Canad. Math. Bull. Vol. **23** (4), 1980

NOETHERIAN RINGS IN WHICH EVERY IDEAL IS A PRODUCT OF PRIMARY IDEALS

BY
D. D. ANDERSON

The classical rings of number theory, Dedekind domains, are characterized by the property that every ideal is a product of prime ideals. More generally, a commutative ring R with identity has the property that every ideal is a product of prime ideals if and only if R is a finite direct sum of Dedekind domains and special principal ideal rings. These rings, called general Z.P.I. rings, are also characterized by the property that every (prime) ideal is finitely generated and locally principal. Some semblance of this factorization into prime ideals was restored by Emmy Noether who proved that every ideal in a Noetherian ring is a *finite intersection of primary ideals.*

The purpose of this note is to characterize the Noetherian rings with the property that every ideal is a *product* of primary ideals. Our main result is that every ideal in a Noetherian ring R is a product of primary ideals if and only if every nonmaximal prime ideal of R is a multiplication ideal (i.e., locally principal).

An ideal I is called a multiplication ideal if for every ideal $J \subseteq I$, there exists an ideal K with $J = KI$. We need the following facts about multiplication ideals:

(1) A multiplication ideal is locally principal [1, 760–761],

(2) A finitely generated ideal is a multiplication ideal if and only if it is locally principal [1, Theorem 3],

(3) A multiplication ideal I with rank $I > 0$ is finitely generated [2, Theorem 3],

(4) The product of multiplication ideals is a multiplication ideal [2, Corollary].

LEMMA 1. *Let R be a commutative ring with identity. Suppose that P is a prime ideal that is a multiplication ideal. If rank $P > 0$, then $\{P^n\}_{n=1}^{\infty}$ is the set of P-primary ideals. If rank $P = 0$, then there is a least positive integer m with $P_P^m = O_P$. In this case, $\{P^n\}_{n=1}^{m}$ is the set of P-primary ideals.*

Proof. First suppose that rank $P > 0$. Then P is finitely generated. Let Q be P-primary. Since P is finitely generated, $P^s \subseteq Q$ for some s. By passing to R/P^s, we see that it suffices to consider the case where rank $P = 0$. But then R_P is a

Received by the editors January 8, 1979 and in revised form April 30, 1979.

https://doi.org/10.4153/CMB-1980-068-2 Published online by Cambridge University Press

special principal ideal ring. Let m be the least positive integer with $P_P^m = O_P$. Since $P_P, P_P^2, \ldots, P_P^m$ are the only P_P-primary ideals, $P, P^{(2)}, \ldots, P^{(m)}$ are the only P-primary ideals. For each i, $1 \le i \le m$, $P^i \subseteq P^{(i)}$. Suppose that k is the largest positive integer with $P^{(i)} \subseteq P^k$. Since P is a multiplication ideal, so is P^k. Hence we can write $P^{(i)} = CP^k$ where $C \nsubseteq P$. (If $P^{(i)} \subseteq P^k$ for all positive integers k, then it is easily seen that $i = m$ and $P^m = P^{(m)}$.) But then $P^k \subseteq P^{(i)}$ since $P^{(i)}$ is P-primary and hence $P^k = P^{(i)}$. Thus $P_P^{(k)} = P_P^k = P_P^{(i)} = P_P^i$ so that $i = k$.

LEMMA 2. *Suppose that an ideal I in a commutative ring R with identity has a normal decomposition involving only prime ideals that are either maximal or multiplication ideals. Then I is a product of primary ideals.*

Proof. Suppose that $I = Q_1 \cap \cdots \cap Q_n$ is a normal decomposition where Q_i is P_i-primary with $P_1, \ldots, P_s$ maximal ideals and $P_{s+1}, \ldots, P_n$ multiplication ideals. Since the result is well-known if all the primes P_i are maximal, we may assume that $s < n$. By Lemma 1, we have $Q_i = P_i^{t_i}$ for some positive integer t_i for $i = s+1, \ldots, n$. We first show that $Q_{s+1} \cap \cdots \cap Q_n = P_{s+1}^{t_{s+1}} \cap \cdots \cap P_{P_n}^{t_n} = P_{s+1}^{t_{s+1}} \cdots P_n^{t_n}$. We note that there are no containment relations between the primes $P_{s+1}, \ldots, P_n$. For if $P_i \subsetneqq P_j$ where $s+1 \le i, j \le n$, then P_j is finitely generated (for rank $P_j > 0$) and hence $P_1 \subseteq \bigcap_{m=1}^{\infty} P_j^m$ [3, Theorem 2.2]. But then $P_j^{t_j} \supsetneqq P_i^{t_i}$ which contradicts the assumption that $I = Q_1 \cap \cdots \cap Q_n$ is a normal decomposition. Let $J = P_{s+1}^{t_{s+1}} \cap \cdots \cap P_n^{t_n}$. Now $P_{s+1}^{t_{s+1}}$ is a multiplication ideal and $J \subseteq P_{s+1}^{t_{s+1}}$; so we can write $J = C_{s+1} P_{s+1}^{t_{s+1}}$. Moreover, since $C_{s+1} P_{s+1}^{t_{s+1}} = J \subseteq P_{s+2}^{t_{s+2}}$ and $P_{s+1}^{t_{s+1}} \nsubseteq P_{s+2}$, we have $C_{s+1} \subseteq P_{s+2}^{t_{s+2}}$ since $P_{s+2}^{t_{s+2}}$ is P_{s+2}-primary. Since $P_{s+2}^{t_{s+2}}$ is a multiplication ideal, $C_{s+1} = C_{s+2} P_{s+2}^{t_{s+2}}$ and hence $J = C_{s+2} P_{s+1} P_{s+2}^{t_{s+2}}$. Continuing in this manner, we have $P_{s+1}^{t_{s+1}} \cap \cdots \cap P_n^{t_n} = J = C_n P_{s+1}^{t_{s+1}} \cdots P_n^{t_n} \subseteq P_{s+1}^{t_{s+1}} \cdots P_{s+1}^{t_{s+1}}$. It follows that $J = P_{s+1}^{t_{s+1}} \cap \cdots \cap P_n^{t_n} = P_{s+1}^{t_{s+1}} \cdots P_n^{t_n}$. Moreover, J being a product of multiplication ideals is a multiplication ideal. Since $I \subseteq J$, we have $I = (I:J)J$. But $(I:J) = (Q_1 \cap \cdots \cap Q_n : J) = (Q_1 : J) \cap \cdots \cap (Q_n : J) = (Q_1 : J) \cap \cdots \cap (Q_s : J)$ since $(Q_i : J) = R$ for $s+1 \le i \le n$. Moreover, each $(Q_i : J)$ is either P_i-primary or R for $i = 1, \ldots, s$. Delete the $(Q_i : J)$ that are equal to R. Since the ideals P_i $(1 \le i \le s)$ are maximal, the P_i-primary ideals $(Q_i : J)$ are comaximal and hence $(Q_1 : J) \cap \cdots \cap (Q_s : J) = (Q_1 : J) \cdots (Q_n : J)$. Thus $I = (Q_1 : J) \cdots (Q_s : J) P_{s+1}^{t_{s+1}} \cdots P_n^{t_n}$ is a product of primary ideals.

LEMMA 3. *Let P be a prime ideal and A an ideal with $A \nsubseteq P$. Suppose that I is an ideal with $AP \subseteq I \subseteq P$ and that I is a product of primary ideals. Then I is a multiple of P. In particular, if every ideal between P and AP is a product of primary ideals, then P/AP is a multiplication ideal in R/AP.*

Proof. Let $I = Q_1 \cdots Q_n$ where Q_i is P_i-primary. Since P is prime, we have, say, $Q_1 \subseteq P$. Now $P_P \supseteq Q_{1P} \supseteq I_P \supseteq (AP)_P = A_P P_P = P_P$. Thus $Q_{1P} = P_P$ and since Q_1 is primary, we must have $Q_1 = P$.

https://doi.org/10.4153/CMB-1980-068-2 Published online by Cambridge University Press

THEOREM. *For a Noetherian ring with identity, the following conditions are equivalent*:

(1) *every ideal of R is a product of primary ideals,*

(2) *every nonmaximal prime ideal of R is a multiplication ideal.*

Proof. (1)$\Rightarrow$(2). Let P be a nonmaximal prime ideal of R. We must show that P is a multiplication ideal. It suffices to show that P is locally principal. Let M be a maximal ideal of R. If $M \not\supseteq P$, then $P_M = R_M$. Thus we may suppose that $M \supsetneqq P$. Then by Lemma 3, P/MP is a multiplication ideal. Thus P_M/M_MP_M is a principal ideal in R_M/M_MP_M. By Nakayama's Lemma, P_M is a principal ideal of R_M.

(2)$\Rightarrow$(1). Lemma 2.

We end the paper by giving some examples of Noetherian rings in which every ideal is a product of primary ideals. Clearly this class of rings is closed under finite direct sums, homomorphic images, and rings of quotients. If we restrict ourselves to the local case, then by the main theorem of this paper these rings are characterized by the property that every nonmaximal prime ideal is principal. Local rings of this type may be put into three classes: (1) $\dim R = 0$, (2) $\dim R = 1$ and the minimal prime ideals are principal, and (3) $\dim R = 2$ and R is a UFD. Homomorphic images of the third type can be used to provide examples of the second type. Outside the local case, two-dimensional locally UFD's and their homomorphic images provide examples of Noetherian rings in which every ideal is a product of primary ideals.

REFERENCES

1. D. D. Anderson, *Multiplication ideals, multiplication rings, and the ring $R(X)$*, Canad. J. Math. XXVIII (1976), 760–768.

2. ——, *Some remarks on multiplication ideals*, (submitted).

3. ——, J. Matijevic and W. Nichols, *The Krull Intersection Theorem II*, Pacific J. Math. **66** (1976), 15–22.

THE UNIVERSITY OF IOWA
IOWA CITY, IOWA 52242