Let

$$w(z, t) := z^3 + \bar{z}^3 - 3z\bar{z}t + t^3,$$

then Ω is exactly $\{z: w(z, 1) \geqq 0\}$. Of course w is nothing but $27\, v_1 v_2 v_3$ expressed in z and t. Also

$$Uf(z, t) = f(\omega z, t) \quad \text{and} \quad Jf(z, t) = f(\bar{z}, t).$$

The differential operator $(1/h)L_h$ (where $h(x) = (x_1 x_2 x_3)^{\alpha - \frac{1}{2}}$) becomes $4L_\alpha$, where

$$L_\alpha := z\bar{\partial}^2 + \bar{z}\partial^2 + t\partial_t^2 + 2\partial_t(z\partial + \bar{z}\bar{\partial})$$
$$+ 2t\bar{\partial}\partial + 3(\alpha + 1)\partial_t,$$

$$\partial := \frac{\partial}{\partial z}, \; \bar{\partial} := \frac{\partial}{\partial \bar{z}}, \; \partial_t := \frac{\partial}{\partial t}.$$

Further, if p is homogeneous of degree n in $z, \bar{z}, t$ (thus degree $2n$ in the x_i) then $L_h p = 0$ if and only if p is an eigenvector of

$$L_\alpha^0 := -(z\partial + \bar{z}\bar{\partial})^2 - (3\alpha + 2)(z\partial + \bar{z}\bar{\partial}) + 2t^2\partial\bar{\partial}$$
$$+ t(z\bar{\partial}^2 + \bar{z}\partial^2)$$

with eigenvalue $-n(n + 3\alpha + 2)$, (that is, $L_\alpha^S = (1/4h)\Delta_{S,h}$ in (z, t)-coordinates; see Proposition 1.4).

Recall H_n^h is the space of polynomials in x_i homogeneous of degree n such that $L_h p = 0$. Define H_n^α to be the space of polynomials in y_i, or $(z, \bar{z}, t)$, homogenous of degree n, such that $L_\alpha p = 0$ (thus H_n^α corresponds to H_{2n}^h). This gives the orthogonal decomposition

$$L^2(E, \mu_\alpha) = \sum_{n=0}^{\infty} \oplus H_n^\alpha.$$

Also dim $H_n^\alpha = n + 1$, and each H_n^α is an eigenmanifold of L_α^S.

For $n = 0, 1, 2, 3 \ldots$ and $\epsilon = 0, 1, 2$ define $P_{n,\epsilon}$ to be the space of polynomials p in $(z, \bar{z}, t)$ homogeneous of degree n satisfying the relation

$$Up = \omega^\epsilon p.$$

A monomial $z^k \bar{z}^l t^{n-k-l} (k, l \geqq 0;\ k + l \leqq n)$ is in $P_{n,\epsilon}$ exactly when

$$k - l \equiv \epsilon \bmod 3.$$

Since L_α commutes with U we can similarly split H_n^α, indeed, define

$$H_{n,\epsilon}^\alpha := P_{n,\epsilon} \cap \ker L_\alpha.$$

We give an algorithm for a basis of $H_{n,\epsilon}^\alpha$, based on a recurrence relation.

2.2. LEMMA.

$$\begin{aligned} L_\alpha(z^l \bar{z}^m t^{n-l-m}) &= l(l-1) z^{l-2} \bar{z}^{m+1} t^{n-l-m} \\ &+ m(m-1) z^{l+1} \bar{z}^{m-2} t^{n-l-m} + 2lm z^{l-1} \bar{z}^{m-1} t^{n-l-m+1} \\ &+ (n-l-m)(l+m+n+3a+2) z^l \bar{z}^m t^{n-l-m-1}. \end{aligned}$$

A convenient indexing for monomials of the same U-orbit as $z^l \bar{z}^m$ is

$$z^{l-2k+j} \bar{z}^{m+k-2j} t^{k+j}$$

subject to $k + j \geqq 0$, $2k - j \leqq l$, $2j - k \leqq m$; (for fixed l, m the possible values of $s = k + j$ satisfy $2s - m \leqq 3k \leqq l + s$ and $0 \leqq s \leqq l + m$).

2.3. THEOREM. *For fixed n and m with $0 \leqq m \leqq n$ there is a unique polynomial $f_{n,m}^\alpha \in H_{n,\epsilon}^\alpha$ with $\epsilon \equiv n - 2m$* mod 3 *whose only term of degree 0 in t is $z^{n-m}\bar{z}^m$. Further let $l = n - m$, then*

$$L_\alpha\left(\sum_{k,j} c_{kj} z^{l-2k+j} \bar{z}^{m+k-2j} t^{k+j}\right) = 0$$

if and only if

$$c_{kj} = \frac{A_{kj}}{(-2n - 3l - 1)_{k+j}}$$

and $\{A_{kj}\}$ *satisfies*

$$A_{kj} = (1/(k + j))\{(l - 2k + j + 2)(l - 2k + j + 1)A_{k-1,j}$$
$$+ (m + k - 2j + 2)(m + k - 2j + 1)A_{k,j-1}$$
$$- 2(l - 2k + j + 1)(m + k - 2j + 1)$$
$$\times (2n - k - j + 3\alpha + 3)A_{k-1,j-1}\};$$

this recurrence is to be computed in order of $k + j = 1, 2, 3 \ldots n$, *with given values for* $A_{k,-k}$ *and* $A_{kj} = 0$ *for* (k, j) *values outside the permitted region. The polynomial* f^{α}_{nm} *is characterized by* $A_{00} = 1$, $A_{k,-k} = 0$ *for* $k \neq 0$.

Proof. Apply the lemma to find the result of applying L_{α} to the given general polynomial (with c_{kj}), and set the coefficient of

$$z^{l-2k+j}\bar{z}^{m+k-2j}t^{k+j-1}$$

equal to zero. This produces a recurrence for c_{kj}, consequently for A_{kj}. The recurrence shows that each A_{kj} is uniquely determined by the values of $A_{k',j}$ for $k' + j' < k + j$, hence the values $A_{k',-k'}$. (Note there are $n + 1$ such values, and $\dim H^{\alpha}_{n} = n + 1$.)

By using Theorem 2.11 in [**5**] we can give another expression for f^{α}_{nm}, indeed

$$f^{\alpha}_{n,m} = \sum_{j=0}^{n} (j!(-2n - 3\alpha - 1)_j)^{-1}t^j(L_{\alpha})^j(z^{n-m}\bar{z}^m),$$

(valid unless $\alpha = -1$ and $n \leqq 1$).

For given n and $\epsilon = 0, 1, 2$ and let $c \equiv 2n + \epsilon \bmod 3$ with $c = 0, 1, 2$ then

$$\{f^{\alpha}_{n,3j+c}:0 \leqq j \leqq [(n - c)/3]\}$$

is a basis for $H^{\alpha}_{n,\epsilon}$. Conjugation maps $H^{\alpha}_{n,1}$ onto $H^{\alpha}_{n,2}$ and

$$\dim H^{\alpha}_{n,1} = \dim H^{\alpha}_{n,2} = [(n + 2)/3]$$

(the cardinality of $\{(j, k):j + k = n, j \geqq 0, k \geqq 0, j - k \equiv 1 \bmod 3\}$); and thus

$$\dim H^{\alpha}_{n,0} = n + 1 - 2[(n + 2)/3].$$

Here are some low degree examples:

$$f^{\alpha}_{0,0} = 1; f^{\alpha}_{1,0} = z, f^{\alpha}_{1,1} = \bar{z}; f^{\alpha}_{2,0} = z^2 - (2/(3\alpha + 5))\bar{z}t,$$

$$f^{\alpha}_{2,1} = z\bar{z} - (1/(3\alpha + 4))t^2, f^{\alpha}_{2,2} = \overline{f^{\alpha}_{2,0}};$$

$$f^{\alpha}_{3,0} = z^3 - (6/(3\alpha + 7))z\bar{z}t + (4/(3\alpha + 5)(3\alpha + 7))t^3,$$

$$f^{\alpha}_{3,1} = z^2\bar{z} - (2/(3\alpha + 7))\bar{z}^2 t - (2/(3\alpha + 7))zt^2,$$

$$f^{\alpha}_{3,2} = \overline{f^{\alpha}_{3,1}}, f^{\alpha}_{3,3} = \overline{f^{\alpha}_{3,0}}.$$

Unfortunately, it must be stated that $\{f_{n,m}\}$ is not an orthogonal basis. Even though most of the functions in this short list are orthogonal to each other for degree and group invariance reasons (that is, $H^{\alpha}_{n,\epsilon} \perp H^{\alpha}_{m\delta}$ unless $n = m$ and $\epsilon = \delta$), $f^{\alpha}_{3,0}$ and $f^{\alpha}_{3,3}$ are both in $H^{\alpha}_{3,0}$ yet

$$\int\int_{\Omega} f^{\alpha}_{3,0}\overline{f^{\alpha}_{3,3}}\, w^{\alpha} dm_2 \neq 0.$$

We will find a recurrence, but not a closed form, for the integral

$$\int\int_{\Omega} z^k \bar{z}^l w^{\alpha} dm_2.$$

2.4. *Definition*. For $k, l \geqq 0, \alpha > -1$, let

$$I_{\alpha}(k, l) = c_{\alpha}\int\int_{\Omega} z^k\bar{z}^l w(z, 1)^{\alpha} dm_2(z)$$

(note the constant c_{α} makes $I_{\alpha}(0, 0) = 1$)).

2.5. PROPOSITION. $I_{\alpha}(k, l) = 0$ *unless* $k \equiv l \bmod 3$, $I_{\alpha}(k, l) = I_{\alpha}(l, k)$ *and*

$$\begin{aligned}(k + l + 3\alpha + 2)I_{\alpha}(k, l) &= lI_{\alpha}(k - 1, l - 1) \\ &+ (k - 1)I_{\alpha}(k - 2, l + 1) \\ &= kI_{\alpha}(k - 1, l - 1) + (l - 1)I_{\alpha}(k + 1, l - 2).\end{aligned}$$

Proof. The set Ω and the measure are invariant under $U(z \mapsto \omega z)$, but

$$U(z^k\bar{z}^l) = \omega^{k-l}z^k\bar{z}^l$$

hence $I_{\alpha}(k, l) = \omega^{k-l}I_{\alpha}(k, l)$; this proves the first statement. Applying J (conjugation) shows

$$I_{\alpha}(k, l) = I_{\alpha}(l, k).$$

Note

$$w(z, 1) = (z^2 - z\bar{z} + \bar{z}^2 - z - \bar{z} + 1)(z + \bar{z} + 1).$$

We use integration by parts to obtain

$$c_\alpha \int\int_\Omega u(z, \bar{z})(\partial - \bar{\partial})((z^2 - z\bar{z} + \bar{z}^2 - z - \bar{z} + 1)^{\alpha+1} \times (z + \bar{z} + 1)^\alpha) dm_2(z)$$

$$= -c_\alpha \int\int_\Omega [(\partial - \bar{\partial})u(z, \bar{z})](z^2 - z\bar{z} + \bar{z}^2 - z - \bar{z} + 1)w^\alpha dm_2(z),$$

and the left side also equals

$$3(\alpha + 1)c_\alpha \int\int_\Omega u(z, \bar{z})(z - \bar{z})w^\alpha dm_2(z);$$

the calculation being valid for $\alpha > -1$. Now set $u = z^{k-1}\bar{z}^l$ (with $k \equiv l$ mod 3 and $k \geqq 1$) and use the relation

$$I_\alpha(k', l') = 0 \quad \text{if } k' \not\equiv l' \bmod 3$$

to simplify both sides. This leads to

$$3(\alpha + 1)I_\alpha(k, l) = -(k - 1)(I_\alpha(k, l) - I_\alpha(k - 2, l + 1)) + l(I_\alpha(k - 1, l - 1) - I_\alpha(k, l)),$$

and so

$$(k + l + 3\alpha + 2)I_\alpha(k, l) = (k - 1)I_\alpha(k - 2, l + 1) + lI_\alpha(k - 1, l - 1).$$

The last identity in the theorem follows from the (k, l)-symmetry.

Put $k = 1$, $l = 3j$ to get

$$(3j + 3\alpha + 4)I_\alpha(1, 3j + 1) = (3j + 1)I_\alpha(0, 3j), \quad (j \geqq 0),$$

and $k = 3j + 3$, $l = 0$ to get

$$(3j + 3\alpha + 5)I_\alpha(3j + 3, 0) = (3j + 2)I_\alpha(3j + 1, 1).$$

From these, we can show

$$I_\alpha(3j, 0) = I_\alpha(0, 3j) = \frac{(2/3)_j(1/3)_j}{(\alpha + 4/3)_j(\alpha + 5/3)_j} \quad \text{and}$$

$$I_\alpha(3j + 1, 1) = \frac{(1/3)_{j+1}(2/3)_j}{(\alpha + 4/3)_{j+1}(\alpha + 5/3)_j}.$$

Finally

$$c_\alpha \int\int_\Omega f_{3,0}^\alpha \overline{f_{3,3}^\alpha} w^\alpha dm_2 = c_\alpha \int\int_\Omega f_{3,0}^\alpha z^3 w^\alpha dm_2(z)$$

(because $f_{3,0}^\alpha$ is perpendicular to terms of lower degree)

$$= I_\alpha(6, 0) - (6/(3\alpha + 7))I_\alpha(4, 1)$$
$$+ (4/(3\alpha + 5)(3\alpha + 7))I_\alpha(3, 0)$$
$$= -72(\alpha + 1)/((3\alpha + 4)(3\alpha + 5)^2(3\alpha + 7)^2(3\alpha + 8)).$$

The limiting situation for $\alpha \to \infty$ is nontrivial but some specific results are possible.

2.6. PROPOSITION. *For each* $n \geqq 0$,

i) $\lim_{\alpha\to\infty} \alpha^{-n}\phi_{nm}(v) = (v_2 + v_3 - 2v_1)^{n-m}(v_3 - v_2)^m$;

ii) $\lim_{\alpha\to\infty} \alpha^{-n}N_{nm}(\alpha) = (2/3)^n 3^{-m} m!(n - m)!$;

iii) $$\lim_{\alpha\to\infty} M_{nmk}(\alpha) = (-1)^{n+k+m}2^{-n}\left(3^{m+k}\binom{n}{m}\binom{n}{k}\right)^{\frac{1}{2}} \times {}_2F_1\left(\begin{matrix}-k, -m; 4/3\\ -n\end{matrix}\right);$$

iv) $\lim_{\alpha\to\infty} f^{\alpha}_{n,m} = z^{n-m}\bar{z}^m$.

Proof. Indeed

$$\alpha^{-n}\phi_{nm}(v) = ((\alpha + 1)_{n-m}(\alpha + 1)_m\alpha^{-n})$$
$$\times {}_2F_1\left(\begin{matrix}m - n, m + n + 3\alpha + 2\\ \alpha + 1\end{matrix}; \frac{v_1}{v_1 + v_2 + v_3}\right)(v_1 + v_2 + v_3)^{n-m}$$
$$\times {}_2F_1\left(\begin{matrix}-m, m + 2\alpha + 1\\ \alpha + 1\end{matrix}; \frac{v_2}{v_2 + v_3}\right)(v_2 + v_3)^m$$
$$\to {}_1F_0(m - n; 3v_1/(v_1 + v_2 + v_3))(v_1 + v_2 + v_3)^{n-m}$$
$$\times {}_1F_0(-m; 2v_2/(v_2 + v_3))(v_2 + v_3)^m,$$

and these are binomial series.

The Krawtchouk polynomial of degree m, parameters n, p (orthogonal for $\binom{n}{x}p^x(1 - p)^{n-x}$) is defined by

$$K_m(x; p, n) := {}_2F_1\left(\begin{matrix}-m, -x\\ -n\end{matrix}; \frac{1}{p}\right);$$

thus in (iii) above, we have $K_k(m; 3/4, n)$.

Define

$$\phi^{\infty}_{nm}(v) := (v_2 + v_3 - 2v_1)^{n-m}(v_3 - v_2)^n.$$

There is an inner product $\langle p, q\rangle$ on polynomials homogeneous of degree n in v_1, v_2, v_3 such that

$$\langle \phi_{nm}^{\infty}, \phi_{nk}^{\infty}\rangle = \delta_{mk}\left(\lim_{\alpha\to\infty} \alpha^{-n}N_{nm}(\alpha)\right),$$

namely

$$\langle p, q\rangle := 3^{-2n}\sum_{(m)} \frac{1}{m_1!m_2!m_3!}\, p_{(m)}\,\overline{q_{(m)}},$$

where (m) is a multi-index (m_1, m_2, m_3) with $\sum_i m_i = n$, and

$$p(v) = \sum_{(m)} p_{(m)} v_1^{m_1}v_2^{m_2}v_3^{m_3}$$

(and similarly q).

There is a simple expression for $z^{n-m}\bar{z}^m$ in terms of ϕ_{nk}^{∞}. Let

$$\xi := (v_2 + v_3 - 2v_1), \quad \eta := v_3 - v_2,$$

then

$$z = -(1/2)(\xi + \sqrt{3}i\eta).$$

2.7. PROPOSITION.

$$z^{n-m}\bar{z}^m = (-1/2)^n \sum_{j=0}^{n}\binom{n}{j}(\sqrt{3}i)^j K_m(j; \tfrac{1}{2}, n)\phi_{nj}^{\infty}(v),$$

and

$$\langle z^{n-m}\bar{z}^m, z^{n-k}\bar{z}^k\rangle = \delta_{mk}3^{-n}m!(n-m)!,$$

so that $\{z^{n-m}\bar{z}^m: 0 \leqq m \leqq n\}$ *is an orthogonal basis for* H_n^{∞}.

Proof. Indeed

$$\begin{aligned}
z^{n-m}\bar{z}^m &= (-1/2)^n(\xi + \sqrt{3}i\eta)^{n-m}(\xi - \sqrt{3}i\eta)^m \\
&= (-1/2)^n\sum_{j,k}\binom{n-m}{j}\xi^{n-m-j}(\sqrt{3}i\eta)^j\binom{m}{k}\xi^{m-k}(-\sqrt{3}i\eta)^k \\
&= (-1/2)^n\sum_{l=0}^{n}(\sqrt{3}i)^l\xi^{n-l}\eta^l\sum_k(-1)^k\binom{m}{k}\binom{n-m}{l-k}
\end{aligned}$$

(where $l = j + k$). The k-sum is known to be

$$\binom{n}{l}K_l(m; \tfrac{1}{2}, n) = \binom{n}{l}K_m(l; \tfrac{1}{2}, n),$$

https://doi.org/10.4153/CJM-1984-040-1 Published online by Cambridge University Press

and thus we have the stated expansion. The inner product

$$\langle z^{n-m}\bar{z}^{m}, z^{n-k}\bar{z}^{k}\rangle = (-1/2)^{2n} \sum_{j=0}^{n} \binom{n}{j}^2 3^j$$

$$\times K_m(j; \tfrac{1}{2}, n)K_k(j; \tfrac{1}{2}, n)(2/3)^n 3^{-j} j!(n-j)!$$

$$= 6^{-n} n! \sum_{j=0}^{n} \binom{n}{j} K_m(j; \tfrac{1}{2}, n)K_k(j; \tfrac{1}{2}, n)$$

$$= 6^{-n} n!\ \delta_{mk} 2^n / \binom{n}{m},$$

by the orthogonality of Krawtchouk polynomials.

The matrix of the isometry U in the normalized ϕ^{∞}_{nm}-basis is

$$U_{jk} = (-1)^j M_{njk}(\infty)$$

$$= (-1)^{n+k} 2^{-n} \left(3^{j+k}\binom{n}{j}\binom{n}{k}\right)^{\frac{1}{2}} K_j(k; 3/4, n), \ (0 \leqq j, k \leqq n).$$

Since U acts diagonally on the functions $z^{n-m}\bar{z}^{m}$, we can obtain an orthogonal diagonalization of U by using Proposition 2.7, indeed

$$4^{-n} \sum_{m=0}^{n} \binom{n}{m} 3^m K_j(m; 3/4, n)(-i/\sqrt{3})^n K_k(m; \tfrac{1}{2}, n)$$

$$= (-\omega/2)^n \omega^k (i/\sqrt{3})^j K_j(k; \tfrac{1}{2}, n), \quad \text{for } 0 \leqq j, k \leqq n.$$

3. The self-adjoint third-order differential operator. We want to find a reasonably natural orthogonal basis for H_n^{α} that diagonalizes the Z_3-action generated by U. As was pointed out, the basis $\{f^{\alpha}_{n,m}\}$ is not orthogonal. The approach will be to construct a self-adjoint differential operator that commutes with the various symmetries (namely, sign-changes of x_j, cyclic permutation). As a starting point we work with the usual surface measure on S (that is, $\alpha = -\frac{1}{2}$), so that the algebra generated by the infinitesimal rotations provides some obvious self-adjoint operators. Each iR_{jk} is self-adjoint, and so $-R_{jk}^2$ is a positive operator which is invariant under sign-changes, indeed

$$-R_{jk}^2 = -4v_j v_k(\partial_j - \partial_k)^2 - 2(v_j - v_k)(\partial_k - \partial_j).$$

There is a similar operator on $L^2(E, \mu_{\alpha})$, for each $\alpha > -1$.

3.1. THEOREM. *For $\alpha > -1$, the operator*

$$T_{jk}: = v_j v_k (\partial_j - \partial_k)^2 + (\alpha + 1)(v_j - v_k)(\partial_k - \partial_j)$$

is self-adjoint on $L^2(E, \mu_\alpha)$. The eigenvectors for T_{23}^α are the polynomials ϕ_{nm}, and

$$T_{23}^\alpha \phi_{nm} = -m(m + 2\alpha + 1)\phi_{nm}, \quad (0 \leqq m \leqq n).$$

Proof. We consider only T_{23}^α. On E we use the variable $v_3 = 1 - v_1 - v_2$ so that

$$\frac{\partial}{\partial v_2} = \partial_2 - \partial_3.$$

Integration by parts in the iterated integral

$$\int_0^1 dv_1 \int_0^{1-v_1} F(v_1, v_2, v_3)dv_2$$

yields

$$\int\int_E v_2 v_3 [(\partial_2 - \partial_3)^2 f]\bar{g}(v_1 v_2 v_3)^\alpha dv_1 dv_2$$

$$= \int\int_E [(\partial_2 - \partial_3)f][(\partial_2 - \partial_3)\bar{g}]v_2 v_3 (v_1 v_2 v_3)^\alpha dv_1 dv_2$$

$$- (\alpha + 1)\int\int_E [(v_3 - v_2)(\partial_2 - \partial_3)f]\bar{g}(v_1 v_2 v_3)^\alpha dv_1 dv_2$$

(where f and g are twice differentiable, the appropriate function is zero on the boundary $v_2 = 0$ or $v_2 = 1 - v_1$ provided $\alpha + 1 > 0$). Move the latter integral to the left side, which then becomes

$$\int\int_E (T_{23}^\alpha f)\bar{g}(v_1 v_2 v_3)^\alpha dv_1 dv_2;$$

whereas the right side becomes symmetric in f, $\bar{g}$.

The operator T_{23}^α acting on ϕ_{nm} reduces to the standard second-order differential equation for Jacobi polynomials (see [**12**], p. 62, eq. (4.21.1)).

For notational convenience, let $R_1: = R_{23}$, $R_2: = R_{31}$, $R_3: = R_{12}$. We use U and J in x-coordinates (that is, $Uf(x_1, x_2, x_3) = f(x_3, x_1, x_2)$ and $Jf(x_1, x_2, x_3) = f(x_1, x_3, x_2)$). They act on differential operators by inner automorphism.

3.2. LEMMA. i) $U^{-1}R_1U = R_2$, $U^{-1}R_2U = R_3$, $U^{-1}R_3U = R_1$,

ii) $JR_1J = -R_1$, $JR_2J = -R_3$, $JR_3J = -R_2$ *(note $J^{-1} = J$).*

It is not hard to see that the only second degree polynomials in $\{R_j\}$ which are invariant under U, J and sign-changes in $\{x_k\}$ are scalar multiples of $R_1^2 + R_2^2 + R_3^2$, the spherical Laplacian. Since it has each

$H_n^{-\frac{1}{2}}$ as an eigenmanifold, we move onward to consider third degree polynomials in $\{R_j\}$. Indeed $R_1R_2R_3$ is invariant under sign-changes, but its factors are permuted by the U and J actions. By use of the commutation relationships

$$[R_j, R_k](:= R_jR_k - R_kR_j) = -R_l,$$

where (jkl) is a cyclic permutation of (123), we see that $R_1R_2R_3$ is U-invariant modulo quadratic terms.

3.3. THEOREM. *Let* $\delta := R_1R_2R_3 + \frac{1}{2}(R_1^2 - R_2^2 + R_3^2)$, *then* δ *is invariant under sign-changes, and* $U^{-1}\delta U = \delta$, $J\delta J = -\delta$ *(relative invariance for* S_3*). Further,* $i\delta$ *is self-adjoint.*

Proof. Let $\rho := R_1R_2R_3$. The idea is to sum $(\text{sgn } \sigma)\ \rho^\sigma$ (where $\rho^\sigma := \sigma^{-1}\rho\sigma$) over $\sigma \in S_3$. We list the values ρ^σ for $\sigma \in S_3 = \{Id, J, UJ, JU, U, U^2\}$:

i) $\sigma = J,\ \rho^\sigma = -R_1R_3R_2 = -R_1R_2R_3 - R_1^2$

(since $R_3R_2 = R_2R_3 + R_1$);

ii) $\sigma = U^2,\ \rho^\sigma = R_3R_1R_2 = R_1R_3R_2 - R_2^2$

$= R_1R_2R_3 + R_1^2 - R_2^2$ (by (i));

iii) $\sigma = UJ,\ \rho^\sigma = -R_3R_2R_1 = -R_3R_1R_2 - R_3^2$

$= -R_1R_2R_3 - R_1^2 + R_2^2 - R_3^2$ (by (ii));

v) $\sigma = JU,\ \rho^\sigma = -R_2R_1R_3 = -R_1R_2R_3 - R_3^2$;

v) $\sigma = U,\ \rho^\sigma = R_2R_3R_1 = R_2R_1R_3 - R_2^2$

$= R_1R_2R_3 - R_2^2 + R_3^2$ (by (iv)).

Then

$$(1/6) \sum_\sigma \rho^\sigma = -(1/6)(R_1^2 + R_2^2 + R_3^2),$$

but

$$(1/6) \sum_\sigma (\text{sgn } \sigma)\ \rho^\sigma = \delta.$$

Since $R_j^* = -R_j$,

$$(R_1R_2R_3)^* = -R_3R_2R_1;$$

further δ is a sum of terms like $R_1R_2R_3 + R_3R_2R_1$ (and permutations), thus $\delta^* = -\delta$.

To express δ in v-coordinates we introduce differential operators of degrees one, two and three.

3.4. *Definition.*

$$\delta_1 := (v_2 - v_3)\partial_1 + (v_3 - v_1)\partial_2 + (v_1 - v_2)\partial_3;$$

$$\delta_2 := v_1(v_2 - v_3)(\partial_1^2 + 2\partial_2\partial_3) + v_2(v_3 - v_1)(\partial_2^2 + 2\partial_3\partial_1) + v_3(v_1 - v_2)(\partial_3^2 + 2\partial_1\partial_2);$$

$$\delta_3 := v_1v_2v_3(\partial_1 - \partial_2)(\partial_2 - \partial_3)(\partial_3 - \partial_1).$$

3.5. PROPOSITION. $\delta = 8\delta_3 + 2\delta_2 + \delta_1$. *Also* $U^{-1}\delta_jU = \delta_j$ *and* $J\delta_jJ = -\delta_j$ *for* $j = 1, 2, 3$.

For the general $L^2(E, \mu_\alpha)$, $\alpha > -1$, we look for a linear combination of δ_1, δ_2, δ_3 which is self-adjoint. The calculations are more manageable in the (z, t)-coordinates.

3.6. PROPOSITION.

$$\delta_1 = -i\sqrt{3}(z\partial - \bar{z}\bar{\partial});$$

$$\delta_2 = -i\sqrt{3}((z^2 - \bar{z}t)\partial^2 - (\bar{z}^2 - zt)\bar{\partial}^2);$$

$$\delta_3 = -(i/(3\sqrt{3}))(z^3 + \bar{z}^3 + t^3 - 3z\bar{z}t)(\partial^3 - \bar{\partial}^3).$$

The construction of the self-adjoint operator for $L^2(E, \mu_\alpha)$ will be described as a list of simple integration-by-parts statements from which we can deduce the coefficients. We use the definitions:

i) differential operators:

$$\tau_1 f(z) := z\partial f(z),$$

$$\tau_2 f(z) := (z^2 - \bar{z}t)\partial^2 f(z),$$

$$\tau_3 f(z) := w(z, 1)\partial^3 f(z),$$

and their conjugates $\bar{\tau}_1 f(z) := \bar{z}\bar{\partial} f(z)$, etc.;

ii) integral kernels:

$$\langle f, g\rangle := c_\alpha \int\int_\Omega f\bar{g}\, w^\alpha dm_2,$$

$$K_1(f, g) := c_\alpha \int\int_\Omega (\partial f)(\bar{\partial} g)^-(z^2 - \bar{z}t)w^{\alpha-1}dm_2,$$

$$K_2(f, g) := c_\alpha \int\int_\Omega f\bar{g}z(z^2 - \bar{z}t)w^{\alpha-1}dm_2,$$

$$K_3(f, g) := c_\alpha \int\int_\Omega f\bar{g}(z^2 - \bar{z}t)^3 w^{\alpha-2}dm_2,$$

where f and g are smooth, and $\alpha > 1$ (we will use analytic continuation on

α). Each of the following equations is the result of one integration by parts applied to the first-named integral (note that $\partial w(z, 1) = 3(z^2 - \bar{z}t)$):

$$(I_1)\ \langle \tau_3 f, g\rangle = -3(\alpha + 1)\langle \tau_2 f, g\rangle - c_\alpha \int\int_\Omega (\partial^2 f)(\bar{\partial} g)^- w^{\alpha+1} dm_2,$$

$$(I_2)\ \langle f, \bar{\tau}_3 g\rangle = -3(\alpha + 1)\langle f, \bar{\tau}_2 g\rangle - c_\alpha \int\int_\Omega (\partial f)(\bar{\partial}^2 g)^- w^{\alpha+1} dm_2,$$

$$(I_3)\ c_\alpha \int\int_\Omega (\partial^2 f)(\bar{\partial} g)^- w^{\alpha+1} dm_2 + c_\alpha \int\int_\Omega (\partial f)(\bar{\partial}^2 g)^- w^{\alpha+1} dm_2 = 3(\alpha + 1)K_1(f, g),$$

$$(I_4)\ \langle \tau_2 f, g\rangle = -2\langle \tau_1 f, g\rangle - K_1(f, g) - 3\alpha c_\alpha \int\int_\Omega (\partial f)\bar{g}(z^2 - \bar{z}t)^2 w^{\alpha-1} dm_2,$$

$$(I_5)\ \langle f, \bar{\tau}_2 g\rangle = -2\langle f, \bar{\tau}_1 g\rangle - K_1(f, g) - 3\alpha c_\alpha \int\int_\Omega f(\bar{\partial} g)^-(z^2 - \bar{z}t)^2 w^{\alpha-1} dm_2,$$

$$(I_6)\ c_\alpha \int\int_\Omega (\partial f)\bar{g}(z^2 - \bar{z}t)^2 w^{\alpha-1} dm_2 + c_\alpha \int\int_\Omega f(\bar{\partial} g)^-(z^2 - \bar{z}t)^2 w^{\alpha-1} dm_2 = -4K_2(f, g) - 3(\alpha - 1)K_3(f, g),$$

$$(I_7)\ \langle \tau_1 f, g\rangle + \langle f, \bar{\tau}_1 g\rangle = -\langle f, g\rangle - 3\alpha K_2(f, g).$$

It is clear that the combined equations $(I_1) + (I_2) - (I_3)$, and $(I_4) + (I_5) - 3\alpha(I_6)$ involve only inner products ($\langle , \rangle$) and the kernels K_i. Add $-3(\alpha + 1)/2$ times the second equation to the first to eliminate K_1. After grouping, obtain

$$(\langle \tau_3 f, g\rangle + \langle f, \bar{\tau}_3 g\rangle) + 9/2(\alpha + 1)(\langle \tau_2 f, g\rangle + \langle f, \bar{\tau}_2 g\rangle) + 3(\alpha + 1)(\langle \tau_1 f, g\rangle + \langle f, \bar{\tau}_1 g\rangle) = 18\alpha(\alpha + 1)K_2(f, g) + (27/2)\alpha(\alpha + 1) \times (\alpha - 1)K_3(f, g).$$

Transform this identity by replacing each τ_i by $\bar{\tau}_i$ and subtract the result from the above. To express this, let

$$\sigma := \tau_3 - \bar{\tau}_3 + (9/2)(\alpha + 1)(\tau_2 - \bar{\tau}_2) + 3(\alpha + 1)(\tau_1 - \bar{\tau}_1),$$

then

$$(*)\ \langle \sigma f, g\rangle - \langle f, \sigma g\rangle = 18\alpha(\alpha + 1)(K_2(f, g) - K_2(g, f)^-) + (27/2)\alpha(\alpha + 1)(\alpha - 1)(K_3(f, g) - K_3(g, f)^-).$$

But

$$K_3(f, g) - K_3(g, f)^-$$
$$= c_\alpha \int\int_\Omega f\bar{g}[(z^2 - \bar{z}t)^3 - (\bar{z}^2 - zt)^3]w^{\alpha-2}dm_2$$
$$= c_\alpha \int\int_\Omega f\bar{g}(z^3 - \bar{z}^3)w^{\alpha-1}dm_2 = K_2(f, g) - K_2(g, f)^-,$$

so the right side of $(*)$ becomes

$$(9/2)\alpha(\alpha + 1)(3\alpha + 1)(K_2(f, g) - K_2(g, f)^-).$$

We can get rid of this term by using (I_7) and its conjugate, that is, the identity

$$\langle(\tau_1 - \bar{\tau}_1)f, g\rangle - \langle f, (\tau_1 - \bar{\tau}_1)g\rangle = -3\alpha(K_2(f, g) - K_2(g, f)^-).$$

Thus, adding $(9/2)\alpha(\alpha + 1)(3\alpha + 1)$ times this identity to $(*)$, we obtain that

$$\sigma + (9/2)\alpha(\alpha + 1)(3\alpha + 1)(\tau_1 - \bar{\tau}_1)$$
$$= (\tau_3 - \bar{\tau}_3) + 9/2(\alpha + 1)(\tau_2 - \bar{\tau}_2) + 9/2(\alpha + 1)^2(\tau_1 - \bar{\tau}_1)$$

is self-adjoint.

3.7. *Definition.* For $\alpha > -1$,

$$D_\alpha := 1/9(z^3 + \bar{z}^3 - 3z\bar{z}t + t^3)(\partial^3 - \bar{\partial}^3)$$
$$+ 1/2(\alpha + 1)((z^2 - \bar{z}t)\partial^2 - (\bar{z}^2 - zt)\bar{\partial}^2)$$
$$+ 1/2(\alpha + 1)^2(z\partial - \bar{z}\bar{\partial}).$$

Equivalently,

$$D_\alpha := (i/\sqrt{3})T_\alpha,$$

where

$$T_\alpha := \delta_3 + 1/2(\alpha + 1)\delta_2 + 1/2(\alpha + 1)^2\delta_1.$$

3.8. THEOREM. *D_α is a self-adjoint operator on $L^2(\mu_\alpha)$, and each H_n^α ($n \geqq 0$) is invariant under D_α and T_α, for $\alpha > -1$.*

Proof. The self-adjointness was proved above for $\alpha > 1$, and can be analytically continued for $\alpha > -1$. Further, D_α preserves the degree of homogeneity of a polynomial, so by the remark following Theorem 1.3 $D_\alpha H_n^\alpha \subset H_n^\alpha$.

The next step is to prove that $D_\alpha | H_n^\alpha$ has no repeated eigenvalues because this forces each eigenvector to be relatively invariant under U (that is, if $f \in H_n^\alpha$, $D_\alpha f = \lambda f$ for some λ then $Uf = \omega^\epsilon f$ for some ϵ). This will be shown by establishing a tridiagonal representation of D_α with respect to the ϕ_{nm}-basis in which there are no zeros on the superdiagonal.

3.9. THEOREM. *For* $\alpha > -1$, $0 \leqq m \leqq n$,

$$T_\alpha \phi_{nm} = -\frac{m(m+\alpha)^2(2\alpha+n+m+1)}{2(2\alpha+2m+1)}\phi_{n,m-1}$$
$$+\frac{(n-m)(2\alpha+m+1)(\alpha+n-m)(3\alpha+n+m+2)}{2(2\alpha+2m+1)}\phi_{n,m+1}.$$

(*The apparent zero division for* $m = 0$, $\alpha = -1/2$ *cancels by the obvious limiting argument, indeed* $T_\alpha\phi_{n0} = (1/2)n(n+\alpha)(n+3\alpha+2)\phi_{n1}$.)

Proof. Observe that the degree of v_1 in ϕ_{nm} is $\leqq n - m$, and T_α increases the v_1-degree by no more than 1. Thus $T_\alpha\phi_{nm}$ is a linear combination of ϕ_{nj} with $j \geqq m - 1$; but $T_\alpha^* = -T_\alpha$ so that

$$\langle T_\alpha\phi_{nm}, \phi_{nm}\rangle = 0 \quad \text{and}$$

$$T_\alpha\phi_{nm} = t_{m-1,m}\phi_{n,m-1} + t_{m+1,m}\phi_{n,m+1}$$

with

$$t_{m+1,m}\|\phi_{n,m+1}\|^2 = -t_{m+1,m}\|\phi_{nm}\|^2.$$

It suffices to find $t_{m-1,m}$, then use the known values of $\|\phi_{nm}\|^2 = N_{nm}(\alpha)$ (see Section 1). The term of highest v_1-degree in ϕ_{nm} is

$$(\alpha+1)_m(-n-2\alpha-m-1)_{n-m}v_1^{n-m}R_m^{(\alpha,\alpha)}(v_2/(v_2+v_3))$$
$$\times (v_2+v_3)^m$$

$$= (\alpha+1)_m(-n-2\alpha-m-1)_{n-m}\sum_{j=0}^{m}\frac{(-m)_j(-\mathrm{m}-\alpha)_j}{(\alpha+1)_j j!}$$
$$\times (-v_2)^j v_3^{m-j}v_1^{n-m}.$$

The terms of highest v_1-degree in T_α are

$$v_1v_2v_3(-\partial_2\partial_3)(\partial_2 - \partial_3)$$
$$+((\alpha+1)/2)(2v_1(v_2-v_3)\partial_2\partial_3 - v_1v_2\partial_2^2 + v_1v_3\partial_3^2)$$
$$+((\alpha+1)^2/2)v_1(\partial_3 - \partial_2),$$

and applying this operator to the typical term $v_2^jv_3^{m-j}$ yields

$$-j(j+\alpha)(m-j+(\alpha+1)/2)v_2^{j-1}v_3^{m-j}$$
$$+(m-j)(m-j+\alpha)(j+(\alpha+1)/2)v_2^{j-1}v_3^{m-1-j}.$$

Transform every term in the sum

$$R_m^{(\alpha,\alpha)}(v_2/(v_2+v_3))(v_2+v_3)^m$$

by this formula, collect the coefficients of $v_2^jv_3^{m-1-j}$ in the result, and obtain

$$m(m + \alpha)^2 R_{m-1}^{(\alpha,\alpha)}(v_2/(v_2 + v_3))(v_2 + v_3)^{m-1}.$$

To obtain $t_{m-1,m}$ divide the coefficient of v_1^{n-m+1} found here by the coefficient of v_1^{n-m+1} in $\phi_{n,m-1}$; indeed

$$t_{m-1,m} = \frac{m(m + \alpha)^2(\alpha + 1)_m(-n - 2\alpha - m - 1)_{n-m}}{(\alpha + 1)_{m-1}(-n - 2\alpha - m)_{n-m+1}}$$
$$= \frac{-m(m + \alpha)^2(2\alpha + n + m + 1)}{2(2\alpha + 2m + 1)}.$$

It remains to calculate

$$t_{m,m-1} = -t_{m-1,m} N_{n,m-1}(\alpha)/N_{nm}(\alpha)$$

(for $m \geqq 1$, using the values from Section 1).

Some machinery is available for eigenvalue and eigenvector computations for tridiagonal symmetric matrices. To exploit this we use the orthonormal basis for H_n^α given by $\{g_{nm}:0 \leqq m \leqq n\}$ where

$$g_{nm}: = i^m \phi_{nm} N_{nm}(\alpha)^{-\frac{1}{2}}.$$

It is straightforward to show that

$$D_\alpha g_{nm} = b_{n,m}(\alpha) g_{n,m-1} + b_{n,m+1}(\alpha) g_{n,m+1},$$

where

$$b_{n,m}(\alpha): = \left(\frac{1}{12}\frac{m(n - m + 1)(m + \alpha)^2(m + 2\alpha)(\alpha + n - m + 1)}{(2\alpha + 2m - 1)(2\alpha + 2m + 1)} \times (2\alpha + n + m + 1)(3\alpha + n + m + 1)\right)^{\frac{1}{2}},$$
$$1 \leqq m \leqq n.$$

The problem of finding the characteristic polynomial of $D_\alpha | H_n^\alpha$ is related to a family of discrete orthogonal polynomials given by a three-term recurrence.

3.10. *Definition*. For $n \geqq 1$, define a family of polynomials $p_m(\lambda; n, \alpha)$ by $p_{-1} = 0, p_0 = 1$,

$$p_{m+1}(\lambda; n, \alpha) = \lambda p_m(\lambda; n, \alpha) - b_{n,m}^2 p_{m-1}(\lambda; n, \alpha), \quad 0 \leqq m \leqq n.$$

Thus each p_m is monic and of the same parity in λ as m.

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3.11. THEOREM. *For* $\alpha > -1$, $n \geqq 1$, $D_\alpha | H_n^\alpha$ *has* $n + 1$ *distinct eigenvalues* $\lambda_0 < \lambda_1 < \lambda_2 \ldots < \lambda_n$ *which are the zeros of* $p_{n+1}(\lambda; n, \alpha)$, *and* $\lambda_{n-j} = -\lambda_j$. *Further* $p_j(\lambda; n, \alpha)$ *is* $(-1)^j$ *times the determinant of the upper left* $j \times j$ *submatrix of* $(D_\alpha - \lambda I)|H_n^\alpha$ (*that is, the projection on span* $\{g_{n0}, \ldots, g_{n,j-1}\}$).

Proof. The distinctness of the eigenvalues follows from $b_{n,m}(\alpha) > 0$ for $1 \leqq m \leqq n$ (see [**9**], p. 124). The other claim is also a standard fact ([**9**], p. 126, (7-8-3)).

3.12. *Definition.* For $\alpha > -1$, $0 \leqq m \leqq n$, let

$$q_{n,m}^\alpha = \sum_{j=0}^{n} u_{jm}(n, \alpha) g_{nj}$$

be the normalized ($\sum_j u_{jm}^2 = 1$) eigenvector D_α with eigenvalue λ_m, and such that

$$(-1)^n u_{0m}(n, \alpha) > 0$$

(this is possible by [**9**], p. 129, (7-9-5)).

Since $\lambda_j = -\lambda_{n-j}$ and $JD_\alpha J = -D_\alpha$ we see that $Jq_{nm}^\alpha = q_{n,n-m}^\alpha$.

The family $\{p_m(\lambda; n, \alpha): 0 \leqq m \leqq n + 1\}$ is closely connected to the eigenvectors of D_α and the products

$$\kappa_{nm}(\alpha) := \prod_{j=1}^{m} b_{n,j}(\alpha)$$

$$= \left[\frac{(\alpha + 1)_m}{4^m (\alpha + 3/2)_m} m!(-n)_m(-n - \alpha)_m(2\alpha + 2)_m(2\alpha + n + 2)_m \times (3\alpha + n + 2)_m(2\alpha + 2m + 1) \cdot 3^{-m}/(2\alpha + m + 1)\right]^{\frac{1}{2}},$$

(for $0 \leqq m \leqq n$, with $\kappa_{n0}(\alpha) := 1$).

3.13. THEOREM.

i) $u_{0m}(n, \alpha)^2 = \kappa_{nn}(\alpha)^2 / (p_n(\lambda_m; n, \alpha) p'_{n+1}(\lambda_m; n, \alpha))$;

ii) $u_{jm}(n, \alpha) = p_j(\lambda_m; n, \alpha) u_{0m}(n, \alpha) / \kappa_{nj}(\alpha)$, *for* $0 \leqq j, m \leqq n$;

iii) *the set* $\{p_m(\lambda; n, \alpha): 0 \leqq m \leqq n\}$ *is a family of orthogonal polynomials, with respect to a discrete measure on* $\{\lambda_j : 0 \leqq j \leqq n\}$; *indeed*

$$\sum_{j=0}^{n} u_{0j}(n, \alpha)^2 p_m(\lambda_j; n, \alpha) p_k(\lambda_j; n, \alpha) = \delta_{km} \kappa_{nm}^2(\alpha), \quad 0 \leqq m, k \leqq n.$$

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Proof. Fix $\alpha > -1$, $n \geqq 1$. Let $r_j(\lambda)$ be the determinant of the lower right $j \times j$ submatrix of $(\lambda I - D_\alpha)|H_n^\alpha$ (the projection on span $\{g_{n,n-j+1}, \dots, g_{nn}\}$). By Paige's theorem (see [**9**], p. 129, (7-9-3a))

$$(*) \quad p'_{n+1}(\lambda_j)u_{mj}u_{kj} = p_m(\lambda_j)(\kappa_{nk}/\kappa_{nm})r_{n-k}(\lambda_j),$$

for $0 \leqq m \leqq k \leqq n$ and $0 \leqq j \leqq n$. Set $m = k = 0$ to get

$$p'_{n+1}(\lambda_j)u_{0j}^2 = \kappa_{nm}r_n(\lambda_j), \quad m = k = n$$

to get

$$p'_{n+1}(\lambda_j)u_{nj}^2 = p_n(\lambda_j), \quad m = 0, k = n$$

to get

$$p'_{n+1}(\lambda_j)u_{0j}u_{nj} = \kappa_{nn}$$

(thus $u_{0j} \neq 0 \neq u_{nj}$). Square the last identity and set it equal to the product of the first two to get

$$\kappa_{nn}^2 = \kappa_{nn}r_n(\lambda_j)p_n(\lambda_j);$$

thus

$$r_n(\lambda_j) = \kappa_{nn}/p_n(\lambda_j) \quad \text{and}$$
$$u_{0j}^2 = \kappa_{nn}^2/(p'_{n+1}(\lambda_j)p_n(\lambda_j)),$$

statement (i). (These are formulas of Gaussian quadrature theory.) For any m, let $k = n$ in $(*)$, so

$$p'_{n+1}(\lambda_j)u_{mj}u_{nj} = p_m(\lambda_j)\kappa_{nn}/\kappa_{nm};$$

now multiply by u_{0j} and get

$$\kappa_{nn}u_{mj} = p_m(\lambda_j)u_{0j}\kappa_{nn}/\kappa_{nm}$$

(statement (ii)). Finally

$$\sum_{j=0}^{n} u_{0j}^2 p_m(\lambda_j)p_n(\lambda_j) = \sum_{j=0}^{n} \kappa_{nm}\kappa_{nk}u_{mj}u_{kj} = \delta_{mk}\kappa_{nm}^2.$$

(Note that $p_n(\lambda_j) \neq 0$ by the interlacing of zeros theorem, and $p'_{n+1}(\lambda_j) \neq 0$ because the zeros λ_j are simple.)

The coefficient u_{0m} in q_{nm}^α has an important interpretation, indeed

$$q_{nm}^\alpha(1, 0, 0) = (-1)^n u_{0m}(2\alpha + 2)_n N_{n0}(\alpha)^{-\frac{1}{2}}$$
$$= \left[\frac{(\alpha + 1)_n^3(2\alpha + 2)_n(3\alpha + 3)_{2n}(2\alpha + 2n + 1)n!}{12^n(\alpha + 3/2)_n(2\alpha + n + 1)p_n(\lambda_m; n, \alpha)p'_{n+1}(\lambda_m; n, \alpha)}\right]^{\frac{1}{2}}.$$

(the point $v = (1, 0, 0)$ corresponds to $z = 1$, $t = 1$). This follows from

$$g_{n,j}(1, 0, 0) = 0 \quad \text{for } j > 0 \quad \text{and}$$

$$g_{n,0}(1, 0, 0) = N_{n0}(\alpha)^{-\frac{1}{2}}(\alpha + 1)_n R_n^{(\alpha, 2\alpha+1)}(1)$$

$$= (-1)^n N_{n0}(\alpha)^{-\frac{1}{2}}(2\alpha + 2)_n.$$

It is possible to find the limiting values of u_{mj} as $\alpha \to \infty$ explicitly.

3.14. THEOREM.

$$\lim_{\alpha\to\infty} u_{mj}(n, \alpha) = (-1)^{m+n} K_m(j; \tfrac{1}{2}, n)\left(2^{-n}\binom{n}{m}\binom{n}{j}\right)^{\frac{1}{2}}$$

(*thus* $\lim_{\alpha\to\infty} u_{0j}(n, \alpha) = (-1)^n\left(2^{-n}\binom{n}{j}\right)^{\frac{1}{2}}$) *and for fixed* λ,

$$\lim_{\alpha\to\infty} \alpha^{-2m} p_m(\alpha^2\lambda; n, \alpha) = \frac{(-n)_m}{2^m} K_m(\lambda + n/2; \frac{1}{2}, n).$$

Proof. Let

$$\tilde{p}_m(\lambda; \alpha) := \alpha^{-2m} p_m(\lambda\alpha^2; n, \alpha)$$

(monic) and note that $\tilde{p}_m$ satisfies $\tilde{p}_0 = 1$, $\tilde{p}_1 = \lambda$,

$$\tilde{p}_{m+1}(\lambda; \alpha) = \lambda\tilde{p}_m(\lambda; \alpha) - (b_{nm}(\alpha)^2/\alpha^4)\tilde{p}_{m-1}(\lambda; \alpha).$$

But

$$\lim_{\alpha\to\infty} (b_{nm}(\alpha)^2/\alpha^4) = m(n - m + 1)/4,$$

so that the limiting polynomials

$$\tilde{p}_m(\lambda) := \lim_{\alpha\to\infty} \tilde{p}_m(\lambda; \alpha)$$

satisfy the three-term recurrence for the monic shifted Krawtchouk polynomials

$$\frac{(-n)_m}{2^m} K_m\left(\lambda + \frac{n}{2}; \frac{1}{2}, n\right).$$

Further

$$\tilde{p}_{n+1}(\lambda) = (\lambda - n/2)(\lambda - n/2 + 1)\ldots(\lambda + n/2)$$

$$= (\lambda - n/2)_{n+1}.$$

(This is the reason for our choice of normalization for D_α.) By continuous dependence of zeros of polynomials on the coefficients we see that

$$\lim_{\alpha\to\infty} \alpha^{-2}\lambda_j = j - n/2$$

(where λ_j is the jth zero of $p_{n+1}(\lambda; n, \alpha)$). The stated results are obtained by using known facts on the identities of Theorem 3.13, for example

$$u_{0j}(n, \alpha)^2 = \frac{b_{n1}^2 b_{n2}^2 \dots b_{nn}^2}{p_n(\lambda_j; n, \alpha) p'_{n+1}(\lambda_j; n, \alpha)} = \frac{(\alpha^{-4} b_{n1}^2) \dots (\alpha^{-4} b_{nn}^2)}{\tilde{p}_n(\lambda_j/\alpha^2; n, \alpha)\tilde{p}'_{n+1}(\lambda_j/\alpha^2; n, \alpha)},$$

and

$$\lim_{\alpha\to\infty} \tilde{p}_n(\lambda_j/\alpha^2; n, \alpha) = (-1)^n \frac{n!}{2^n} K_n\left(j; \frac{1}{2}, n\right) = (-1)^{n+j} n! \, 2^{-n},$$

and so on.

The asymptotic analysis of the λ_j as well as the U-action on a given eigenfunction q_{nm}^{α} can be easily derived from the matrix representation of D_{α} in the f_{nm}^{α}-basis. This is not an orthogonal basis, but the matrix is tridiagonal and splits into three pieces (on $H_{n,\epsilon}^{\alpha}$, $\epsilon = 0, 1, 2$).

Recall that f_{nm}^{α} is that element of H_n^{α} whose single term of highest $(z, \bar{z})$-degree is $z^{n-m}\bar{z}^m$. By the D_{α}-invariance of H_n^{α}, $D_{\alpha} f_{nm}^{\alpha}$ is a linear combination of f_{nj}^{α} $(0 \leqq j \leqq n)$. To establish the coefficients of the expansion it suffices to consider the terms of highest $(z, \bar{z})$-degree.

3.15. *Proposition.*

$$D_{\alpha} f_{nm}^{\alpha} = \frac{2}{3}\binom{n-m}{3}\left(f_{n,m+3}^{\alpha} + f_{n,m}^{\alpha}\right) - \frac{2}{3}\binom{m}{3}(f_{n,m-3}^{\alpha} + f_{n,m}^{\alpha}) + \frac{1}{2}(n - 2m)(\alpha + 1)(\alpha + n) f_{n,m}^{\alpha}.$$

Proof. The highest $(z, \bar{z})$-degree terms in D_{α} are

$$(1/9)(z^3 + \bar{z}^3)(\partial^3 - \bar{\partial}^3) + ((\alpha + 1)/2)(z^2\partial^2 - \bar{z}^2\bar{\partial}^2) + ((\alpha + 1)^2/2)(z\partial - \bar{z}\bar{\partial}).$$

Apply this to $z^{n-m}\bar{z}^m$ to get the stated coefficients.

A basis for $H_{n,\epsilon}^{\alpha}$ is given by

$$\{f_{n,c+3j}: 0 \leqq j \leqq [(n - c)/3]\}$$

where $c \equiv 2n + \epsilon \bmod 3$ (with $\epsilon, c = 0, 1, 2$). Each $H_{n,\epsilon}^{\alpha}$ is invariant under D_{α} and $D_{\alpha}|H_{n,\epsilon}^{\alpha}$ is tridiagonal with all the super- and subdiagonal entries being nonzero. Restating Proposition 3.15, we see

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$$D_\alpha f^{\alpha}_{n,c+3j} = -2/3\binom{c+3j}{3} f_{n,c+3(j-1)} + \left(\frac{2}{3}\left(\binom{n-c-3j}{3} - \binom{c+3j}{3}\right) + 1/2(n-2c-6j)(\alpha+1)(\alpha+n)\right) f_{n,c+3j} + 2/3\binom{n-c-3j}{3} f_{n,c+3(j+1)}.$$

Thus $\alpha^{-2}D_\alpha|H^{\alpha}_{n,\epsilon}$ is represented by a fixed diagonal matrix with a tridiagonal perturbation of $O(\alpha^{-1})$. In the limit we have

$$\lim_{\alpha\to\infty} \alpha^{-2}\lambda = \left(\frac{n}{2} - c - 3j\right)$$

for each eigenvalue λ of D_α, for some j. The union over $\epsilon = 0, 1, 2$ of the eigenvalues of $D_\alpha|H^{\alpha}_{n,\epsilon}$ is the set $\{\lambda_j: 0 \leqq j \leqq n\}$ and since the ordering is preserved (by pairwise distinctness of the λ_j) we deduce that

$$\lim_{\alpha\to\infty} \alpha^{-2}\lambda_m = m - n/2$$

and λ_m is an eigenvalue of $D_\alpha|H^{\alpha}_{n,\epsilon}$ for $\epsilon \equiv n - 2m \bmod 3$. The latter argument is based on the continuous dependence of λ_m on $\alpha > -1$.

3.16. THEOREM. *For $\alpha > -1$, $0 \leqq m \leqq n$, the eigenvector q^{α}_{nm} (for λ_m) is in $H^{\alpha}_{n,\epsilon}$ with $\epsilon \equiv n - 2m$* mod 3.

Koornwinder [**8**] has studied polynomials on a region in **C** bounded by Steiner's hypocycloid (this has an S_3-symmetry) and found a self-adjoint third-order differential operator for that situation. However, the analogue of our f^{α}_{nm}-basis is actually orthogonal.

The author studied cubic harmonics [**4**], these being spherical harmonics on $\mathbf{R}^3$ which are invariant under the octahedral group (generated by sign-changes and U, J). Let $\alpha = -\frac{1}{2}$ then

$$\{q^{\alpha}_{2m,m}: m \geqq 0\} \cup \{(1/\sqrt{2})(q^{\alpha}_{2m+3j,m} + q^{\alpha}_{2m+3j,m+3j}): m \geqq 0, j \geqq 1\}$$

is an orthogonal basis for the cubic harmonics.

4. Consequences and further problems. The opposite of $\alpha \to \infty$ is $\alpha \to -1$. What can be said about the limiting behavior of the eigenvalues of D_α? The calculations in Theorem 2.3 are still valid when $\alpha = -1$ so we will use the basis $f^{-1}_{n,m}$ to represent D_α as a matrix. It is true that D_{-1} is no longer self-adjoint since the positive-definite inner product degenerates. Indeed the central 3 or 4 eigenvalues of $D_\alpha|H^{\alpha}_{n}$ collapse to zero. For $n \geqq$

3 there are three linearly independent eigenvectors for $\lambda = 0$, so when n is odd, there is a degenerate eigenvalue.

4.1. LEMMA. *For $\alpha \geqq -1$, any n, the geometric multiplicity of any eigenvalue of $D_\alpha H^\alpha_{n,\epsilon}$ is one.*

Proof. Since $(D_\alpha - \lambda I)\,|H^\alpha_{n,\epsilon}$ has all nonzero elements on the superdiagonal the solution (u_j) of

$$(D_\alpha - \lambda I)\sum_j u_j f_{n,c+3j} = 0$$

is determined by u_0.

4.2. LEMMA. *For fixed $n \geqq 0$, $c = 0, 1, 2$,*

$$D_{-1}\sum_{j\geqq 0}\binom{n}{c+3j}f_{n,c+3j} = 0.$$

Proof. The coefficient of $f_{n,c+3k}$ in

$$\frac{3}{2}D_{-1}\sum_j\binom{n}{c+3j}f_{n,c+3j}$$

is

$$-\binom{n}{c+3k+3}\binom{c+3k+3}{3} + \binom{n-c-3k}{3}\binom{n}{c+3k}$$
$$-\binom{c+3k}{3}\binom{n}{c+3k} + \binom{n-c-3k+3}{3}\binom{n}{c+3k+3} = 0$$

because the first two and the last two terms cancel out by the identity

$$\binom{n}{l}\binom{n-l}{3} = \binom{n}{l+3}\binom{l+3}{3}.$$

4.3. LEMMA. *For $n \leqq 3$ the determinant of $(\lambda I - D_{-1})\,|H^\alpha_n$ is $\lambda^3 s_{n-2}(\lambda)$ where s_j is monic and of the same parity as j, and has j distinct zeros.*

Proof. By continuity we can use the matrix of D_α in the g_{nm}-basis and let $\alpha \to -1$. Recall

$$b_{n1}(\alpha)^2 = \frac{1}{12}\,\frac{n(\alpha+1)^2(n+\alpha)(n+2\alpha+2)(n+3\alpha+2)}{(2\alpha+3)}$$
$$\to 0 \text{ as } \alpha \to -1,$$

and for $m \geqq 2$,

$$b_{nm}(-1)^2 = \frac{m(n-m+1)(m-1)^2(m-2)(n-m)(n+m-1)(n+m-2)}{12(2m-3)(2m-1)}$$

which is zero for $m = 2$ or $m = n$ but nonzero otherwise. Thus the central $(2 \leqq m \leqq n - 1)$ block of $\lambda I - D_{-1}$ is hermitian tridiagonal with zeros on the diagonal and nonzero elements on the superdiagonal. Hence its determinant, denoted by $s_{n-2}(\lambda)$, has $n - 2$ distinct zeros, and further

$$p_{n+1}(\lambda; n, -1) = \lambda^3 s_{n-2}(\lambda).$$

Clearly

$$s_{n-2}(\lambda) = (-1)^n s_{n-2}(\lambda).$$

Observe that D_{-1} annihilates all polynomials of degree $\leqq 2$ so there are no degeneracies for $n \leqq 2$.

4.4. THEOREM. *For $n \geqq 3$, D_{-1} has exactly one eigenvector for $\lambda = 0$ in each $H^{-1}_{n,\epsilon}$, $\epsilon = 0, 1, 2$. The algebraic multiplicity of the eigenvalue $\lambda = 0$ of $D_{-1}|H^{-1}_n$ is 3 when n is even, 4 when n is odd.*

Proof. Lemmas 4.1 and 4.2 show that D_{-1} has exactly one eigenvector for $\lambda = 0$ in each $H^{-1}_{n,\epsilon}$. Further

$$p_{n+1}(\lambda; n, -1) = \lambda^3 s_{n-2}(\lambda)$$

and s_{n-2} has a simple zero at 0 if n is odd, and no zero at 0 if n is even; thus we obtain the stated multiplicities.

Let $\tilde{q}^{\alpha}_{nm}$ be the multiple of q^{α}_{nm} which has 1 as the coefficient of $f^{\alpha}_{n,c}$ (where $c \equiv n - 2m \bmod 3$) (possible by the argument in Lemma 4.1). Then the coefficients of $f^{\alpha}_{n,c+3j}$ in $\tilde{q}^{\alpha}_{nm}$ are polynomials in λ and α.

4.5. COROLLARY. *For fixed n, $\lambda_j(\alpha) \to \lambda_j(-1)$ as $\alpha \to -1$;*

i) *when $n = 2k + 1$ $(k \geqq 1)$ then $\lambda_0 < \ldots < \lambda_{k-1} = \lambda_k = \lambda_{k+1} = \lambda_{k+2} = 0 < \ldots < \lambda_{2k+1}$ (for $\alpha = -1$), further the limits of $\tilde{q}^{\alpha}_{nm}$ are all distinct except*

$$\lim_{\alpha \to -1} \tilde{q}^{\alpha}_{n,k-1} = \lim_{\alpha \to -1} \tilde{q}^{\alpha}_{n,k+2} \in H^{-1}_{2k+1,0};$$

ii) *when $n = 2k$ $(k \geqq 2)$ then $\lambda_0 < \ldots < \lambda_{k-1} = \lambda_k = \lambda_{k+1} = 0 < \ldots < \lambda_{2k}$, and the limits of the eigenvectors $\tilde{q}^{\alpha}_{nm}$ are all distinct.*

Proof. For the simple eigenvalues of $D_{-1}|H^{-1}_{n,\epsilon}$ perturbation theory asserts that the respective eigenvectors converge to $\tilde{q}^{-1}_{nm}$ (note that the chosen normalization for $\tilde{q}^{\alpha}_{nm}$ makes the coefficients continuous functions of α). The only degeneracy occurs for $n = 2k + 1$, $\epsilon = 0$ for $\lambda = 0$.

Although we do not give an explicit diagonalization for the matrix of U in the ϕ_{nm}-basis ($\alpha > -1$), the commutation $D_\alpha U = UD_\alpha$ does lead to an interesting contiguity relation for certain balanced ${}_4F_3$-series. For convenience, let

$$F(k, m; n, \alpha) := {}_4F_3\left(\begin{matrix} -k, k + 2\alpha + 1, -m, m + 2\alpha + 1 \\ \alpha + 1, -n, n + 3\alpha + 2 \end{matrix}; 1\right)$$

$$(\alpha > -1;\ k, m \leqq n).$$

4.6. THEOREM. *For $\alpha > -1$, $0 \leqq k, m \leqq n$ the function*

$$(k, m) \mapsto \frac{1}{2\alpha + 2m + 1}[(n - m)(2\alpha + m + 1)$$

$$\times (3\alpha + n + m + 2)(\alpha + m + 1)$$

$$\times F(k, m + 1; n, \alpha) - m(m + \alpha)(2\alpha + n + m + 1)$$

$$\times (\alpha + n - m + 1)F(k, m - 1; n, \alpha)]$$

is symmetric in (k, m).

Proof. We use T_α instead of D_α (see Theorem 3.9). Suppose

$$U\phi_{nm} = \sum_j u_{jm}\phi_{nj}$$

then for fixed k, m we have

$$(T_\alpha U)_{km} = (UT_\alpha)_{km}$$

(matrices for the ϕ_{nm}-basis). This identity becomes

$$t_{k,k-1}u_{k-1,m} + t_{k,k+1}u_{k+1,m} = u_{k,m-1}t_{m-1,m} + u_{k,m+1}t_{m+1,m}$$

(for the values of t_{ij} see Theorem 3.9). Further $U\phi_{nm} = \theta_{nm}$ so by the adjoint of the transformation in Theorem 1.7 (iii)

$$u_{km} = (-1)^k M_{nmk}(\alpha)(N_{nm}(\alpha)/N_{nk}(\alpha))^{\frac{1}{2}}$$

$$= (-1)^{n+m}\binom{n}{k}$$

$$\times \frac{(\alpha + 1)_{n-m}(\alpha + 1)_m(n + 3\alpha + 2)_k(2\alpha + 2k + 1)}{(\alpha + 1)_k(2\alpha + k + 2)_n(2\alpha + k + 1)}$$

$$\times F(k, m; n, \alpha).$$

Substitute this in the commutation relation, and cancel out common factors to obtain the invariance of the stated expression under the interchange of k and m.

Wilson has some contiguity relations for balanced ${}_4F_3$-series ([**13**], p. 48) which bear a family resemblance to Theorem 4.6, but it appears that his multipliers are of lower degree (this does not rule out the possibility that 4.6 can be deduced from his formulas). In 4.6 divide by α^2 and let $\alpha \to \infty$ to obtain that the function

$$(k, m) \mapsto 3(n - m)K_{m+1}(k; 3/4, n) - mK_{m-1}(k; 3/4, n)$$

is symmetric in (k, m).

There is another geometric interpretation for H_n^α when $\alpha = \frac{k}{2} - 1$, $k = 1, 2, 3, \dots$, namely that for $x \in \mathbf{R}^{3k}$, $p(v) \in H_n^\alpha$, the polynomial

$$X \mapsto p\left(\sum_1^k x_j^2, \sum_{k+1}^{2k} x_j^2, \sum_{2k+1}^{3k} x_j^2\right)$$

is harmonic of degree $2n$; such functions are essentially intertwining functions for

$$O(3k - 1)\backslash O(3k)/(O(k)xO(k)xO(k)),$$

(see [**11**]).

Further problems. It may be that the determinant-related polynomials $p_m(\lambda; n, \alpha)$ have a closed form (hypergeometric series). One would like to have approximations for the eigenvalues λ_j of D_α (not just asymptotic results for $\alpha \to \infty$). From numerical experimentation it appears that the eigenfunctions of $D_\alpha | H_{n,\epsilon}^\alpha$ have all positive coefficients (of f_{nm}^α), but this is not as yet proven. It may be true that the eigenfunctions q_{nm}^α achieve their maximum on the boundary of E (or Ω). This does hold for each ϕ_{nm} when $\alpha > -1$. It is not generally true that

$$|q_{nm}^\alpha(z)| \leqq q_{nm}^\alpha(1) \quad \text{for } z \in \Omega:$$

indeed consider

$$f_{2,0}^\alpha(z) = z^2 - (2/(3\alpha + 5))\bar{z}t$$

(see Section 2) which is an eigenfunction of D;

$$f_{2,0}^\alpha(1) = 3(\alpha + 1)/(3\alpha + 5)$$

but

$$f_{2,0}^\alpha((1 + \omega)/2) = \omega(3(\alpha + 3)/(4(3\alpha + 5)))$$

which is of larger absolute value than $f_{2,0}^\alpha(1)$ for $-1 \leqq \alpha < -1/3$.

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