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APPROXIMATION OF SMOOTH MAPS BY REAL ALGEBRAIC MORPHISMS

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ABSTRACT. Let $\mathbb{G}_{p,q}(\mathbb{F})$ be the Grassmann space of all q-dimensional $\mathbb{F}$-vector subspaces of $\mathbb{F}^p$, where $\mathbb{F}$ stands for $\mathbb{R}$, $\mathbb{C}$ or $\mathbb{H}$ (the quaternions). Here $\mathbb{G}_{p,q}(\mathbb{F})$ is regarded as a real algebraic variety. The paper investigates which C^∞ maps from a nonsingular real algebraic variety X into $\mathbb{G}_{p,q}(\mathbb{F})$ can be approximated, in the C^∞ compact-open topology, by real algebraic morphisms.

1. **Introduction and results.** Let us recall that the term *affine real algebraic variety* designates a locally ringed space isomorphic to an algebraic set in $\mathbb{R}^n$, for some n, endowed with the Zariski topology and the sheaf of $\mathbb{R}$-valued regular functions [2, Section 3.2]. *Real algebraic varieties* (not necessarily affine) are defined in the usual way [2]. It is well known that every Zariski locally closed subvariety of the real projective space $\mathbb{P}^n(\mathbb{R})$ is actually affine [2, Theorem 3.4.4, Proposition 3.2.10]. *Real algebraic morphisms* are called *regular maps*. Important examples of affine real algebraic varieties are Grassmannians. More precisely, let $\mathbb{G}_{p,q}(\mathbb{F})$ be the Grassmann space of all q-dimensional $\mathbb{F}$-vector subspaces of $\mathbb{F}^p$, where $\mathbb{F}$ stands for $\mathbb{R}$, $\mathbb{C}$, or $\mathbb{H}$ ($\mathbb{H}$ denotes the quaternions). As in [2, Sections 3.4, 13.3], we regard $\mathbb{G}_{p,q}(\mathbb{F})$ as an affine real algebraic variety.

Every real algebraic variety can be equipped with the topology determined by the usual metric topology on $\mathbb{R}$. Unless explicitly stated otherwise, all topological terms related to real algebraic varieties will refer to this topology.

Let X be an affine real algebraic variety. An algebraic $\mathbb{F}$-vector bundle ξ on X is called *strongly algebraic* if ξ is algebraically isomorphic to an algebraic $\mathbb{F}$-vector subbundle of the trivial $\mathbb{F}$-vector bundle with total space $X \times \mathbb{F}^p$ for some p (*cf.* [1] or [2, Sections 12.1, 12.6, 13.3] for equivalent definitions). The universal $\mathbb{F}$-vector bundle $\gamma_{p,q}(\mathbb{F})$ on $\mathbb{G}_{p,q}(\mathbb{F})$ is strongly algebraic [2]. Every strongly algebraic $\mathbb{F}$-vector bundle of rank q on X is algebraically isomorphic to the pullback $\mathbb{F}$-vector bundle $f^*\gamma_{p,q}(\mathbb{F})$ for some $p \geq q$ and some regular map $f: X \longrightarrow \mathbb{G}_{p,q}(\mathbb{F})$ [2, Theorem 12.1.7]. A topological $\mathbb{F}$-vector bundle on X is said to *admit an algebraic structure* if it is topologically isomorphic to a strongly algebraic $\mathbb{F}$-vector bundle (*cf.* [3]).

Given two real algebraic varieties X and Y, we let $R(X, Y)$ denote the set of all regular maps from X into Y. If X and Y are nonsingular, which will always be the case in this paper, we regard $R(X, Y)$ as a subset of the space $C^\infty(X, Y)$ of all C^∞ maps from X

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into Y; the latter space is equipped with the C^∞ compact-open topology (the weak C^∞ topology in the terminology used in [5]).

Let us set $d(\mathbb{F}) = \dim_{\mathbb{R}} \mathbb{F}$. The aim of this paper is to prove the following.

THEOREM 1.1. *Let X be a nonsingular affine real algebraic variety and let $f: X \longrightarrow \mathbb{G}_{p,q}(\mathbb{F})$ be a C^∞ map with $p > q \geq 1$. If*

$$\dim X \leq \big(p + 1 - \min(q, p - q)\big) d(\mathbb{F}),$$

then the following conditions are equivalent:

(a) f belongs to the closure of $R\big(X, \mathbb{G}_{p,q}(\mathbb{F})\big)$ in $C^\infty\big(X, \mathbb{G}_{p,q}(\mathbb{F})\big)$;
(b) f is homotopic to a regular map from X into $\mathbb{G}_{p,q}(\mathbb{F})$;
(c) $f^*\gamma_{p,q}(\mathbb{F})$ admits an algebraic structure.

We do not know if any assumption which puts constraints on $\dim X$, p, and q is necessary. No such assumption is needed if X is compact [2, Theorem 13.3.1]. Furthermore, no such assumption is required for the equivalence of conditions (a) and (b) [6]. Condition (c) is of interest since in many cases it can be directly verified, while there is no reasonable way of explicitly verifying either (a) or (b).

Theorem 1.1 implies immediately the following.

COROLLARY 1.2. *Let X be a nonsingular affine real algebraic variety and let $f: X \longrightarrow \mathbb{G}_{p,q}(\mathbb{F})$ be a C^∞ map with $p > q \geq 1$. If $\dim X \leq 2d(\mathbb{F})$, then the following conditions are equivalent:*

(a) f belongs to the closure of $R\big(X, \mathbb{G}_{p,q}(\mathbb{F})\big)$ in $C^\infty\big(X, \mathbb{G}_{p,q}(\mathbb{F})\big)$;
(b) f is homotopic to a regular map from X into $\mathbb{G}_{p,q}(\mathbb{F})$;
(c) $f^*\gamma_{p,q}(\mathbb{F})$ admits an algebraic structure. ∎

Let us observe that if either $\dim X < d(\mathbb{F})$ or $\dim X = d(\mathbb{F})$ and X has no compact connected component, then every topological $\mathbb{F}$-vector bundle of constant rank on X is trivial, and hence Corollary 1.2 implies that $R\big(X, \mathbb{G}_{p,q}(\mathbb{F})\big)$ is dense in $C^\infty\big(X, \mathbb{G}_{p,q}(\mathbb{F})\big)$ for all $p > q \geq 1$.

We shall now specialize to $\dim X = 1$ and $\mathbb{F} = \mathbb{R}$.

COROLLARY 1.3. *If X is a nonsingular affine real algebraic variety with $\dim X = 1$, then $R\big(X, \mathbb{G}_{p,q}(\mathbb{R})\big)$ is dense in $C^\infty\big(X, \mathbb{G}_{p,q}(\mathbb{R})\big)$ for all $p > q \geq 1$.*

PROOF. By Theorem 1.1, it suffices to show that every topological $\mathbb{R}$-vector bundle ξ of constant rank on X admits an algebraic structure. We may assume that X is a Zariski open and dense subset of a compact nonsingular affine real algebraic variety $\tilde{X}$. Since $\dim \tilde{X} = 1$, there exists a topological $\mathbb{R}$-vector bundle $\tilde{\xi}$ on $\tilde{X}$ whose restriction to X is isomorphic to ξ. By [2, Theorem 12.5.1], $\tilde{\xi}$ admits an algebraic structure, and hence so does ξ. ∎

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REMARK 1.4. Denote by S^d the unit sphere in $\mathbb{R}^{d+1}$,

$$S^d = \{(x_0, \dots, x_d) \in \mathbb{R}^{d+1} \mid x_0^2 + \dots + x_d^2 = 1\}.$$

It is well known that $\mathbb{G}_{2,1}(\mathbb{F})$ and $S^{d(\mathbb{F})}$ are isomorphic as real algebraic varieties, and that $\gamma_{2,1}(\mathbb{F})$ corresponds to the Hopf $\mathbb{F}$-line bundle on $S^{d(\mathbb{F})}$. In particular, Theorem 1.1 and Corollary 1.2 yield the obvious results on maps with values in S^1, S^2 and S^4.

One easily sees that every topological $\mathbb{R}$-line bundle on $S^1 \times \mathbb{R}$ (up to isomorphism there is only one such $\mathbb{R}$-line bundle) admits an algebraic structure. Hence, in view of Corollary 1.2, $\mathcal{R}(S^1 \times \mathbb{R}, S^1)$ is dense in $\mathcal{C}^\infty(S^1 \times \mathbb{R}, S^1)$.

On the other hand, $\mathcal{R}(X, S^1)$ is not dense in $\mathcal{C}^\infty(X, S^1)$, where $X = \mathbb{R}^2 \setminus \{0\}$. Indeed, the $\mathcal{C}^\infty$ function $f\colon X \longrightarrow S^1$, defined by $f(x_1, x_2) = (1/\sqrt{x_1^2 + x_2^2})(x_1, x_2)$ for all (x_1, x_2) in X, does not belong to the closure of $\mathcal{R}(X, S^1)$ in $\mathcal{C}^\infty(X, S^1)$ (*cf.* [6, Remark 1.5]). ■

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2. **Proof of Theorem 1.1.** It is known that (a) and (b) are equivalent [6], while (b) obviously implies (c). Thus it remains to prove that (c) implies (a). Since $\mathbb{G}_{p,q}(\mathbb{F})$ and $\mathbb{G}_{p,p-q}(\mathbb{F})$ are isomorphic as real algebraic varieties, we may assume, and do so below, that $q \leq p - q$.

Suppose that (c) holds, that is, there exists a strongly algebraic $\mathbb{F}$-vector bundle ξ and a topological isomorphism $\varphi\colon \xi \longrightarrow f^*\gamma_{p,q}(\mathbb{F})$. Since f is of class $\mathcal{C}^\infty$, we may assume that φ is also of class $\mathcal{C}^\infty$ [5, p. 101, Theorem 3.5]. Denote by ε the trivial $\mathbb{F}$-vector bundle on X with total space $X \times \mathbb{F}^p$; thus $\varepsilon_x = \{x\} \times \mathbb{F}^p$ is the fiber of ε at x in X. The image $f(x)$ is a q-dimensional $\mathbb{F}$-vector subspace of $\mathbb{F}^p$ and the fiber $\big(f^*\gamma_{p,q}(\mathbb{F})\big)_x$ of $f^*\gamma_{p,q}(\mathbb{F})$ at x satisfies

$$\big(f^*\gamma_{p,q}(\mathbb{F})\big)_x = \{x\} \times f(x) \subseteq \{x\} \times \mathbb{F}^p = \varepsilon_x.$$

It follows that $\sigma\colon X \longrightarrow \mathrm{Hom}(\xi, \varepsilon)$, defined by

$$\sigma(x)(e) = \varphi(e)$$

for all x in X and all e in the fiber ξ_x of ξ at x, is a $\mathcal{C}^\infty$ section. Obviously, the $\mathbb{F}$-linear transformation $\sigma(x)\colon \xi_x \longrightarrow \varepsilon_x$ is injective for every x in X. Let $\mathcal{N}$ be a neighborhood of σ in the $\mathcal{C}^\infty$ compact-open topology on the space of all global $\mathcal{C}^\infty$ sections of $\mathrm{Hom}(\xi, \varepsilon)$.

ASSERTION. There exists a regular section $s\colon X \longrightarrow \mathrm{Hom}(\xi, \varepsilon)$ such that s belongs to $\mathcal{N}$. Furthermore, if $\dim X \geq 2$, then s can be chosen in such a way that the $\mathbb{F}$-linear transformation $s(x)\colon \xi_x \longrightarrow \varepsilon_x$ is injective for every x in X.

Suppose for a moment that the assertion is proved. Let

$$D(s) = \{x \in X \mid s(x) \text{ is not injective}\}$$

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and define $g\colon X \setminus D(s) \longrightarrow \mathbb{G}_{p,q}(\mathbb{F})$ by

$$g(x) = \rho\big(s(x)(\xi_x)\big)$$

for x in $X \setminus D(s)$, where $\rho\colon X \times \mathbb{F}^p \longrightarrow \mathbb{F}^p$ is the canonical projection (g is well defined, $s(x)$ being injective for all x in $X \setminus D(s)$). Clearly, $D(s)$ is a Zariski closed subset of X. Since ξ is strongly algebraic and s is regular, it follows that g is a regular map on $X \setminus D(s)$ (*cf.* [2, Proposition 3.4.9 and Theorem 12.1.7]). If $\dim X \geq 2$, then $D(s)$ is empty and hence g is defined on X. If $\dim X \leq 1$, then g can be extended to a regular map on X, which we, abusing notation, also denote by g (indeed, existence of such an extension is obvious for $\dim X = 0$, while for $\dim X = 1$ it follows from the fact that $\mathbb{G}_{p,q}(\mathbb{F})$ is a projective real algebraic variety). Thus we have a regular map $g\colon X \longrightarrow \mathbb{G}_{p,q}(\mathbb{F})$ defined for $\dim X \geq 0$. Obviously, g is arbitrarily close to f in the C^∞ compact-open topology, provided that $\mathcal{N}$ is a sufficiently small neighborhood of σ. Hence the assertion implies (a).

Now we proceed to proving the Assertion. We begin with the observation that there exists a regular section $s\colon X \longrightarrow \mathrm{Hom}(\xi, \varepsilon)$ that belongs to $\mathcal{N}$. The argument is straightforward. Since $\mathrm{Hom}(\xi, \varepsilon)$ is a strongly algebraic $\mathbb{F}$-vector bundle (*cf.* [2, Proposition 12.1.8]), there exists a strongly algebraic $\mathbb{F}$-vector bundle η on X such that the $\mathbb{F}$-vector bundle $\zeta = \mathrm{Hom}(\xi, \varepsilon) \oplus \eta$ is algebraically isomorphic to a trivial $\mathbb{F}$-vector bundle on X (*cf.* [2, Theorem 12.1.7]). It therefore follows from the classical Weierstrass approximation theorem that there exists a regular section $t\colon X \longrightarrow \zeta$, arbitrarily close in the C^∞ compact-open topology to the C^∞ section $\sigma_\eta\colon X \longrightarrow \zeta$ defined by $\sigma_\eta(x) = \big(\sigma(x), 0\big)$ for all x in X. If t is sufficiently close to σ_η, then the regular section $s = \pi \circ t$ belongs to $\mathcal{N}$, where $\pi\colon \zeta \longrightarrow \mathrm{Hom}(\xi, \varepsilon)$ is the canonical projection. This completes the proof of the observation.

In particular, the observation implies the Assertion for $\dim X \leq 1$. Henceforth we assume $\dim X \geq 2$. Some extra care is now necessary because for a regular section $s\colon X \longrightarrow \mathrm{Hom}(\xi, \varepsilon)$, selected at random in $\mathcal{N}$, the $\mathbb{F}$-linear transformation $s(x)\colon \xi_x \longrightarrow \varepsilon_x$ need not be injective for every x in X. There is no difficulty if X is compact, since then every regular section s sufficiently close to σ satisfies the injectivity condition; recall that $\sigma(x)$ is injective for all x in X. Therefore we also assume that X is not compact.

Now we prove the Assertion (with X noncompact and $\dim X \geq 2$) in six steps.

STEP 1. We show that there exist a compact nonsingular affine real algebraic variety $\tilde{X}$ and a strongly algebraic $\mathbb{F}$-vector bundle $\tilde{\xi}$ on $\tilde{X}$ such that X is a Zariski open and dense subset of $\tilde{X}$, the algebraic subvariety $\tilde{X} \setminus X$ of $\tilde{X}$ is a divisor with normal crossings, and ξ is algebraically isomorphic to the restriction $\tilde{\xi}|X$ of $\tilde{\xi}$ to X.

Since ξ is a strongly algebraic $\mathbb{F}$-vector bundle, we can find a positive integer n and a regular map $h\colon X \longrightarrow \mathbb{G}_{n,q}(\mathbb{F})$ such that $h^*\gamma_{n,q}(\mathbb{F})$ is algebraically isomorphic to ξ. We may assume that X is a Zariski locally closed subvariety of the k-dimensional projective space $\mathbb{P}^k(\mathbb{R})$ for some positive integer k. Let $\overline{X}$ be the Zariski closure of X in $\mathbb{P}^k(\mathbb{R})$. Let us also recall that $\mathbb{G}_{n,q}(\mathbb{F})$ is a projective real algebraic variety. Considering h as a rational map of $\overline{X}$ into $\mathbb{G}_{n,q}(\mathbb{F})$ and applying Hironaka's theorems on resolution of singularities

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and resolution of points of indeterminacy [4], we obtain $\tilde{X}$ having the desired properties and a regular map $\tilde{h}\colon \tilde{X} \longrightarrow \mathbb{G}_{n,q}(\mathbb{F})$ that is an extension of h. Setting $\tilde{\xi} = \tilde{h}^*\gamma_{n,q}(\mathbb{F})$ we complete the proof of Step 1.

STEP 2. Denote by $\tilde{\varepsilon}$ the trivial $\mathbb{F}$-vector bundle on $\tilde{X}$ with total space $\tilde{X} \times \mathbb{F}^p$. Identify ε with $\tilde{\varepsilon}|X$ and ξ with $\tilde{\xi}|X$. Let K be a compact subset of X and let $\lambda\colon \tilde{X} \longrightarrow \mathbb{R}$ be a $\mathcal{C}^\infty$ function such that $\lambda = 1$ in a neighborhood of K in X and $\lambda = 0$ in a neighborhood of $\tilde{X} \setminus X$ in $\tilde{X}$. Then $\tilde{\sigma}\colon \tilde{X} \longrightarrow \mathrm{Hom}(\tilde{\xi}, \tilde{\varepsilon})$, defined by

$$\tilde{\sigma}(x) = \begin{cases} \lambda(x)\sigma(x) & \text{for } x \in X \\ 0 & \text{for } x \in \tilde{X} \setminus X \end{cases},$$

is a $\mathcal{C}^\infty$ section satisfying $\tilde{\sigma} = \sigma$ in a neighborhood of K in X. In particular, $\tilde{\sigma}(x)\colon \tilde{\xi}_x \longrightarrow \tilde{\varepsilon}_x$ is injective for all x in K. If the compact K is sufficiently large, then K contains all the compact connected components of X and there exists a neighborhood $\tilde{\mathcal{N}}$ of $\tilde{\sigma}$ in the $\mathcal{C}^\infty$ compact-open topology on the space of all global $\mathcal{C}^\infty$ sections of $\mathrm{Hom}(\tilde{\xi}, \tilde{\varepsilon})$ such that for every section τ in $\tilde{\mathcal{N}}$, the restriction $\tau|X$ belongs to $\mathcal{N}$. Select K and $\tilde{\mathcal{N}}$ with these properties so that, in addition, each connected component of $\tilde{X} \setminus K$ contains a connected component of $\tilde{X} \setminus X$ and every $\mathcal{C}^\infty$ section $v\colon \tilde{X} \longrightarrow \mathrm{Hom}(\tilde{\xi}, \tilde{\varepsilon})$, with $v|K = u|K$ for some section u in $\tilde{\mathcal{N}}$, is also in $\tilde{\mathcal{N}}$.

STEP 3. Let H be the total space of $\mathrm{Hom}(\tilde{\xi}, \tilde{\varepsilon})$. Every element of H is an $\mathbb{F}$-linear transformation from $\tilde{\xi}_x$ into $\tilde{\varepsilon}_x$ for some x in $\tilde{X}$. Given an integer r satisfying $0 \le r \le q$, set

$$H^r = \{\ell \in H \mid \mathrm{rank}\, \ell = r\}.$$

It is well known that H^r is a $\mathcal{C}^\infty$ submanifold of H of (real) codimension $(p-r)(q-r)d(\mathbb{F})$. Since $q \le p - q$, we have $\dim \tilde{X} \le (p - q + 1)d(\mathbb{F})$, and hence

$$\dim \tilde{X} < \mathrm{codim}\, H^r \quad \text{for } 0 \le r \le q-2,$$
$$\dim \tilde{X} \le \mathrm{codim}\, H^{q-1}.$$

Set

$$H^* = \bigcup_{r=0}^{q-1} H^r.$$

By a standard transversality result (*cf.* [5, p. 83, Exercise 13]), there exists a $\mathcal{C}^\infty$ section $\alpha\colon \tilde{X} \longrightarrow \mathrm{Hom}(\tilde{\xi}, \tilde{\varepsilon})$, arbitrarily close to $\tilde{\sigma}$ in the $\mathcal{C}^\infty$ compact-open topology, such that α is transverse to H^r for $0 \le r \le q$. Obviously,

$$\alpha^{-1}(H^*) = \alpha^{-1}(H^{q-1})$$

and this set is finite (possibly empty). Note that H^* is a closed subset of H and $\tilde{\sigma}^{-1}(H^*) \subseteq \tilde{X} \setminus K$. We may therefore assume that α is chosen sufficiently close to $\tilde{\sigma}$ so that

$$\alpha \in \tilde{\mathcal{N}} \quad \text{and} \quad \alpha^{-1}(H^*) \subseteq \tilde{X} \setminus K.$$

Recall that each connected component of $\tilde{X} \setminus K$ contains a connected component of the hypersurface $\tilde{X} \setminus X$. Since $\alpha^{-1}(H^*)$ is a finite set contained in $\tilde{X} \setminus K$ and $\dim(\tilde{X} \setminus X) \geq 1$, it follows from [5, p. 181, Theorem 1.7] that there exists a C^∞ diffeotopy $\theta_t: \tilde{X} \longrightarrow \tilde{X}$, $t \in [0, 1]$, such that θ_0 is the identity map of $\tilde{X}$, $\theta_t(x) = x$ for x in a neighborhood of K and all t in $[0, 1]$, and $\theta_1\big(\alpha^{-1}(H^*)\big) \subseteq \tilde{X} \setminus X$. Set $\psi = \theta_1^{-1}$ and consider the C^∞ section

$$\psi^* \alpha: \tilde{X} \longrightarrow \psi^* \operatorname{Hom}(\tilde{\xi}, \tilde{\varepsilon}).$$

The total space of $\psi^* \operatorname{Hom}(\tilde{\xi}, \tilde{\varepsilon})$ is

$$\psi^* H = \{(x, \ell) \in \tilde{X} \times H \mid \psi(x) = \operatorname{pr}(\ell)\},$$

where $\operatorname{pr}: H \longrightarrow \tilde{X}$ is the projection of the bundle $\operatorname{Hom}(\tilde{\xi}, \tilde{\varepsilon})$, and

$$(\psi^* \alpha)(x) = \Big(x, \alpha\big(\psi(x)\big)\Big) \quad \text{for } x \text{ in } \tilde{X}.$$

Since θ_t^{-1} is a diffeotopy and $\psi = \theta_1^{-1}$, there exists an isomorphism

$$\mu: \psi^* \operatorname{Hom}(\tilde{\xi}, \tilde{\varepsilon}) \longrightarrow \operatorname{Hom}(\tilde{\xi}, \tilde{\varepsilon})$$

such that $\mu(x, \ell) = \ell$ for all (x, ℓ) in $(\psi^* H) \cap (K \times H)$ (*cf.* the proof of Theorem 2.4, p. 97 in [5]). Set

(1) $$\beta = \mu \circ (\psi^* \alpha).$$

Note that $\beta: \tilde{X} \longrightarrow \operatorname{Hom}(\tilde{\xi}, \tilde{\varepsilon})$ is a C^∞ section satisfying $\beta|K = \alpha|K$, and hence

(2) $$\beta \in \tilde{N}$$

(recall that α is in $\tilde{N}$ and see Step 2). Furthermore, by construction,

(3) $$\beta^{-1}(H^*) = \psi^{-1}\big(\alpha^{-1}(H^*)\big) \subseteq \tilde{X} \setminus X.$$

It is convenient to observe that α in the above construction can be chosen to be real analytic. Indeed, by a standard transversality result (*cf.* [5, p. 84, Exercise 15]), every global C^∞ section of $\operatorname{Hom}(\tilde{\xi}, \tilde{\varepsilon})$ sufficiently close to α is transverse to H^r for $0 \leq r \leq q$, and hence α can be replaced by such a section. Since α can be approximated by regular sections (*cf.* the beginning of the proof of the Assertion), we may assume that α itself is a regular, thus a real analytic, section.

The next step is a general remark.

STEP 4. Let $B = \{y \in \mathbb{R}^d \mid \|y\| \leq 1\}$, where $\| - \|$ stands for the Euclidean norm on $\mathbb{R}^d$. Denote by $M(m \times n, \mathbb{R})$ the space of all $m \times n$ real matrices. Given a C^∞ map $F: B \longrightarrow M(m \times n, \mathbb{R})$, where $n \leq m$, and a point y in B, we may regard $F(y)$ as an $\mathbb{R}$-linear transformation from $\mathbb{R}^n$ into $\mathbb{R}^m$. Set

$$D(F) = \{y \in B \mid F(y) \text{ is not injective}\}.$$

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Obviously, if $F^{\#}(y)$ is the sum of squares of all the $n \times n$ minors of the matrix $F(y)$, then $F^{\#}: B \longrightarrow \mathbb{R}$ is a C^∞ function and

$$D(F) = (F^{\#})^{-1}(0).$$

Assume now that F is a real analytic map. We claim that if

$$D(F) = \{0\},$$

then there exist a positive integer k and a neighborhood $\mathcal{M}$ of F in the C^∞ compact-open topology on $C^\infty\big(B, M(m \times n, \mathbb{R})\big)$ such that each map G in $\mathcal{M}$ with the same k-jet as F at 0 satisfies

$$D(G) = \{0\}.$$

Indeed, since $F^{\#}$ is a nonnegative analytic function and $(F^{\#})^{-1}(0) = \{0\}$, it follows from the Łojasiewicz inequality [7] that one can find a positive real number c and a positive integer k such that

$$F^{\#}(y) \geq c\|y\|^k \quad \text{for } y \in B.$$

Now, if $G: B \longrightarrow M(m \times n, \mathbb{R})$ is a C^∞ map sufficiently close to F in the C^∞ compact-open topology and if G has the same k-jet as F at 0, then Taylor's theorem implies

$$G^{\#}(y) \geq \frac{c}{2}\|y\|^k \quad \text{for } y \in B.$$

In particular, $D(G) = (G^{\#})^{-1}(0) = \{0\}$, and hence the claim is proved.

We return to the proof of the Assertion. We shall use the sections α and β constructed in Step 3.

STEP 5. We show that there exist a positive integer k and a neighborhood $\mathcal{U}$ of α in the C^∞ compact-open topology such that for each section u in $\mathcal{U}$ with the same k-jet as α at every point of $\alpha^{-1}(H^*)$, one has

$$u^{-1}(H^*) = \alpha^{-1}(H^*).$$

To this end let us first recall that α is a real analytic section and

$$\alpha^{-1}(H^*) = \{x \in \tilde{X} \mid \alpha(x): \tilde{\xi}_x \longrightarrow \tilde{\varepsilon}_x \text{ is not injective}\}.$$

Consider $\tilde{\xi}$ and $\tilde{\varepsilon}$ as real analytic $\mathbb{R}$-vector bundles, and the $\alpha(x)$ as $\mathbb{R}$-linear transformations, $x \in X$. Since $\alpha^{-1}(H^*)$ is a finite set, the conclusion follows from Step 4.

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STEP 6. We complete the proof of the Assertion and hence finish the proof of Theorem 1.1.

Let k and $\mathcal{U}$ be as in Step 5. It follows from (1) that

$$\mathcal{V} = \{\mu \circ (\psi^* u) \mid u \in \mathcal{U}\}$$

is a neighborhood of β in the C^∞ compact-open topology. By construction, if v is in $\mathcal{V}$ and v has the same k-jet as β at every point of $\beta^{-1}(H^*)$, then

$$v^{-1}(H^*) = \beta^{-1}(H^*).$$

Furthermore, in view of (2), if we take $\mathcal{U}$ sufficiently small, then

$$\mathcal{V} \subseteq \tilde{\mathcal{N}}.$$

Since $\beta^{-1}(H^*)$ is a finite set, arguing as at the beginning of the proof of the Assertion, we can find a regular section $b: \tilde{X} \longrightarrow \mathrm{Hom}(\tilde{\xi}, \tilde{\varepsilon})$ such that b is in $\mathcal{V}$ and b has the same k-jet as β at every point of $\beta^{-1}(H^*)$. In particular, $b^{-1}(H^*) = \beta^{-1}(H^*)$. Note that $s = b|X$ is a global regular section of $\mathrm{Hom}(\xi, \varepsilon)$ that belongs to $\mathcal{N}$ (see Step 2 and use the fact that b is in $\tilde{\mathcal{N}}$) and, by virtue of (3), the $\mathbb{F}$-linear transformation $s(x): \xi_x \longrightarrow \varepsilon_x$ is injective for every x in X. Hence the Assertion is proved. ∎

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