ON THE ABSOLUTE CESARO SUMMABILITY FACTORS OF TRIGONOMETRIC SERIES

by NIRANJAN SINGH

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1.1 Let $\sum_0^\infty a_n$ be any given infinite series with s_n as its n-th partial sum. We write

$$S_n^\alpha = \sum_{\nu=0}^{n} A_{n-\nu}^{\alpha-1} s_\nu, \quad \alpha > -1,$$

and

$$\sigma_n^\alpha = \frac{S_n^\alpha}{A_n^\alpha},$$

where

$$A_n^\alpha = \binom{n+\alpha}{n}.$$

If $\{\sigma_n^\alpha\}$ is a sequence of bounded variation, that is to say,

$$\sum_1^\infty | \sigma_n^\alpha - \sigma_{n-1}^\alpha | < \infty, \tag{1.1.1}$$

then we say that $\sum_0^\infty a_n$ is summable $| C, \alpha |$. By virtue of the identity (1)

$$t_n^\alpha = n(\sigma_n^\alpha - \sigma_{n-1}^\alpha),$$

where t_n^α is the (C, α) mean of the sequence $\{n\, a_n\}$, the condition (1.1.1) becomes

$$\sum_1^\infty \frac{| t_n^\alpha |}{n} < \infty.$$

1.2 Concerning the almost everywhere summability $| C, \alpha | (\alpha > \frac{1}{2})$ of the trigonometric series

$$\sum_1^\infty (a_n \cos nx + b_n \sin nx) \equiv \sum_1^\infty A_n(x),$$

Wang (4) in 1941 proved the following theorem.

Theorem A. *If*

$$\sum_{n=2}^\infty (a_n^2 + b_n^2)(\log n)^{1+\varepsilon} \quad (\varepsilon > 0)$$

converges, then the trigonometric series $\sum_{n=1}^\infty A_n(x)$ *is summable* $| C, \alpha | (\alpha > \frac{1}{2})$ *almost everywhere.*

This result of Wang has been subsequently generalized by Ul'yanov (6). He proved the following theorem.

Theorem B. *If*

$$\sum_{1}^{\infty}(a_n^2+b_n^2)W_n<\infty,$$

then $\sum_{n=1}^{\infty} A_n(x)$ *is summable* $|C, \alpha|$ $(\alpha>\frac{1}{2})$, *where* $\{W_n\}$ *is a positive monotonic increasing sequence of numbers such that* $\sum_{1}^{\infty}\frac{1}{nW_n}<\infty$.

It is clear from the condition of Theorem B that W_n cannot be taken to be $\log n$. It has also been shown by Wang (5) that in his theorem ε cannot be taken to be zero.

In this note we obtain a further generalization of the above theorem. We shall prove the following in which it is possible to take W_n to be $\log n$.

2.1. Theorem. *Let* $\{W_n\}$ *and* $\{\lambda_n\}$ *be two positive sequences such that* $\left(\frac{W_n}{\lambda_n^2}\right)$ *is a monotonic increasing sequence * and*

$$\sum_{1}^{\infty}\frac{\lambda_n^2}{nW_n}<\infty.$$

If

$$\sum_{1}^{\infty}(a_n^2+b_n^2)W_n<\infty,$$

then $\sum_{1}^{\infty} A_n(x)\lambda_n$ *is summable* $|C, \alpha|$ $(\alpha>\frac{1}{2})$ *almost everywhere.*

It may be remarked that Theorem B is a particular case $\lambda_n=1$ of our theorem. Also, if we take $W_n=1$ and $\lambda_n=\frac{1}{(\log n)^{\frac{1}{2}+\varepsilon}}$, then our theorem includes the following theorem of Pati (2).

Theorem C. *If* $\sum_{1}^{\infty}(a_n^2+b_n^2)<\infty$, *then*

$$\sum_{2}^{\infty}\frac{A_n(x)}{(\log n)^{\frac{1}{2}+\varepsilon}}\quad(\varepsilon>0)$$

is summable $|C, \alpha|$ $(\alpha>\frac{1}{2})$ *almost everywhere.*

2.2 We require the following lemmas for the proof of our theorem.

Lemma 1 (3). *If* $\{F_n(t)\}$ *is a sequence of positive, monotonic increasing functions, defined in the interval* (a, b), *and* $\Sigma F_n(b)$ *is convergent, then* $\Sigma F_n'(t)$ *converges almost everywhere in* (a, b).

* In the original draft the monotonicity of $\{W_n\}$ and $\{\lambda_n\}$ was assumed separately. The author is grateful to the referee for this improvement.

Lemma 2 [6, p. 40]. *Let* $\{W(n)\}$ *be a positive and non-decreasing function in* $[n_0, \infty]$. *Then the series* $\sum_{m=n_0}^{\infty} \frac{1}{mW(m)}$ *and* $\sum_{m=n_0^2}^{\infty} \frac{1}{mW(m^{\frac{1}{2}})}$ *converge or diverge simultaneously.*

3.1 Proof of the Theorem. Let

$$T_n^{\alpha}(x) = \frac{1}{A_n^{\alpha}} \sum_{\nu=1}^{n} A_{n-\nu}^{\alpha-1} \nu A_\nu(x) \lambda_\nu.$$

Then we have to prove that

$$\sum_{n=1}^{\infty} \frac{1}{n} \mid T_n^{\alpha}(x) \mid$$

converges almost everywhere.

By virtue of Lemma 1, it is sufficient to show that the series

$$\sum_{n=1}^{\infty} \frac{1}{n} \int_0^{2\pi} \mid T_n^{\alpha}(x) \mid dx$$

converges.

Now *

$$\begin{aligned}
\int_0^{2\pi} \mid T_n^{\alpha}(x) \mid dx &\leqq C\left(\int_0^{2\pi} \{T_n^{\alpha}(x)\}^2 dx\right)^{\frac{1}{2}} \\
&\leqq \frac{C}{n^{\alpha}}\left(\int_0^{2\pi} \left\{\sum_{\nu=1}^{n} A_{n-\nu}^{\alpha-1} \lambda_\nu . \nu . A_\nu(x)\right\}^2 dx\right)^{\frac{1}{2}} \\
&\leqq \frac{C}{n^{\alpha}}\left(\sum_{\nu=1}^{n} \binom{\alpha+n-\nu-1}{n-\nu}^2 \nu^2 \lambda_\nu^2 (a_\nu^2 + b_\nu^2)\right)^{\frac{1}{2}} \\
&\leqq \frac{C}{n^{\alpha}}\left(\sum_{\nu=1}^{n} (n-\nu)^{2\alpha-2} \nu^2 \lambda_\nu^2 (a_\nu^2 + b_\nu^2)\right)^{\frac{1}{2}} \\
&\leqq \frac{C}{n^{\alpha}}\left(\sum_{\nu=1}^{[n^{1/2}]} + \sum_{\nu=[n^{1/2}]+1}^{n}\right)^{\frac{1}{2}} \\
&\leqq \frac{C}{n^{\alpha}}\left\{\left(\sum_{\nu=1}^{[n^{1/2}]}\right)^{\frac{1}{2}} + \left(\sum_{\nu=[n^{1/2}]+1}^{n}\right)^{\frac{1}{2}}\right\}
\end{aligned}$$

Hence

$$\sum_{n=1}^{\infty} \frac{1}{n} \int_0^{2\pi} \mid T_n^{\alpha}(x) \mid dx \leqq C \sum_{n=1}^{\infty} \frac{1}{n^{\alpha+1}} \left\{\left(\sum_{\nu=1}^{[n^{1/2}]}\right)^{\frac{1}{2}} + \left(\sum_{\nu=[n^{1/2}]+1}^{n}\right)^{\frac{1}{2}}\right\}$$

$$= \Sigma_1 + \Sigma_2, \text{ say.}$$

* We denote by C a positive constant independent of n but not necessarily the same at each occurrence.

We have

$$\Sigma_1 \leqq C \sum_{n=1}^{\infty} \frac{1}{n^{\alpha+1}} \left(\sum_{\nu=1}^{[n^{1/2}]} (n-\nu)^{2\alpha-2} \frac{\nu^2 \lambda_\nu^2}{W_\nu} \right)^{\frac{1}{2}}$$

$$\leqq C \sum_{n=1}^{\infty} \frac{1}{n^{\alpha+1}} n^{\alpha-\frac{1}{2}} \leqq C,$$

by virtue of the fact that $\left\{\dfrac{\lambda_\nu^2}{W_\nu}\right\}$ is a decreasing sequence.

$$\Sigma_2 \leqq C \sum_{n=1}^{\infty} \frac{\lambda_{[n^{1/2}]}}{n^{\frac{1}{2}}(W_{[n^{1/2}]})^{\frac{1}{2}}} \frac{1}{n^{\alpha+\frac{1}{2}}} \left\{ \sum_{\nu=[n^{1/2}]+1}^{n} (n-\nu)^{2\alpha-2} W_\nu \, . \, \nu^2(a_\nu^2+b_\nu^2) \right\}^{\frac{1}{2}}$$

$$\leqq C \left(\sum_{n=1}^{\infty} \frac{\lambda_{[n^{1/2}]}^2}{n W_{[n^{1/2}]}} \right)^{\frac{1}{2}} \left\{ \sum_{n=2}^{\infty} \frac{1}{n^{2\alpha+1}} \sum_{\nu=[n^{1/2}]+1}^{n} (n-\nu)^{2\alpha-2} \nu^2 W_\nu (a_\nu^2+b_\nu^2) \right\}^{\frac{1}{2}}$$

$$\leqq C \left(\sum_{n=1}^{\infty} \frac{\lambda_n^2}{n W_n} \right)^{\frac{1}{2}} \left\{ \sum_{n=1}^{\infty} \frac{1}{n^{2\alpha+1}} \sum_{\nu=[n^{1/2}]+1}^{n} (n-\nu)^{2\alpha-2} \nu^2 W_\nu (a_\nu^2+b_\nu^2) \right\}^{\frac{1}{2}}$$

$$\leqq C \left(\sum_{n=1}^{\infty} \frac{1}{n^{2\alpha+1}} \sum_{\nu=1}^{n} (n-\nu)^{2\alpha-2} \nu^2 W_\nu (a_\nu^2+b_\nu^2) \right)^{\frac{1}{2}},$$

by lemma 2 and the hypothesis of the theorem.
And therefore

$$\Sigma_2^2 \leqq C \sum_{\nu=1}^{\infty} \nu^2 W_\nu (a_\nu^2+b_\nu^2) \sum_{n=\nu}^{\infty} (n-\nu)^{2\alpha-2} n^{-2\alpha-1}$$

$$\leqq C \sum_{\nu=1}^{\infty} \nu^2 W_\nu (a_\nu^2+b_\nu^2) \left(\frac{1}{\nu^2}\right)$$

$$\leqq C \sum_{\nu=1}^{\infty} W_\nu (a_\nu^2+b_\nu^2)$$

$$\leqq C,$$

by the hypothesis of the theorem and the fact that

$$\sum_{n=\nu}^{\infty} (n-\nu)^{2\alpha-2} n^{-2\alpha-1} = O\left(\frac{1}{\nu^2}\right)$$

for $\alpha > \frac{1}{2}$.

This completes the proof of the theorem.

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Department of Mathematics and Statistics,
Aligarh Muslim University,
Aligarh (UP)
India.