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A ONE-DIMENSIONAL WAVE EQUATION WITH NONLINEAR DAMPING

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Abstract. We consider a one-dimensional weakly damped wave equation, with a damping coefficient depending on the displacement. We prove the existence of a regular connected global attractor of finite fractal dimension for the associated dynamical system, as well as the existence of an exponential attractor.

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1. The Equation. For $u = u(x, t) : [0, \pi] \times \mathbb{R}^+ \to \mathbb{R}$, we consider the following one-dimensional nonlinear wave equation

$$\begin{cases} u_{tt} - u_{xx} + \sigma(u)u_t + g(u) = f, \quad x \in (0, \pi), \ t > 0, \\ u(0, t) = u(\pi, t) = 0, \quad t \in \mathbb{R}^+, \\ u(x, 0) = u_0(x), \quad x \in (0, \pi), \\ u_t(x, 0) = v_0(x), \quad x \in (0, \pi). \end{cases} \tag{1.1}$$

Here, $f \in L^2(0, \pi)$ is independent of time, whereas $\sigma \in C^1(\mathbb{R})$ is such that

$$\sigma(r) > 0, \qquad \forall r \in \mathbb{R}. \tag{1.2}$$

Finally, $g \in C^1(\mathbb{R})$ fulfills the dissipation inequality

$$\liminf_{|r|\to\infty} \frac{g(r)}{r} > -1. \tag{1.3}$$

Observe that 1 is the first eigenvalue of the self-adjoint (strictly) positive operator $-\frac{d^2}{dx^2}$, acting on $L^2(0, \pi)$, with Dirichlet boundary conditions.

This problem describes, for instance, the motion of a vibrating string with fixed endpoints in a viscous medium. In particular, u represents the displacement from equilibrium, while u_t is the velocity. The term $\sigma(u)u_t$ is a resistance force. The coefficient of viscous resistance σ is usually a positive constant; however, if the viscous medium embedding the string is stratified, then σ depends on u. The term $g(u) - f$ may correspond to a (nonlinear) elastic force. A similar model has been considered in [**1**], in connection with a quenching problem.

Notation. We denote by $\langle\cdot,\cdot\rangle$ and $\|\cdot\|$ the inner product and the norm in $L^2(0,\pi)$, and we consider the Hilbert spaces

$$\mathcal{H} = H_0^1(0,\pi)\times L^2(0,\pi) \qquad \text{and} \qquad \mathcal{V} = \big[H^2(0,\pi)\cap H_0^1(0,\pi)\big]\times H_0^1(0,\pi),$$

normed by, respectively,

$$\|(u,v)\|_{\mathcal{H}}^2 = \|u_x\|^2 + \|v\|^2 \qquad \text{and} \qquad \|(u,v)\|_{\mathcal{V}}^2 = \|u_{xx}\|^2 + \|v_x\|^2.$$

Then, we have the compact inclusion $\mathcal{V}\Subset\mathcal{H}$, along with the inequalities

$$\begin{aligned} \|u\| &\le \|u_x\|, && \forall u\in H_0^1(0,\pi),\\ \|u_x\| &\le \|u_{xx}\|, && \forall u\in H^2(0,\pi)\cap H_0^1(0,\pi). \end{aligned}$$

The quantities

$$E_0(u,v) = \frac{1}{2}\|(u,v)\|_{\mathcal{H}}^2 \qquad \text{and} \qquad E_1(u,v) = \frac{1}{2}\|(u,v)\|_{\mathcal{V}}^2$$

represent the energy and the higher-order energy, respectively, associated with the vector (u,v).

2. Preliminaries. With standard methods (see e.g. [**14**]), problem (1.1) is easily seen to generate a strongly continuous semigroup $S(t)$ acting on $\mathcal{H}$. In particular, the continuous dependence estimate

$$\|S(t)z_1 - S(t)z_2\|_{\mathcal{H}} \le e^{ct}\|z_1 - z_2\|_{\mathcal{H}} \tag{2.1}$$

holds for every $t\in\mathbb{R}^+$, for some $c\ge 0$ depending only on the norms of z_1, z_2. Let us just briefly mention how to obtain (2.1). If u^1, u^2 are two solutions to (1.1) corresponding to the initial values $z_1, z_2\in\mathcal{H}$, we multiply the difference of the respective equations by $\bar u_t$, where $\bar u = u^1 - u^2$ (indeed, this has to be done within a proper approximation scheme). Then, we end up with a differential inequality for the function $\|S(t)z_1 - S(t)z_2\|_{\mathcal{H}}^2$. Once suitable energy estimates are available (cf. the next section), the only problematic term to control is

$$\big\langle \sigma(u^1)u_t^1 - \sigma(u^2)u_t^2, u_t^1 - u_t^2\big\rangle.$$

For that, the continuous embedding $H_0^1(0,\pi)\subset L^\infty(0,\pi)$ and the assumptions on σ furnish

$$\big|\big\langle \sigma(u^1)u_t^1 - \sigma(u^2)u_t^2, \bar u_t\big\rangle\big| \le |\langle \sigma(u^1)\bar u_t, \bar u_t\rangle| + \big|\big\langle [\sigma(u^1)-\sigma(u^2)]u_t^2, \bar u_t\big\rangle\big| \le cE_0(\bar u, \bar u_t),$$

for some c depending only on the size of the initial data. An application of the Gronwall lemma completes the argument.

Our main theorem reads as follows (see [**3**, **11**, **12**, **15**] for definitions and related results on the subject).

THEOREM 2.1. *The strongly continuous semigroup $S(t)$ possesses a (unique) connected global attractor $\mathcal{A}\subset\mathcal{H}$, which coincides with the unstable manifold of the set $\mathcal{S}$ of stationary points of $S(t)$. In addition, $\mathcal{A}$ is contained and bounded in $\mathcal{V}$.*

As it is customary in the study of the asymptotic dynamics of semigroups, the first step towards the global attractor is the existence of a bounded set (in $\mathcal{H}$) absorbing in finite time all the trajectories departing from bounded sets (the so-called absorbing set). This, in particular, ensures that the damping coefficient $\sigma(u)$, accounting for the dynamical friction, cannot blow up. It is indeed clear from a physical viewpoint that a very large σ plays against energy dissipation: just consider the damped linear wave equation with a large constant σ, whose dissipation exponent is related to $1/\sigma$.

The way to produce uniform in time energy estimates has been suggested by Babin and Vishik in their pioneering work [**2**], namely, to multiply the equation by $u_t + \varepsilon u$ in $L^2(0, \pi)$, for some $\varepsilon > 0$ small. In the present case, this strategy can be pursued by introducing a suitable energy functional, and making use of [**4**, Lemma 2.7]. On the other hand, because of the hyperbolic nature of the equation, it is possible to prove quite easily the existence of a global Lyapunov functional. Then, exploiting an abstract result (cf. [**11**, **13**]), we can directly demonstrate the existence of the global attractor (of the desired regularity) without passing through the absorbing set. Of course, once we have the attractor, we also recover the absorbing set.

A quite interesting (and much harder) problem would be to consider (1.1) in dimension three. In that case, even assuming the natural bound

$$\sigma(r) \leq c(1 + |r|^2), \qquad \forall r \in \mathbb{R},$$

so that $\sigma(u)u_t \in H^{-1}(\Omega)$, our argument does not apply. Indeed, as it will be clear from the forthcoming proofs, we heavily rely on the continuous embedding $H_0^1(0, \pi) \subset L^\infty(0, \pi)$, which is false in dimensions greater than one.

We conclude this introductory notes reporting a modified form of the Gronwall lemma that will be needed in the sequel (see e.g. [**5**] for the proof).

LEMMA 2.2. *Let $\psi : \mathbb{R}^+ \to \mathbb{R}$ be an absolutely continuous function, which fulfills for some $\varepsilon > 0$ and almost every $t \geq 0$ the differential inequality*

$$\frac{d}{dt}\psi(t) + 2\varepsilon\psi(t) \leq h_1(t)\psi(t) + h_2(t),$$

where $\int_\tau^t h_1(y)\,dy \leq m_1 + \varepsilon(t - \tau)$, for all $\tau \in [0, t]$, and $\sup_{t\geq 0} \int_t^{t+1} |h_2(y)|dy \leq m_2$, for some constants $m_1, m_2 \geq 0$. Then there exist $M_1, M_2 \geq 0$ such that

$$\psi(t) \leq M_1|\psi(0)|e^{-\varepsilon t} + M_2, \qquad \forall t \in \mathbb{R}^+.$$

Moreover, if $m_2 = 0$ (that is, if $h_2 \equiv 0$), it follows that $M_2 = 0$.

3. The Lyapunov Functional.

We begin to prove that $S(t)$ is a gradient system, that is, there exists a global Lyapunov functional. For $u \in H_0^1(0, \pi)$, we define

$$G(u) = \int_0^\pi \int_0^{u(x)} g(r)\,dr\,dx \qquad \text{and} \qquad F(u) = -\langle f, u\rangle.$$

Then, we consider the function $\Phi_0 \in C(\mathcal{H}, \mathbb{R})$ given by

$$\Phi_0(u, v) = E_0(u, v) + G(u) + F(u).$$

It is apparent from (1.3) that Φ_0 satisfies the two inequalities

$$\nu E_0(u, v) - c \leq \Phi_0(u, v) \leq \Upsilon(E_0(u, v)),$$

for some $\nu > 0$, $c \geq 0$, and some increasing function $\Upsilon : \mathbb{R}^+ \to \mathbb{R}^+$. Besides, Φ_0 is decreasing along the trajectories of $S(t)$. Indeed, setting $(u(t), u_t(t)) = S(t)(u_0, v_0)$, we have the equality

$$\frac{d}{dt}\Phi_0(u, u_t) + \langle \sigma(u)u_t, u_t\rangle = 0. \tag{3.1}$$

In particular, because of (1.2), if $\Phi_0(u, u_t)$ is constant, then $u_t(x, t) = 0$ almost everywhere, so that

$$(u(t), u_t(t)) = (u^\star, 0) \in \mathcal{H}, \qquad \forall t \in \mathbb{R}^+,$$

where $u^\star$ solves the elliptic problem

$$\begin{cases} -u^\star_{xx} + g(u^\star) = f, & x \in (0, \pi) \\ u^\star(0) = u^\star(\pi) = 0. \end{cases}$$

This means that $(u^\star, 0) \in \mathcal{S}$, the set of stationary points of $S(t)$.

REMARK 3.1. It is also clear by (1.3) that the set $\mathcal{S}$ is bounded in $\mathcal{H}$.

Let us summarize the above results in the following proposition.

PROPOSITION 3.2. *There exists a global Lyapunov functional Φ_0 for $S(t)$, namely, a function $\Phi_0 \in C(\mathcal{H}, \mathbb{R})$ such that*

- $\Phi_0(z) \to \infty$ *if and only if* $\|z\|_{\mathcal{H}} \to \infty$*;*
- $\Phi_0(S(t)z)$ *is nonincreasing for any* $z \in \mathcal{H}$*;*
- *if* $\Phi_0(S(t)z) = \Phi_0(z)$ *for all* $t > 0$*, then* $z \in \mathcal{S}$.

REMARK 3.3. The Lyapunov functional yields, for any fixed initial data $(u_0, v_0) \in \mathcal{H}$, the existence of a positive constant C, depending (increasingly) only on the norm of (u_0, v_0), such that the corresponding solution satisfies

$$\sup_{t\geq 0}[\|u_x(t)\| + \|u_t(t)\|] \leq C.$$

In particular, using (1.2) and the continuous embedding $H_0^1(0, \pi) \subset L^\infty(0, \pi)$, we learn that

$$\sigma(r) \geq \sigma_0, \qquad \forall r \in \mathbb{R}, \tag{3.2}$$

for some $\sigma_0 = \sigma_0(C) > 0$. Then, integrating (3.1) on $\mathbb{R}^+$, we also obtain the integral control (redefining C accordingly)

$$\int_0^\infty \|u_t(t)\|^2 dt \leq C. \tag{3.3}$$

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4. Proof of Theorem 2.1. In light of Remark 3.1 and Proposition 3.2, by applying [**11**, Theorem 3.8.5] (cf. also [**13**]) we will reach the desired conclusion if we can show that, for every fixed bounded set $\mathcal{B} \subset \mathcal{H}$, the semigroup $S(t)$ admits the decomposition (possibly depending on $\mathcal{B}$)

$$S(t) = L(t) + N(t),$$

such that

$$\lim_{t\to\infty} [\sup_{z\in\mathcal{B}} \|L(t)z\|_{\mathcal{H}}] = 0 \qquad \text{and} \qquad \sup_{t\geq 0} \sup_{z\in\mathcal{B}} \|N(t)z\|_{\mathcal{V}} < \infty.$$

Hence, let $\mathcal{B} \subset \mathcal{H}$ be a *fixed* bounded set. For given initial data $(u_0, v_0) \in \mathcal{B}$, set

$$S(t)(u_0, v_0) = (u(t), u_t(t)), \quad L(t)(u_0, v_0) = (v(t), v_t(t)), \quad N(t)(u_0, v_0) = (w(t), w_t(t)),$$

with v and w solutions to the problems

$$\begin{cases} v_{tt} - v_{xx} + \sigma(u)v_t = 0, \\ v(0, t) = v(\pi, t) = 0, \\ v(0) = u_0, \\ v_t(0) = v_0, \end{cases} \tag{4.1}$$

and

$$\begin{cases} w_{tt} - w_{xx} + \sigma(u)w_t + g(u) = f, \\ w(0, t) = w(\pi, t) = 0, \\ w(0) = 0, \\ w_t(0) = 0. \end{cases} \tag{4.2}$$

We now state and prove some lemmata. All the following results hold uniformly as $(u_0, v_0) \in \mathcal{B}$. Until the end of this section, $c \geq 0$ will be a generic constant depending only on $\mathcal{B}$.

LEMMA 4.1. *There exists* $\nu = \nu(\mathcal{B}) > 0$ *such that*

$$E_0(v(t), v_t(t)) \leq ce^{-\nu t}.$$

The quite standard proof of this lemma, based on the multiplication of (4.1) by $v_t + \varepsilon v$ with $\varepsilon > 0$ small, is left to the reader. The only new ingredient here is the fact that, due to Remark 3.3, $\sigma(u)$ is bounded (from above and from below), with positive bounds depending on $\mathcal{B}$. As a byproduct, we obtain the boundedness of $E_0(w, w_t)$.

LEMMA 4.2. *For every* $\varepsilon > 0$*, there exists* $C_\varepsilon = C_\varepsilon(\mathcal{B})$ *such that*

$$\int_\tau^t [\|u_t(y)\| + \|v_x(y)\| + \|w_t(y)\|]\, dy \leq \varepsilon(t - \tau) + C_\varepsilon,$$

for all $t > \tau \geq 0$.

Proof. It is clear from Lemma 4.1 that

$$\int_\tau^t [\|v_x(y)\|^2 + \|v_t(y)\|^2]\, dy \leq c.$$

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Since, due to (3.3), the same inequality holds for u_t, we conclude that

$$\int_\tau^t \|w_t(y)\|^2 dy \leq c.$$

Hence, the Young inequality yields the thesis. □

LEMMA 4.3. *There holds*

$$\sup_{t\geq 0} E_1(w(t), w_t(t)) \leq c.$$

Proof. For $\eta \in (0, \frac{1}{4})$, we define the functional

$$\Phi_1(w, w_t) = E_1(w, w_t) - \langle g(u), w_{xx}\rangle - F(w_{xx}) + \eta\langle w_{xt}, w_x\rangle.$$

Then, in view of Remark 3.3, we have the two inequalities

$$\frac{1}{2}E_1(w, w_t) - c \leq \Phi_1(w, w_t) \leq 2E_1(w, w_t) + c. \tag{4.3}$$

Multiplying (4.2) by $-w_{xxt} - \eta w_{xx}$, on account of (3.2), we obtain

$$\begin{aligned}\frac{d}{dt}&\Phi_1(w, w_t) + (\sigma_0 - \eta)\|w_{xt}\|^2 + \eta\|w_{xx}\|^2 \\ &\leq \eta\langle g(u) + \sigma(u)w_t - f, w_{xx}\rangle - \langle g'(u)u_t, w_{xx}\rangle - \langle \sigma'(u)u_x w_t, w_{xt}\rangle.\end{aligned}$$

It is straightforward to check that

$$\eta\langle g(u) + \sigma(u)w_t - f, w_{xx}\rangle - \langle g'(u)u_t, w_{xx}\rangle \leq \frac{\eta}{2}\|w_{xx}\|^2 + c.$$

At this stage however, c depends on η, but this will be fixed eventually. The remaining term is controlled as

$$\begin{aligned}-\langle \sigma'(u)u_x w_t, w_{xt}\rangle &= -\langle \sigma'(u)v_x w_t, w_{xt}\rangle - \langle \sigma'(u)w_x w_t, w_{xt}\rangle \\ &\leq c\|v_x\|\|w_{xt}\|^2 + c\|w_t\|\|w_{xx}\|\|w_{xt}\|.\end{aligned}$$

Hence, using (4.3) and setting η small enough, we are led to

$$\frac{d}{dt}\Phi_1(w, w_t) + \varepsilon\Phi_1(w, w_t) \leq c(\|v_x\| + \|w_t\|)\Phi_1(w, w_t) + c,$$

for some $\varepsilon > 0$. Due to Lemma 4.2, the generalized Gronwall Lemma 2.2 yields

$$\Phi_1(w(t), w_t(t)) \leq c,$$

which, by (4.3), gives the result. □

Collecting Lemma 4.1 and Lemma 4.3, the proof of Theorem 2.1 is completed.

REMARK 4.4. In fact, Theorem 2.1 also holds for the semigroup associated to a generalized version of (1.1) (cf. [**8**]); namely,

$$\begin{cases} u_{tt} - u_{xx} + \sigma(u)h(u_t) + g(u) = f, & x \in (0, \pi), \quad t > 0, \\ u(0, t) = u(\pi, t) = 0, & t \geq 0, \\ u(x, 0) = u_0(x), & x \in (0, \pi), \\ u_t(x, 0) = v_0(x), & x \in (0, \pi), \end{cases}$$

where $h \in C^1(\mathbb{R})$ is globally Lipschitz continuous with $h(0) = 0$, and satisfies the following condition: for any $\varepsilon > 0$, there exists $\rho_\varepsilon > 0$ such that

$$[h(r_1) - h(r_2)](r_1 - r_2) \geq \rho_\varepsilon |r_1 - r_2|^2,$$

whenever $|r_1 - r_2| \geq \varepsilon$. We will not enter into the details of these calculations in this work.

5. Regular Exponentially Attracting Sets. We now show that there exists a closed ball $\mathcal{B}_1$ in $\mathcal{V}$ which is exponentially attracting in $\mathcal{H}$ and absorbs itself under the action of $S(t)$. First, as a byproduct of Theorem 2.1, we have a corollary.

COROLLARY 5.1. *The semigroup $S(t)$ possesses a bounded absorbing set $\mathcal{B}_0 \subset \mathcal{H}$.*

Once the existence of a bounded absorbing set $\mathcal{B}_0$ is established, Lemma 4.1 and Lemma 4.3 imply, in particular, the existence of a closed ball $\mathcal{K} \subset \mathcal{V}$ (hence, compact in $\mathcal{H}$) which is exponentially attracting for $S(t)$.

REMARK 5.2. For the generalized problem considered in Remark 4.4, such a set $\mathcal{K}$ is still attracting, but not exponentially attracting, since the condition $h'(r) \geq c > 0$ is not assumed.

To obtain the desired set $\mathcal{B}_1$, we possibly have to modify $\mathcal{K}$, in order to have the required absorbing property. To this aim, we need a lemma.

LEMMA 5.3. *There exist $\nu_1 > 0$, $R_1 \geq 0$, and an increasing positive function Π such that*

$$\sup_{t \geq 0} E_1(u(t), u_t(t)) \leq \Pi(R)e^{-\nu_1 t} + R_1,$$

whenever $E_1(u_0, v_0) \leq R$.

The proof of this lemma, based on a multiplication of (1.1) by $-u_{xxt} - \eta u_{xx}$ for $\eta > 0$, parallels to the one of Lemma 4.3, and is therefore omitted. Note that the exponent ν_1 does not depend on $\mathcal{B}$, due to the existence of the absorbing set $\mathcal{B}_0$.

It is then clear that, up to properly enlarging $\mathcal{K}$, we obtain the following proposition.

PROPOSITION 5.4. *There exists a closed ball $\mathcal{B}_1 \subset \mathcal{V}$ such that*

(i) *there is a positive increasing function M such that, for every bounded set $\mathcal{B} \subset \mathcal{H}$ with $R = \sup_{z \in \mathcal{B}} \|z\|_{\mathcal{H}}$, there holds*

$$\text{dist}_{\mathcal{H}}(S(t)\mathcal{B}, \mathcal{B}_1) \leq M(R)e^{-\nu t},$$

with $\nu = \nu(\mathcal{B}_0) > 0$ as in Lemma 4.1;

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(ii) *there is a time* $t_1 \geq 0$ *such that*

$$S(t)\mathcal{B}_1 \subset \mathcal{B}_1, \qquad \forall t \geq t_1.$$

Here, $\text{dist}_{\mathcal{H}}$ denotes the usual Hausdorff semidistance in $\mathcal{H}$.

6. Exponential Attractors. Finally we state and prove a result on the existence of an exponential attractor.

THEOREM 6.1. *Assuming* g' *locally Lipschitz continuous, the semigroup* $S(t)$ *possesses a regular exponential attractor, namely, a bounded set* $\mathcal{E} \subset \mathcal{V}$, *closed and of finite fractal dimension in* $\mathcal{H}$, *positively invariant for* $S(t)$, *and satisfying the following exponential attraction property:*

(EA) *there exist* $\omega > 0$ *and a positive increasing function* J *such that, for every bounded set* $\mathcal{B} \subset \mathcal{H}$ *with* $R = \sup_{z\in\mathcal{B}} \|z\|_{\mathcal{H}}$, *there holds*

$$\text{dist}_{\mathcal{H}}(S(t)\mathcal{B}, \mathcal{E}) \leq J(R)e^{-\omega t}.$$

Since the exponential attractor is, in particular, a compact attracting set, it necessarily contains the global attractor $\mathcal{A}$, which is the minimal compact attracting set.

COROLLARY 6.2. *If* g' *is locally Lipschitz continuous, then the global attractor* $\mathcal{A}$ *of* $S(t)$ *has finite fractal dimension in* $\mathcal{H}$.

The proof of Theorem 6.1 leans on the next abstract result from [**6**, **9**], adapted to the present case. We use the notation of Proposition 5.4.

LEMMA 6.3. *Let there exist* $t^\star \geq t_1$ *such that the following conditions hold.*

(C1) *The map*

$$(t, z) \mapsto S(t)z : [t^\star, 2t^\star] \times \mathcal{B}_1 \to \mathcal{B}_1$$

is Lipschitz continuous when $\mathcal{B}_1$ *is endowed with the* $\mathcal{H}$*-topology.*

(C2) *Setting* $S = S(t^\star)$, *there are* $\lambda \in (0, \frac{1}{2})$ *and* $\Lambda \geq 0$ *such that, for every* $z_1, z_2 \in \mathcal{B}_1$,

$$Sz_1 - Sz_2 = D(z_1, z_2) + K(z_1, z_2),$$

where

$$\|D(z_1, z_2)\|_{\mathcal{H}} \leq \lambda\|z_1 - z_2\|_{\mathcal{H}}$$

and

$$\|K(z_1, z_2)\|_{\mathcal{V}} \leq \Lambda\|z_1 - z_2\|_{\mathcal{H}}.$$

Then there exists a set $\mathcal{E} \subset \mathcal{B}_1$, *closed and of finite fractal dimension in* $\mathcal{H}$, *positively invariant for* $S(t)$, *such that*

$$\text{dist}_{\mathcal{H}}(S(t)\mathcal{B}_1, \mathcal{E}) \leq J_0 e^{-\omega_0 t},$$

for some $\omega_0 > 0$ *and* $J_0 \geq 0$.

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Hence, provided that (C1)-(C2) are verified, we have the thesis of Theorem 6.1, except that the basin of exponential attraction is $\mathcal{B}_1$, and not the whole space $\mathcal{H}$, as required. To fill this gap, we shall exploit the transitivity of the exponential attraction, devised in [**7**, Theorem 5.1]. Namely, if (2.1) holds, and

$$\mathrm{dist}_{\mathcal{H}}(S(t)\mathcal{B}, \mathcal{B}_1) \leq M(R)e^{-\nu t} \qquad \text{and} \qquad \mathrm{dist}_{\mathcal{H}}(S(t)\mathcal{B}_1, \mathcal{E}) \leq J_0 e^{-\omega_0 t},$$

then the desired property (EA) follows.

Thus, we are left to prove conditions (C1)-(C2) true. This will be done in the next section.

7. Verifying Conditions (C1)-(C2) of Lemma 6.3. Throughout this section, $c \geq 0$ will denote a generic constant depending only on $\mathcal{B}_1$.

• Since

$$\|S(t)z_1 - S(\tau)z_2\|_{\mathcal{H}} \leq \|S(t)z_1 - S(t)z_2\|_{\mathcal{H}} + \|S(t)z_2 - S(\tau)z_2\|_{\mathcal{H}},$$

condition (C1) follows directly from (2.1) and the bound

$$\sup_{t \geq t_1} \sup_{z \in \mathcal{B}_1} \|\partial_t S(t)z\|_{\mathcal{H}} \leq c.$$

Indeed, for $t \geq t_1$, $\|u_{xt}(t)\| \leq c$ by (ii) of Proposition 5.4, and $\|u_{tt}(t)\| \leq c$ by comparison in (1.1).

• To prove (C2), for $i = 1, 2$, consider initial data $z_i = (u_{0i}, v_{0i}) \in \mathcal{B}_1$, and denote the difference of the corresponding solutions to (1.1) by

$$\bar{u}(t) = u^1(t) - u^2(t).$$

Also, introduce the function

$$\phi(t) = \int_0^1 \sigma'(su^1(t) + (1-s)u^2(t))\, ds.$$

Notice that

$$\sup_{t \geq 0} \|\phi_x(t)\| \leq c. \tag{7.1}$$

Then, we can write

$$S(t)z_1 - S(t)z_2 = (\bar{v}(t), \bar{v}_t(t)) + (\bar{w}(t), \bar{w}_t(t)),$$

where $\bar{v}$ and $\bar{w}$ are the solutions to the systems

$$\begin{cases} \bar{v}_{tt} - \bar{v}_{xx} + \sigma(u^1)\bar{v}_t + \phi u_t^2 \bar{v} = 0, \\ \bar{v}(0,t) = \bar{v}(\pi, t) = 0, \\ \bar{v}(x,0) = u_{01} - u_{02}, \\ \bar{v}_t(x,0) = v_{01} - v_{02}, \end{cases} \tag{7.2}$$

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and

$$\begin{cases} \bar{w}_{tt} - \bar{w}_{xx} + \sigma(u^1)\bar{w}_t + \phi u_t^2 \bar{w} + g(u^1) - g(u^2) = 0, \\ \bar{w}(0,t) = \bar{w}(\pi,t) = 0, \\ \bar{w}(x,0) = 0, \\ \bar{w}_t(x,0) = 0. \end{cases} \tag{7.3}$$

LEMMA 7.1. *There exists* $\varpi > 0$ *such that*

$$E_0(\bar{v}(t), \bar{v}_t(t)) \le c e^{-\varpi t} \|z_1 - z_2\|_{\mathcal{H}}^2.$$

Proof. For $\eta > 0$, let us set

$$\Psi(\bar{v}, \bar{v}_t) = E_0(\bar{v}, \bar{v}_t) + \eta \langle \bar{v}, \bar{v}_t \rangle.$$

It is apparent that, for η small enough,

$$\frac{1}{2} E_0(\bar{v}, \bar{v}_t) \le \Psi(\bar{v}, \bar{v}_t) \le 2E_0(\bar{v}, \bar{v}_t).$$

Multiplying (7.2) by $\bar{v}_t + \eta \bar{v}$, thanks to 3.2, we have

$$\frac{d}{dt}\Psi(\bar{v}, \bar{v}_t) + (\sigma_0 - \eta)\|\bar{v}_t\|^2 + \eta\|\bar{v}_x\|^2 \le -\big\langle \phi u_t^2 \bar{v}, \bar{v}_t \big\rangle - \eta \langle \sigma(u^1)\bar{v}_t, \bar{v} \rangle - \eta \big\langle \phi u_t^2 \bar{v}, \bar{v} \big\rangle.$$

On account of (7.1), we easily obtain

$$-\big\langle \phi u_t^2 \bar{v}, \bar{v}_t \big\rangle - \eta \big\langle \phi u_t^2 \bar{v}, \bar{v} \big\rangle \le c \big\| u_t^2 \big\| E_0(\bar{v}, \bar{v}_t),$$

while

$$-\eta \langle \sigma(u^1)\bar{v}_t, \bar{v} \rangle \le \frac{\eta}{2} \|\bar{v}_x\|^2 + \eta c \|\bar{v}_t\|^2.$$

Hence, if η is small enough, we are led to

$$\frac{d}{dt}\Psi(\bar{v}, \bar{v}_t) + 2\varpi \Psi(\bar{v}, \bar{v}_t) \le c \big\| u_t^2 \big\| \Psi(\bar{v}, \bar{v}_t),$$

for some $\varpi > 0$. The thesis is then a consequence of (3.3) and Lemma 2.2. □

LEMMA 7.2. *There holds*

$$E_1(\bar{w}(t), \bar{w}_t(t)) \le c e^{ct} \|z_1 - z_2\|_{\mathcal{H}}^2.$$

Proof. Multiplying (7.3) by $-\bar{w}_{xxt}$, and using 1.2, we see that

$$\begin{aligned} \frac{d}{dt} E_1(\bar{w}, \bar{w}_t) + \sigma_0 \|\bar{w}_{xt}\|^2 \ &\le -\big\langle \phi_x u_t^2 \bar{w}, \bar{w}_{xt} \big\rangle - \big\langle \phi u_{xt}^2 \bar{w}, \bar{w}_{xt} \big\rangle - \big\langle \phi u_t^2 \bar{w}_x, \bar{w}_{xt} \big\rangle \\ &\quad - \big\langle g'(u^1)u_x^1 - g'(u^2)u_x^2, \bar{w}_{xt} \big\rangle - \big\langle \sigma'(u^1)u_x^1 \bar{w}_t, \bar{w}_{xt} \big\rangle. \end{aligned}$$

Notice that

$$\begin{aligned} -\big\langle g'(u^1)u_x^1 - g'(u^2)u_x^2, \bar{w}_{xt} \big\rangle \ &= -\langle g'(u^1)\bar{u}_x, \bar{w}_{xt} \rangle - \big\langle [g'(u^1) - g'(u^2)]u_x^2, \bar{w}_{xt} \big\rangle \\ &\le c\|\bar{u}_x\| \|\bar{w}_{xt}\| + c \big\| u_{xx}^2 \big\| \|\bar{u}_x\| \|\bar{w}_{xt}\| \\ &\le \sigma_0 \|\bar{w}_{xt}\|^2 + c\|\bar{u}_x\|^2, \end{aligned}$$

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and

$$-\langle \sigma'(u^1)u^1_x \bar{w}_t, \bar{w}_{xt}\rangle \leq c\|\bar{w}_{xt}\|^2.$$

Moreover, from (7.1),

$$\begin{aligned}
&-\langle \phi_x u^2_t \bar{w}, \bar{w}_{xt}\rangle - \langle \phi u^2_{xt} \bar{w}, \bar{w}_{xt}\rangle - \langle \phi u^2_t \bar{w}_x, \bar{w}_{xt}\rangle \\
&\leq c\|u^2_{xt}\|\|\bar{w}_x\|\|\bar{w}_{xt}\| + c\|u^2_t\|\|\bar{w}_{xx}\|\|\bar{w}_{xt}\| \\
&\leq cE_1(\bar{w}, \bar{w}_t).
\end{aligned}$$

In conclusion, we have

$$\frac{d}{dt}E_1(\bar{w}, \bar{w}_t) \leq cE_1(\bar{w}, \bar{w}_t) + c\|\bar{u}_x\|^2,$$

and the assertion follows from (2.1) and the Gronwall lemma. □

Exploiting Lemma 7.1, we choose $t^\star \geq t_1$ large enough such that

$$E_0(\bar{v}(t), \bar{v}_t(t)) \leq \frac{\lambda^2}{2}\|z_1 - z_2\|^2_{\mathcal{H}},$$

for a fixed $\lambda < \frac{1}{2}$. Choosing

$$D(z_1, z_2) = (\bar{v}(t^\star), \bar{v}_t(t^\star))$$

and

$$K(z_1, z_2) = (\bar{w}(t^\star), \bar{w}_t(t^\star)),$$

by virtue of Lemma 7.2, we obtain (C2).

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